数理解析研究所講究録 1264 巻 2002 年 159-172

PERIODIC SOLUTIONS FOR FORCED VAN DER POL TYPE EQUATIONS

CHIKAHIRO EGAMI AND NORIMICHI HIRANO

ABSTRACT. In the present paper, we will see that a Van der Pol type equations has a periodic solution when the forcing term is periodic. By showing the results of some simulations, we illustrate periodicity of the solutions. We also prove that a periodic solution found near the origin is a repellor.

Key Words

Van der Pol type equation, Forcing term, Periodic solution, Banach space, Topological (Leray-Schauder) degree, Repellor, Attractor, Subharmonic solution

1 INTRODUCTION

Let $f, g, e : \mathbb{R} \to \mathbb{R}$ be continuous functions. The Liénard type equation with forcing term

$$(L) \qquad u_{tt} + f(u)u_t + g(u) = e(t) \qquad t \in \mathbb{R}$$

has been studied by many authors due to its adoption to a wide variety of mechanical, electrical, biological and economical systems. The equation (L) is usually called Duffing type when $f(u) = \zeta$, or Van der Pol type when $g(u) = \eta u$, where ζ and η are constants.

In economics, there have been a number of elegant mathematical treatments of some of the traditional business cycle theories e.g. the treatment of Kaldor's model by Chang and Smyth (1971) and Schinasi (1982) and of a complete Keynesian system by Torre (1977). Because of one-dimensional relaxation oscillator just like Van der Pol type without an forcing term, their treatments concentrated on the question of the existence of limit cycles and consequently made use of the planar properties such as Poincaré-Bendixson theory and Jordan curve (cf. C. Chiarella [14, §2, §3, §7], K. Kawamata [15, pp.131-148]).

Recently, N-dimensional extension of the equation (L) has been studied by several authors. However, it is not easy for $N \geq 2$ to obtain similar results as one-dimensional cases. N-dimensional existence results for periodic solutions of the forced Van der Pol type are not yet established until now in comparison with those of the forced Duffing type (e.g., see [1] by J. Mawhin).

Throughout this paper, unless otherwise explicitly stated, $F, g : \mathbb{R}^N \to \mathbb{R}^N$ are continuous functions, and $e \in L^2(\mathbb{R}; \mathbb{R}^N)$ is a periodic function with period T. We consider the existence problem for periodic solutions of the forced N-dimensional Van der Pol type equation of the form

$$(V) \qquad u_{tt} + \frac{d}{dt}F(u) + g(u) = e(t) \qquad t \in \mathbb{R},$$

where $u(t) = (u_1(t), u_2(t), \dots, u_N(t))$, and $u_i(t) \in \mathbb{R}$ for each $t \in \mathbb{R}$ and each integer $i \in [1, N]$.

In the following section, we shall prove our main result for (V). We make use of the Leray-Schauder degree theory (cf. N. G. Lloyd [12, §4], K. Masuda [21, §23, §24]). In section

3, we give the results of the simulations for some concrete models. In section 4, we prove that the periodic solution found near the origin by simulations in section 3 is repellor.

2 Existance of Periodic Solutions

In this section, we establish an existence result for periodic solutions of (V). To state our result, we need some preliminaries.

For $x, y \in L^2([0,T];\mathbf{R}^N)$, let us define that

$$|x| = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}, \quad \|x\| = \left(\int_0^T |x(t)|^2\,dt\right)^{1/2}, \quad \langle x, y\rangle = \int_0^T \sum_{i=0}^{N} x_i(t) y_i(t)\,dt.$$

In the following, we put

$$H = \left\{u \in L^2([0,T];\mathbf{R}^N) : u(0) = u(T), u_t \in L^2([0,T];\mathbf{R}^N)\right\},$$

with the norm

$$\|u\|_H = \left(\|u\|^2 + \|u_t\|^2\right)^{1/2}.$$

We also put

$$\widetilde{H} = \left\{u \in H : \int_0^T u(t)\,dt = 0\right\}.$$

Further, given a set Ω, its closure is written $\overline{\Omega}$, its boundary $\partial\Omega$.

Our main result is the following,

Theorem 2.1 *Let $e \in \widetilde{H}$. If $F(u)$ and $g(u)$ satisfy the following conditions (F1), (F2) and (G1), then the differential equation (V) has at least one periodic solution with period T.*

$$\text{(F1)} \qquad F(u_1, u_2, \dots, u_N) = \begin{pmatrix} F_1(u_1) \\ F_2(u_2) \\ \vdots \\ F_N(u_N) \end{pmatrix},$$

where $F_i \in C^1(\mathbf{R};\mathbf{R})$ for each integer $i \in [1, N]$;

$$\text{(F2)} \qquad f_i(0) < 0 \quad \text{and} \quad \liminf_{|s|\to\infty} \frac{f_i(s)}{s^2} > 0,$$

where f_i denotes F_i';

$$\text{(G1)} \qquad g(u) = Au \quad \text{and} \quad \det A \neq 0,$$

where A is a $N \times N$ constant matrix.

To prove Theorem 2.1 we need some lemmata. However, we omit those proofs here.

Lemma 2.2 *Suppose that (F1), (F2) and (G1). Let $e \in \widetilde{H}$. If $u(t)$ is a T-periodic solution of the differential equation (V), then $u \in \widetilde{H}$ and*

$$\|u\| \leq 2T\|u_t\|.$$

Remark. For the inequality in Lemma 2.2, which is called Poincaré's inequality, we can have still stricter evaluation (cf. C. P. Gupta, J. J. Nieto and L. Sanchez [3]). However, the present evaluation is enough for lemmata in this paper.

Lemma 2.3 *Let $\lambda_0 \in (0, \min(\frac{2\pi}{T}, 1))$. Then the set*

$$S_1 = \left\{ u \in H : u_{tt} + \frac{d}{dt}F(u) + \lambda A u = e(t) \qquad \text{for some} \quad \lambda \in [\lambda_0, 1] \right\}$$

is bounded in H.

Lemma 2.4 *The set*

$$S_2 = \left\{ u \in H : u_{tt} + \delta\frac{d}{dt}F(u) + \lambda_0 A u = \delta e(t) \qquad \text{for some} \quad \delta \in [0, 1] \right\}$$

is bounded in H.

Here, let us introduce the topological (Leray-Schauder) degree and its properties.

Definition 2.5 *Let X be Banach space, D be a bounded open subset of X and $\mathcal{L}$ be a compact mapping from $\overline{D}$ into X. If $\mathcal{L}x \neq p$ for any $x \in \partial D$, then we define the Leray-Schauder degree of $\mathcal{L}$ at $p \in X$ relative to D to be* $\deg(\mathcal{L}, D, p)$.

The Leray-Schauder degree is known to have the following three properties.

(i) $\deg(I, D, p) = 1$ for every $p \in D$, where I denotes identity mapping.

(ii) $\deg(\mathcal{L}, D, p) \neq 0$ implies $\mathcal{L}x = p$ for some $x \in D$.

(iii) If $\mathcal{H}(\xi)$ is a homotopy of compact mapping with $\mathcal{H}(\xi)x \neq p$ for any $x \in \partial D$ and any $\xi \in [0, 1]$, then $\deg(\mathcal{H}(\xi), D, p)$ is independent of ξ.

Proof of Theorem 2.1 Let $B_r(0)$ be the open ball in $\widetilde{H}$ centered at 0 with radius $r > 0$. For each $\lambda \in [0, 1]$ and $\delta \in [0, 1]$, we define a operator $\mathcal{T}(\lambda, \delta) : \widetilde{H} \to \widetilde{H}$ by

$$\mathcal{T}(\lambda, \delta)u = -\delta \int_0^t F(u(s))\, ds - \lambda \int_0^t\!\!\int_0^s Au(\tau)\, d\tau\, ds + \delta \int_0^t\!\!\int_0^s e(\tau)\, d\tau\, ds + C,$$

where

$$C = \delta \int_0^T F(u(s))\, ds + \lambda \int_0^T\!\!\int_0^s Au(\tau)\, d\tau\, ds - \delta \int_0^T\!\!\int_0^s e(\tau)\, d\tau\, ds.$$

Because $\mathcal{T}(\lambda, \delta)$ is a integral operator, $\mathcal{T}(\lambda, \delta)U$ is equicontinuous for any bounded subset U of $\widetilde{H}$. Then $\mathcal{T}(\lambda, \delta)U$ is relatively compact by Ascoli-Arzela theorem. Hence $\mathcal{T}(\lambda, \delta)$ is a compact operator. Furthermore, we put

$$\mathcal{H}(\xi)u = \begin{cases} \mathcal{T}(1 - 3(1 - \lambda_0)\xi, 1)u & \text{for } \xi \in [0, \frac{1}{3}] \text{ and } u \in \widetilde{H}, \\ \mathcal{T}(\lambda_0, 2 - 3\xi)u & \text{for } \xi \in [\frac{1}{3}, \frac{2}{3}] \text{ and } u \in \widetilde{H}, \\ \mathcal{T}(3\lambda_0(1 - \xi), 0)u & \text{for } \xi \in [\frac{2}{3}, 1] \text{ and } u \in \widetilde{H}. \end{cases}$$

Then $\mathcal{H}(\xi)$ is a homotopy of compact mapping on $\widetilde{H}$. It is obvious from the definition of $\mathcal{H}(\xi)$ that u is a fixed points of $\mathcal{H}(0)$ if and only if u is a solution of (V). We have by the definition of $\mathcal{H}(\xi)$ that

$$S_1 = \left\{ u \in \widetilde{H} : u = \mathcal{H}(\xi)u \qquad \text{for some} \quad \xi \in [0, \frac{1}{3}] \right\}$$

$$S_2 = \left\{ u \in \widetilde{H} : u = \mathcal{H}(\xi)u \qquad \text{for some} \quad \xi \in [\frac{1}{3}, \frac{2}{3}] \right\}.$$

Then by Lemma 2.3 and Lemma 2.4, we have that there exists large M_0 such that

$$u \neq \mathcal{H}(\xi)u \qquad \text{for any } u \in \partial B_{M_0}(0) \text{ and any } \xi \in [0, \frac{2}{3}].$$

For $\xi \in [\frac{2}{3}, 1]$, we can see that each fixed point $u \in \widetilde{H}$ of $\mathcal{H}(\xi)$ satisfies

$$u = -(3\lambda_0(1-\xi)) \int_0^t \int_0^s Au(\tau)\, d\tau\, ds + C. \tag{2.1}$$

Then we immediately have that $u = 0$ is the unique solution of (2.1), we also have that

$$u \neq \mathcal{H}(\xi)u \qquad \text{for any } u \in \partial B_{M_0}(0) \text{ and any } \xi \in [\frac{2}{3}, 1].$$

While, we should just calculate $\deg(I - \mathcal{H}(0), B_{M_0}(0), 0)$ to consider the fixed point problem $u = \mathcal{H}(0)u$. Since $u \neq \mathcal{H}(\xi)u$ for any $u \in \partial B_{M_0}(0)$ and any $\xi \in [0,1]$, then by the property (iii) of degree,

$$\deg(I - \mathcal{H}(0), B_{M_0}(0), 0) = \deg(I - \mathcal{H}(1), B_{M_0}(0), 0) = \deg(I, B_{M_0}(0), 0).$$

Using the property (i) of degree, $\deg(I, B_{M_0}(0), 0) = 1$. Therefore by the property (ii) of degree, we obtain that $H(0)$ has at least one fixed point in $\widetilde{H}$, and Theorem 2.1 is proved. □

3 Van der Pol Oscillator

In section 3, We showed existence of periodic solution of (V). Then we would like to find out where a periodic solution with period T exits. In this section, we introduce three concrete examples and illustrate the results of simulations for each model.

3.1 Preliminary

We define three kinds of periodic solutions with mutually different character.

Definition 3.1 *Let $u \in L^2([0,T]; R^N)$ be a periodic solution with period T of (V). If there exists a neighborhood U of $\Gamma = \{(u(t), u_t(t)) : t \in [0,T]\}$ such that*

$$\lim_{t \to +\infty} d((v(t), v_t(t)), \Gamma) = 0 \qquad \textit{for each} \quad (v_0, v_{t0}) \in U,$$

where $v(t)$ is a solution of (V) with initial value $(v(0), v_t(0)) = (v_0, v_{t0}) \in R^N \times R^N$ and $d(\cdot,\cdot)$ denotes usual Euclid distance, then u is said to be attractive or attractor.

Definition 3.2 *In contrast of Definition 3.1, if there exists U of Γ such that*

$$\lim_{t \to -\infty} d((v(t), v_t(t)), \Gamma) = 0 \qquad \textit{for each} \quad (v_0, v_{t0}) \in U,$$

then u is said to be repellor.

Definition 3.3 *If u is a periodic solution with the integral multiple of T, then u is said to be subharmonic solution.*

3.2 Normal Model

We first introduce the most basic oscillating circuit with a negative resistor devised by Van der Pol. On Fig.3.1, L, C and R stands for inductance, capacitance and resistance,

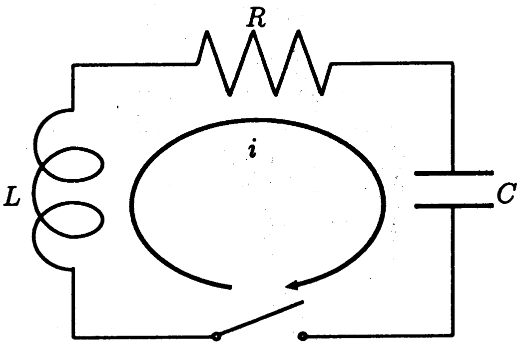

Fig. 3.1: Van der Pol Oscillator

respectively. This R, is called negative resistor, has the nonlinear property for current i such that

$$R(i) = -r_0 + r_1 i + r_2 i^2, \tag{3.1}$$

where r_0, r_1 and r_2 is nonnegative. Let τ be time. Then the circuit equation corresponding to Fig.3.1 is formalized by the following

$$Li_\tau + R(i)i + \frac{1}{C}\int i(\tau)\,d\tau = 0.$$

Differentiating above equation by τ, we have

$$i_{\tau\tau} - \frac{1}{L}\left(r_0 - 2r_1 i - 3r_2 i^2\right) i_\tau + \frac{1}{LC} i = 0.$$

Here, transforming τ into $\sqrt{LC}t$ and putting $i = \sqrt{\frac{r_0}{3r_2}}x$, we have

$$x_{tt} - r_0\sqrt{\frac{C}{L}}\left(1 - \frac{2r_1}{\sqrt{3r_0 r_2}}x - x^2\right)x_t + x = 0. \tag{3.2}$$

Put $\varepsilon = r_0\sqrt{\frac{C}{L}}$ now. If we suppose $r_1 = 0$, then we obtain the Van der Pol type equation

$$x_{tt} + \varepsilon(x^2 - 1)x_t + x = 0, \tag{M1}$$

where $x(t) \in \mathbb{R}$. ε represents the nonlinearity of the system (M1) and (3.1) holds the essence of Self-induced oscillation. In case that ε is equal to zero, (M1) is actually just a linear oscillator. It is known as a relaxation oscillation that the solutions of the autonomous system (M1) have the unique limit cycle for each ε on (x, x_t) plane by Poincaré-Bendixson Theorem (cf. F. Verhulst [10, §4.3]). Fig.3.2 gives the ω-limit set of the orbits. Where ε change from 0 to 20 by 2 step. The horizontal axis and the vertical axis indicates $x(t)$ and $x_t(t)$, respectively. We can see in Fig.3.2 the amplitude of limit cycle becomes large as ε grows. For example, the period of limit cycle for $\varepsilon = 2, \varepsilon = 6.5$ and $\varepsilon = 15$ is about $7.63, 13.79$ and 26.80.

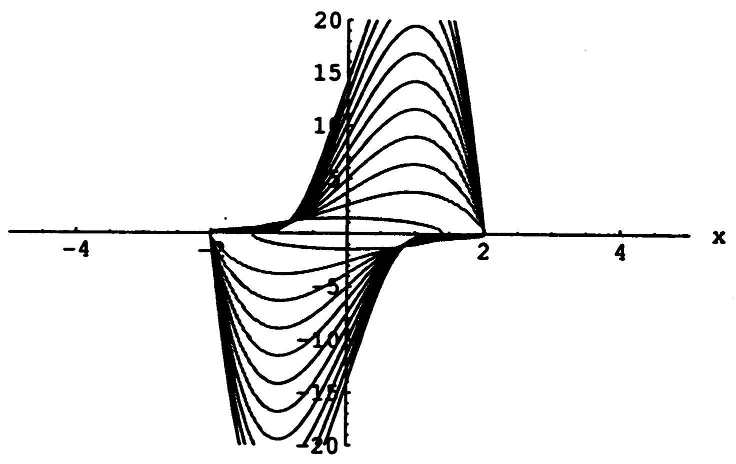

Fig. 3.2: Limit Cycle

3.3 Forced Model

We next introduce the Van der Pol oscillator with external power source. On Fig.3.3, E is the voltage of external power source. Using transformation similar to (3.2), we can formulate the circuit equation corresponding to Fig.3.3 as follows

$$x_{tt} - r_0\sqrt{\frac{C}{L}}\left(1 - \frac{2r_1}{\sqrt{3r_0r_2}}x - x^2\right)x_t + x = \sqrt{\frac{3r_2}{r_0}}\sqrt{\frac{C}{L}}E\cos t \tag{3.3}$$

We put $\varepsilon = r_0\sqrt{\frac{C}{L}}$ and $B = \sqrt{\frac{3r_2}{r_0}}\sqrt{\frac{C}{L}}E$. We call B forcing coefficient. If we assume $r_1 = 0$, then we have the forced Van der Pol type equation

$$x_{tt} + \varepsilon(x^2 - 1)x_t + x = B\cos t. \tag{M2}$$

We show three results of simulations of the model (M2). Then we have to set a initial value on $[0, 2\pi] \times \mathbf{R}^N \times \mathbf{R}^N$. Let us set four initial values $(t_0, x(t_0), x_t(t_0))$ at $(0,0,0), (0,0,3), (0,-1,1)$ and $(0,1,5)$. Let ε be fixed at 10.0 and B be set at 2.0, 6.5

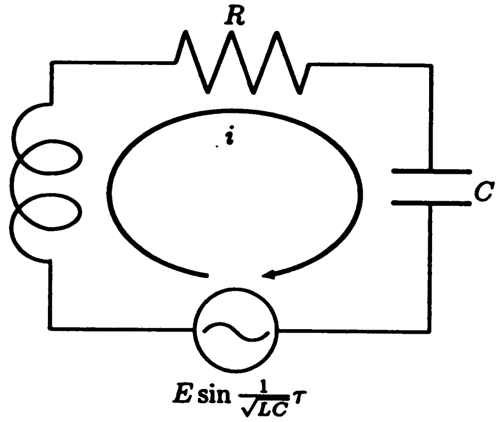

Fig. 3.3: Van der Pol Oscillator with External power source

and 15.0. It is rather more important for us than the relation between initial values and orbits whether a periodic solution exists or where a periodic solution was observed or its period. Fig.3.4, Fig.3.5 and Fig.3.6 are 3-dimensional plots and projections onto $t = 0$ of time series of x and x_t of (M2) in case that the amplitude of external force B is weak 2.0, strong 15.0 and middle 6.5, respectively. In Fig.3.4, we can find a 2π-periodic repellor in the region of $t < 0$ which oscillate near the origin with small amplitude, and a 6π-periodic subharmonic solution in the region of $t > 0$ in which all four orbits reach. In Fig.3.5, we can observe a 2π-periodic attractor in the region of $t > 0$ in which all four orbits was attracted as time progresses. In the region of $t < 0$, we can hardly calculate because the orbits diverge instantly to infinity for all four initial values. Fig.3.6 gives a complicated state that we have a repellor in the region of $t < 0$ and both an attractor and a subharmonic solution in the region of $t > 0$.

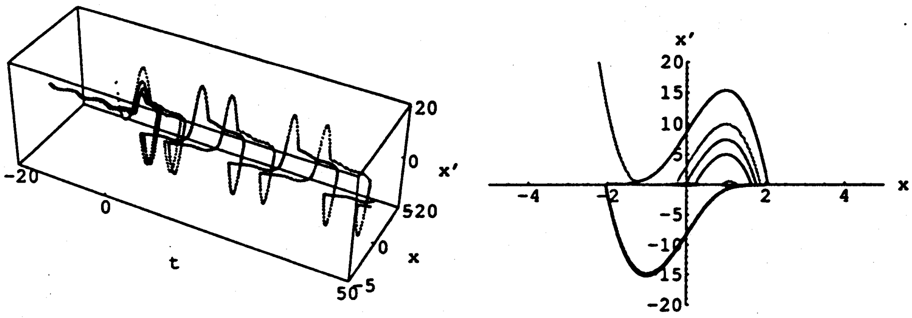

Fig. 3.4: Case of $B = 2$

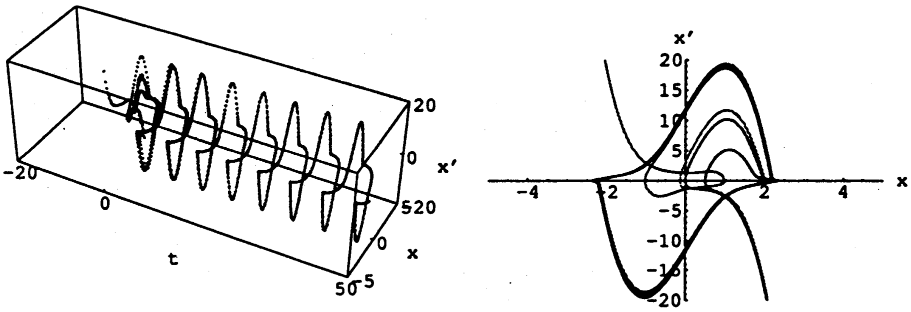

Fig. 3.5: Case of $B = 15$

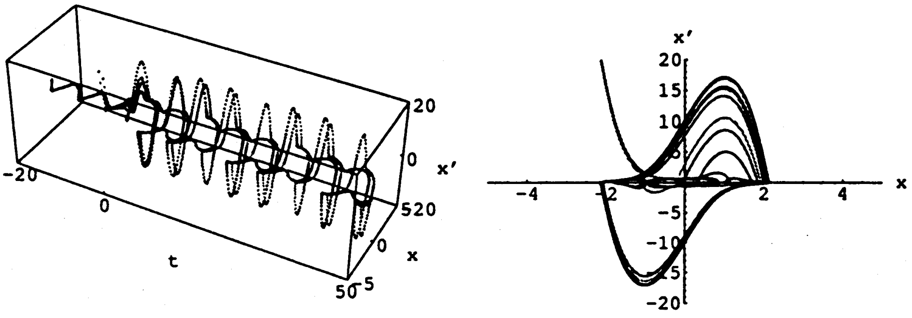

Fig. 3.6: Case of $B = 6.5$

Fig.3.7 and Fig.3.8 are drawn in order to give clearly the orbits of three periodic solutions. Fig.3.7 is the 3-dimensional plot of $(t \bmod 2\pi, x(t), x_t(t))$ for large $|t|$ sufficiently, and Fig.3.8 is its projection onto $t = 0$. In Fig.3.8, we can see that the oscillation with the smallest amplitude is repellor, the orbit is attractor which oscillate by the inner side of the largest swing, and the orbit is subharmonic solution which also has the small amplitude crossing x axis while oscillating with the largest amplitude.

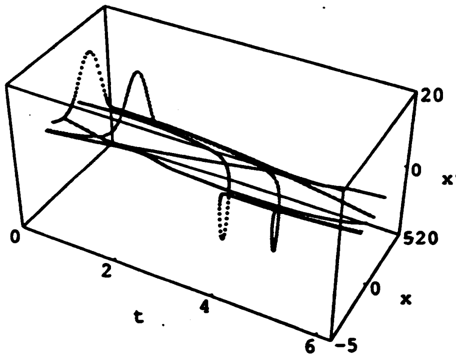

Fig. 3.7: Invariant Set for $|t| \gg 1$

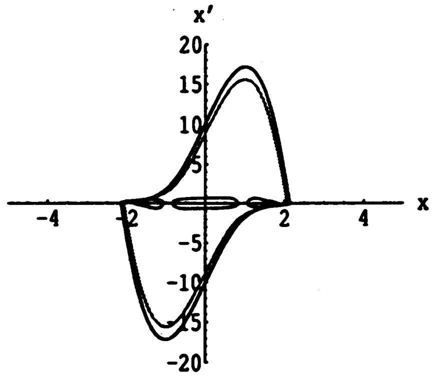

Fig. 3.8: Periodic Solutions

In Fig.3.9, Fig.3.10 and Fig.3.11, $(t \bmod T, x(t))$ are drawn for $t \in [100, 200]$, $t \in [100, 200]$ and $t \in [-200, -100]$ to investigate the exact periods of periodic solutions, where $T = 2\pi, 2\pi$ and 6π, respectively. In order to get the exact period, we have to choose T appropriately so that the curves $(t \bmod T, x(t))$ may be overlapped at one.

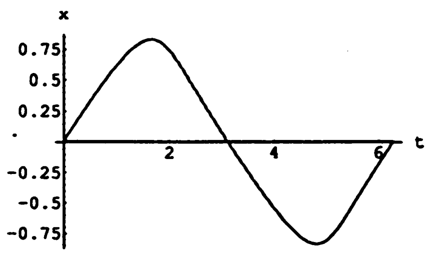

Fig. 3.9: Repellor

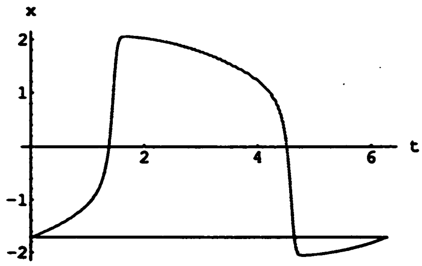

Fig. 3.10: Attractor

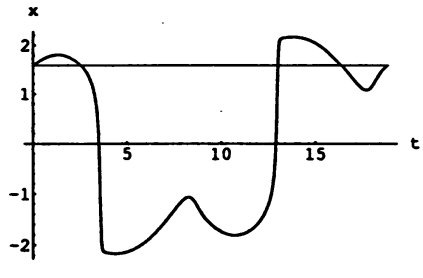

Fig. 3.11: Subharmonic Solution

3.4 Coupled Model with Forcing term

We introduce the coupling model connected two oscillating circuits with external power source.

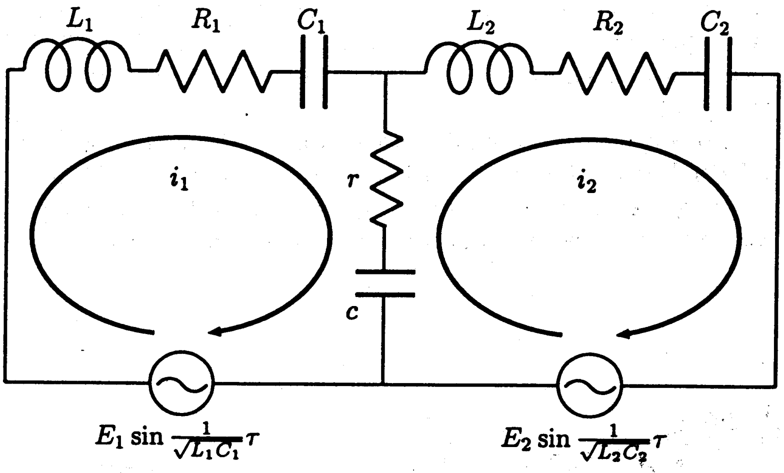

Fig. 3.12: Coupled Van der Pol Oscillator with External power source

On the Fig.3.12, c and r represents the small capacitance added artificially and the minute resistance included in the lead portion of the circuit, respectively. The characteristic of negative resistance is described by

$$R_n(i) = -r_{n0} + r_{n1}i + r_{n2}i^2 \qquad \text{for} \quad n = 1, 2.$$

Now assume $\sqrt{L_1C_1} = \sqrt{L_2C_2}$ and put $i_1 = \sqrt{\frac{r_{10}}{3r_{12}}}x$ and $i_2 = \sqrt{\frac{r_{20}}{3r_{22}}}y$. By using same transformation as (3.3), we have the following circuit equation corresponding to Fig.3.12

$$\begin{aligned}
x_{tt} - r_{10}\sqrt{\frac{C_1}{L_1}}\left(1 - \frac{2r_{11}}{\sqrt{3r_{10}r_{12}}}x - x^2\right)x_t + x &= \sqrt{\frac{3r_{12}}{r_{10}}}\sqrt{\frac{C_1}{L_1}}E_1 \cos t \\
&\quad + \frac{C_1}{c}(y - x) + r\sqrt{\frac{C_1}{L_1}}(y_t - x_t), \\
y_{tt} - r_{20}\sqrt{\frac{C_2}{L_2}}\left(1 - \frac{2r_{21}}{\sqrt{3r_{20}r_{22}}}y - y^2\right)y_t + y &= \sqrt{\frac{3r_{22}}{r_{20}}}\sqrt{\frac{C_2}{L_2}}E_2 \cos t \\
&\quad + \frac{C_2}{c}(x - y) + r\sqrt{\frac{C_2}{L_2}}(x_t - y_t).
\end{aligned}$$

Here we put $k_n = C_n/c$, $\varepsilon_n = r_{n0}\sqrt{\frac{C_n}{L_n}}$ and $B_n = \sqrt{\frac{3r_{n2}}{r_{n0}}}\sqrt{\frac{C_n}{L_n}}E_n$. If we suppose $r_{n1} = 0$, $r \langle\langle 1$, then we can build up our main model

$$\text{(M3)} \qquad \begin{aligned}
x_{tt} - \varepsilon_1\left(1 - x^2\right)x_t + x &= B_1 \cos t + k_1(y - x), \\
y_{tt} - \varepsilon_2\left(1 - y^2\right)y_t + y &= B_2 \cos t + k_2(x - y),
\end{aligned}$$

where we call k_n coupling coefficient.

We show two results of simulations of the model (M3). Then we have to set a initial value on $[0, 2\pi] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$. Let $\varepsilon_1, \varepsilon_2$ and k_1, k_2 be fixed at $2.0, 10.0$ and $0.5, 0.3$, respectively. We suppose $B_1 = B_2$.

We first show case of weak external force $B_1 = B_2 = 1.5$. Let us set four initial values $(t_0, x(t_0), x_t(t_0), y(t_0), y_{(}t_0))$ at $(0,0,0,0,0)$, $(0,0,-1,0,-1)$, $(0,-2,-5,-1,-10)$ and $(0,-2,5,-2.5,-10)$.

Fig.3.13 and Fig.3.14 are time series plotting $(t, x(t), x_t(t))$ and $(t, y(t), y_t(t))$, respectively. In both Fig.3.13 and Fig.3.14, we can observe a 2π-periodic repellor in the region of $t < 0$ and a 6π-periodic subharmonic solution in the region of $t > 0$. Then $x(t)$ of this subharmonic solution in Fig.3.13 appears like 2π-periodic. But watching $y(t)$ in Fig.3.14, we have its period is 6π.

Furthermore, we expect that, even if a model was multidimensionalize, the amplitude of each variable depend on ε_i of each dimension. In this case, the amplitude of $x(t)$ and $y(t)$ depend on ε_1 and ε_2, respectively. However, if we suppose that coupling coefficients k_i is larger, then we may have a more complicated situation of the orbits.

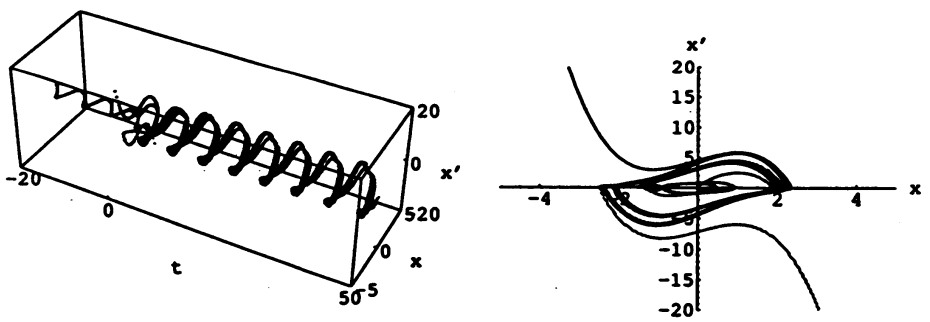

Fig. 3.13: $(t, x(t), x_t(t))$ in case of $B_1 = B_2 = 1.5$

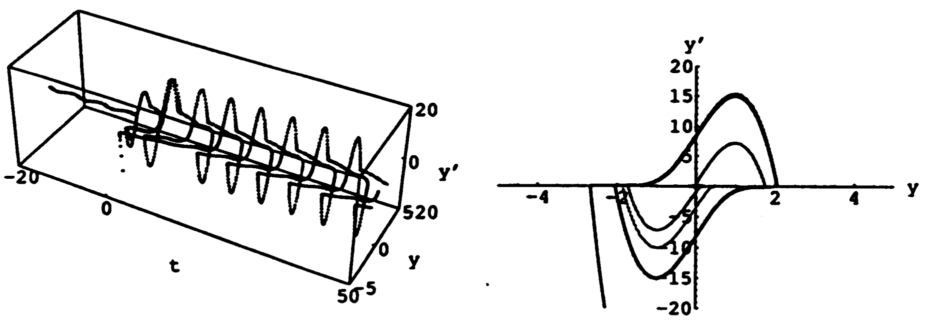

Fig. 3.14: $(t, y(t), y_t(t))$ in case of $B_1 = B_2 = 1.5$

Fig.3.15, Fig.3.16 and Fig.3.17, Fig.3.18 give the period of repellor and subharmonic solution, respectively. We can check that each period is 2π and 6π.

We next show case of strong external force $B_1 = B_2 = 15$. Let us set four initial values $(t_0, x(t_0), x_t(t_0), y(t_0), y_t(t_0))$ at $(0,0,0,0,0)$, $(0,-2,-5,-1,-10)$, $(0,-2,5,-2.5,-10)$ and $(0,3,-1,3,1)$.

In Fig.3.19 and Fig.3.20, we can observe a 2π-periodic attractor in the region of $t > 0$ like $B = 15$ of case in the last section.In the region of $t < 0$, the orbits diverge instantly to infinity for all four initial values.Fig.3.21 and Fig.3.22 give the period of attractor is 2π.

In the end, by the results of some simulations, we can expect that forced Van der Pol system has an attractor in case that the external force is strong or a repellor in case that the external force is weak. But it is not easy to investigate completely setting of parameters for

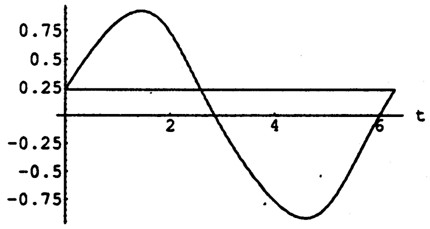

Fig. 3.15: Repellor ($t \bmod 2\pi, x(t)$)

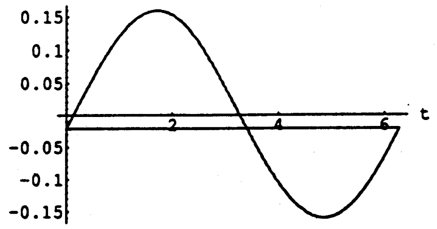

Fig. 3.16: Repellor ($t \bmod 2\pi, y(t)$)

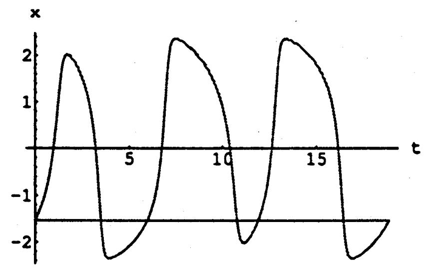

Fig. 3.17: Subharmonic($t \bmod 6\pi, x(t)$)

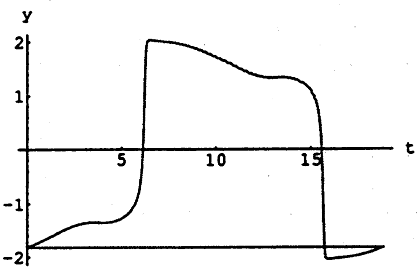

Fig. 3.18: Subharmonic($t \bmod 6\pi, y(t)$)

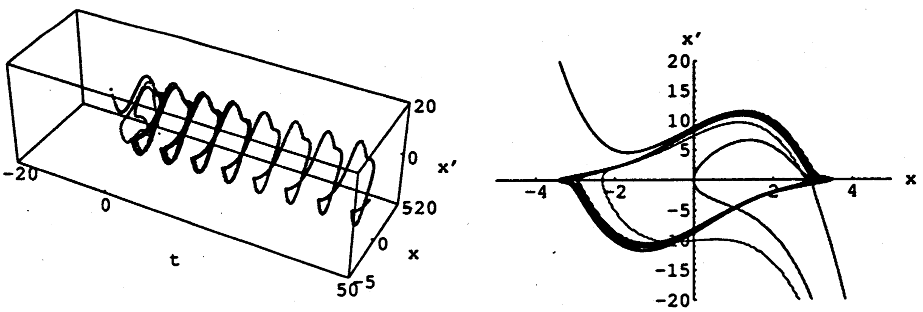

Fig. 3.19: ($x(t), x_t(t)$) in case of $B_1 = B_2 = 15$

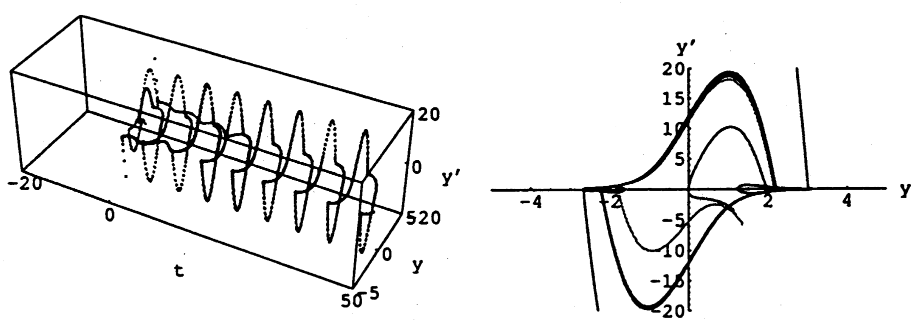

Fig. 3.20: ($y(t), y_t(t)$) in case of $B_1 = B_2 = 15$

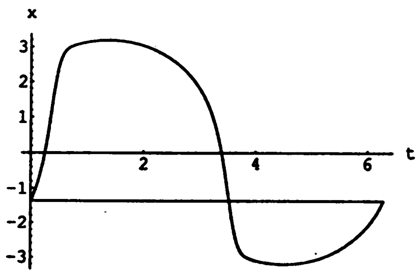

Fig. 3.21: Attractor $(t \bmod 2\pi, x(t))$

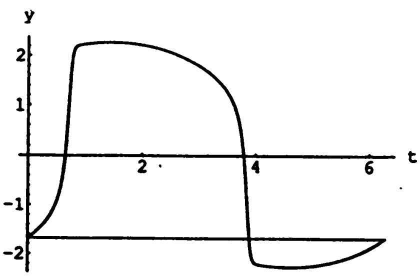

Fig. 3.22: Attractor $(t \bmod 2\pi, y(t))$

all cases that a subharmonic solution exists (e.g. regarding the results for one-dimensional system, see J. E. Flaherty and F. C. Hoppensteadt [2]).

4 Repellor

In this section , we prove that the periodic solution found near the origin by simulations in section 3 is repellor.

Proposition 4.1 *Let ρ be a small constant, $u \in L^2(R; R^N)$ be a T-periodic solution of (V) with $\|u\| \le \rho$ and A is a Hermite conjugate matrix. Then u is a repellor.*

Proof. Using condition (F2), we have that, there exists $\rho_i > 0$ for $i \in [1, N]$ such that

$$f_i(u_i) < 0 \qquad \text{for} \quad |u_i| < \rho_i,$$

then put $\rho = \min_{i \in [1,N]} \rho_i$. Putting

$$f(u) = \begin{pmatrix} f_1(u_1) & 0 & \dots & 0 \\ 0 & f_2(u_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f_N(u_N) \end{pmatrix},$$

we can rewrite (V) to

$$u_{tt} + f(u)u_t + Au = e(t). \tag{4.1}$$

Let $u \in L^2(R; R^N)$ be a T-periodic solution of (4.1) with $\|u\| \le \rho$. Further let h be a constant and $u + h\phi$ be a solution (4.1). Then we have

$$(u + h\phi)_{tt} + f(u + h\phi)(u + h\phi)_t + A(u + h\phi) = e(t). \tag{4.2}$$

By (4.1) and (4.2), we have

$$\phi_{tt} + \frac{f(u + h\phi) - f(u)}{h\phi} u_t \phi + f(u + h\phi)\phi_t + A\phi = 0.$$

By letting h go to 0 for the equation above, we get the linearized equation

$$\phi_{tt} + f(u)\phi_t + (f'(u)u_t + A)\phi = 0, \tag{4.3}$$

where $f'(u)$ is Hessian of $f(u)$. Here we rewrite the equation above as a system of first order ordinary equations of the form

$$\begin{pmatrix} \phi_t \\ \chi_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f'(u)u_t - A & -f(u) \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{4.4}$$

We consider the initial value problem of autonomous system (4.4) with $(\phi(0), \chi(0)) = (\phi_0, \chi_0) \in \mathbb{R}^N \times \mathbb{R}^N$. We put $\Phi(t) = (\phi(t), \chi(t))$ for $t \geq 0$. Then by Floquet's theorem (e.g., see [9, §1.4] by F. C. Hoppensteadt), we have that the fundamental solution $\Phi(t)$ of (4.4) can be written in the form

$$\Phi(t) = Q(t)\exp(\Lambda t),$$

where $Q(t) = \{q_{ij}\}$ is a matrix such that each element $q_{ij}(t)$ is a T-periodic function, and Λ is a Jordan matrix. To prove that u is a repellor, it is sufficient to see that each eigenvalue λ_i of Λ is positive. Let φ is a solution of (4.3). Then we have

$$\varphi_{tt} + f(u)\varphi_t + (f'(u)u_t + A)\varphi = 0.$$

Integrating the equation above over $[0, t]$, we have

$$\varphi_t(t) + (f(u)\varphi)(t) + A\int_0^t \varphi(s)\,ds = \varphi_t(0) + (f(u)\varphi)(0)$$

Here we put $\sigma = \varphi_t(0) + (f(u)\varphi)(0)$. We also put $\phi(t) = \int_0^t \varphi(s)\,ds$. Then we have

$$\psi_{tt} + f(u)\psi_t + A\psi = \sigma \qquad \text{for } t \geq 0.$$

Then multiplying the equality above by ψ_t and integrating over $[0, t]$, we get

$$\begin{aligned} |\psi_t(t)|^2 + \langle\!\langle A\psi(t), \psi(t)\rangle\!\rangle \geq\ & |\psi_t(0)|^2 + \langle\!\langle A\psi(0), \psi(0)\rangle\!\rangle \\ & + \frac{1}{2}\int_0^t |\psi_t(s)|^2\,ds + \langle\!\langle \sigma, \psi(t) - \psi(0)\rangle\!\rangle. \end{aligned} \tag{4.5}$$

where we put $\langle\!\langle x, y\rangle\!\rangle = \sum_{i=1}^N x_i y_i$ for $x, y \in \mathbb{R}^N$. Suppose that there exists a negative eigenvalue λ_i of Λ. Then by choosing the initial value $(\varphi(0), \varphi_t(0))$ appropriately, we have that for some $D > 0$,

$$|\psi_t| \leq De^{\lambda_i t}, |\psi| \leq De^{\lambda_i t} \qquad \text{for all } t \geq 0.$$

This implies that $\lim_{t\to\infty} |\psi_t(t)|^2 + \langle\!\langle A\psi(t), \psi(t)\rangle\!\rangle = 0$ and $\lim_{t\to\infty}\langle\!\langle \sigma, \psi(t) - \psi(0)\rangle\!\rangle = 0$. This contradicts to (4.5). This completes the proof. □

References

[1] Jean Mawhin, *An Extension of a Theorem of A.C.Lazer on Forced Nonlinear Oscillations*, Journal of Mathematical Analysis and Applications, Vol.40, pp.20-29, (1972).

[2] J.E.Flaherty, F.C.Hoppensteadt, *Frequency Entrainment of a Forced van der Pol Oscillator*, Studies in Applied Mathematics, Vol.58, pp.5-15, (1978).

[3] Duane W.Storti, Per G.Reinhall, *Stability of in-phase and out-of-phase modes for a pair of linearly coupled van der Pol oscillators*, Nonlinear dynamics, pp.1-23, Series on Stability, Vibration and Control Systems. Series B, Vol.2, World Scientific Publishing, River Edge, NJ, (1997).

[4] Nguyen Phuong Các, *Periodic Solutions of a Liénard equation with forcing term*, Nonlinear Analysis, Vol.43, pp.403-415, (2001).

[5] Chaitan P.Gupta, Juan J.Nieto, Luis Sanchez, *Periodic Solutions of Some Lienard and Duffing Equations*, Journal of Mathematical Analysis and Applications, Vol.140, pp.67-82, (1989).

[6] Norman Levinson, *A Second Order Differential Equation with Singular Solutions*, Annals of Mathematics, Vol.50, No.1, pp.127-153, (1949).

[7] M.L.Cartwright, J.E.Littlewood, *Some Fixed Point Theorems*, Annals of Mathematics, Vol.54, No.1, pp.1-37, (1951).

[8] John Guckenheimer, Philip Holmes, "Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields", Applied Mathematical Sciences, 42, Springer-Verlag, (1983).

[9] Frank C. Hoppensteadt, "Analysis and Simulation of Chaotic Systems", Applied Mathematical Sciences, 94, Springer-Verlag, (2000).

[10] Ferdinand Verhulst "Nonlinear Differential Equations and Dynamical Systems", Springer-Verlag, (1990).

[11] Wolfgang Walter "Ordinary Differential Equations ", Springer-Verlag, (1998).

[12] N.G.Lloyd, "Degree Theory", Cambridge University Press, (1978).

[13] G.Hardy, J.E.Littlewood, G.Pólya, "INEQUALITIES", Cambridge University Press, (1934).

[14] Carl Chiarella, "The Elements of a Nonlinear Theory of Economic Dynamics", LectureNotes in Economics and Mathematical Systems, 343, Springer-Verlag, (1990).

[15] Toru Maruyama, Wataru Takahashi, "Nonlinear and Convex Analysis in Economic Theory", LectureNotes in Economics and Mathematical Systems, 419, Springer-Verlag, (1995).

[16] Gianni Ricci, Kumaraswamy Velupillai, "Growth Cycles and Multisectoral Economics: the Goodwin Tradition", LectureNotes in Economics and Mathematical Systems, 309, Springer-Verlag, (1988).

[17] 宮寺功, "関数解析", 理工学社, (1972).

[18] 青木利夫, 高橋渉, 平野載倫, "演習集合位相空間", 培風館, (1985).

[19] 浅田統一郎, "成長と循環のマクロ動学", 日本経済評論社, (1997).

[20] 佐藤力, "非線形振動論", 基礎物理科学シリーズ, 2, 朝倉書店, (1970).

[21] 増田久弥, "非線形数学", 新数学講座, 15, 朝倉書店, (1985).

CHIKAHIRO EGAMI† AND NORIMICHI HIRANO‡

Division of Information Media Environment Sciences,
Graduate School of Environment and Information Sciences,
Yokohama National University,
79-5 Tokiwadai, Hodogaya, Yokohama, Kanagawa, 240-8501
E-mail address: †chika@hiranolab.jks.ynu.ac.jp, ‡hirano@math.sci.ynu.ac.jp