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A Percolation Model of Stock Price Fluctuations

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Abstract

It is widely known that distributions of stock-price fluctuations show "fat tails." This report explains the fat-tail distributions as a result of local interaction between traders of bounded sight. Percolation theory, a theory of statistical physics, is employed to model the markets. It enables us not only to reproduce the fat-tail distribution, but to define and calculate the floor and ceiling for stock prices.

1 Introduction

Most financial theories, such as the Black-Sholes model (1973), usually assume stock price returns are normally distributed. Mandelbrot (1963) and Fama (1965), however, had already pointed out that stock return distributions usually show higher kurtosis or fatter tails than the normal distribution. In other words, stock prices move more frequently and to a greater degree than the Gaussian models predict.

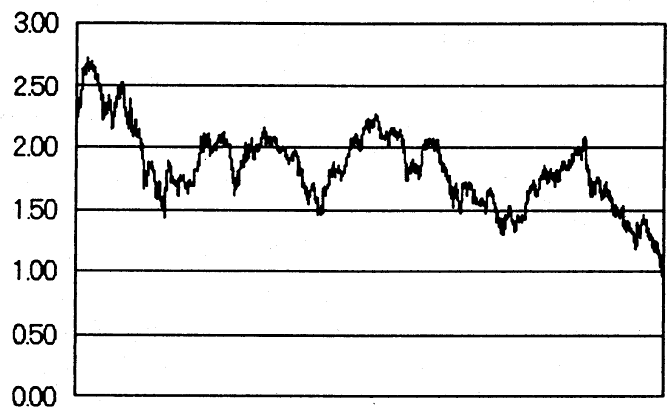

Fig.1.1: NIKKEI 225 from Jan.1991 to Sep.2001

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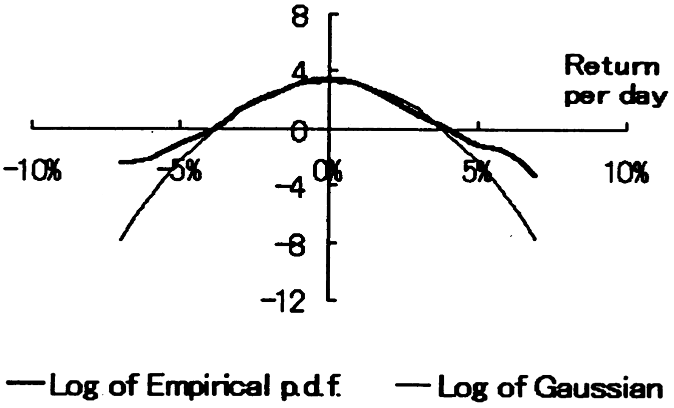


Fig.1.2: The log of the empirical p.d.f. for NIKKEI 225 returns per day

Fig.1.2 shows the log of the empirical probability density function (p.d.f.) for NIKKEI 225 returns per day and the log of the Gaussian with the same mean and variance as the empirical data. For the figure, the empirical frequency of returns larger than 5% per day is higher than the Gaussian predicts. Actually, NIKKEI 225 has changed by more than 7% per day on six occasions in these 11 years, although the Gaussian model would need 2,000 years to experience such a fluctuation six times.

Many research papers have investigated the statistical properties of stock price fluctuations, however, the focus has not been on the reasons why the distributions have these fat tails. This paper shows that the fat tails can be explained by local interaction between traders in stock markets. The main assumption is that traders have bounded sight, which means that each of them has no interest in price changes of stocks he does not hold. This behavior causes localised interaction between the traders because buying or selling has an effect only on other traders sharing the same stocks.

Percolation theory, a theory of statistical physics, is employed to model the local interaction. Stauffer and Penna (1998) also applied percolation theory to stock markets. As illustrated in Section 2, however, their model was based on the herding effect by which traders follow trends without considering any economic data. Stauffer and Penna assumed people go to a neighborhood bank or broker for investment advice, so that they form a cluster sharing the same advice. On the other hand, this paper models the stock market as a network of locally interacting traders. It is a different way from Stauffer-Penna, and it enables us to calculate the "floor" or "ceiling" for stock prices.

This paper is constructed as follows. Section 2 provides a short summary of percolation theory, and gives the Stauffer-Penna model as an example. Section 3, the main part of this paper, gives the model and some concepts. It will be shown that distributions generated by numerical simulations of the model have fat tails. In section 4, parameters of the model are estimated using NIKKEI 225 Index from 1991 to 2001. Moreover, the floor and ceiling for NIKKEI 225 are defined and calculated. Section 5 gives concluding

2 Percolation Theory and the Stauffer-Penna Model

In 1957, Broadbent and Hammersley developed the **percolation theory** as a means to design gas masks (Broadbent-Hammersley, 1957). The essence of their idea can be explained by the following example. Suppose that a fire breaks out in a large orchard, where trees are planted at the vertices of a square lattice $\boldsymbol{Z}^2$. Trees in the neighborhood of a burning tree will catch fire with probability $p \in [0,1]$. The probability p is sometimes called the **influence rate**.

The fire is supposed to be fierce enough to burn trees out instantaneously, so that the spread of fire happens only once for each tree (Fig.2.1).

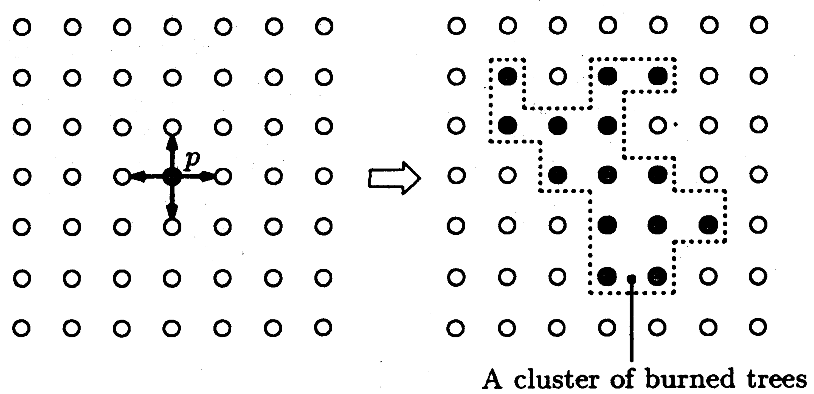


Fig.2.1: Percolation of a forest fire

The fire results in a cluster of burned trees. Let C be a formed cluster, that is a set of vertices of $\boldsymbol{Z}^2$ which belong to the cluster, and $|C|$ be a size of the cluster C.

Definition 1 *The* ***percolation probability*** $\theta(p)$ *is the probability that the origin belongs to an infinite cluster for each* $p \in [0,1]$*, and defined by*

$$\theta(p) = \boldsymbol{P}_p(\,|C| = \infty\,) = 1 - \sum_{n=1}^{\infty} \boldsymbol{P}_p(\,|C| = n\,), \tag{1}$$

where $\boldsymbol{P}_p(\,|C| = n\,)$ *denotes the probability that the cluster size* $|C|$ *equals* n *when the influence rate* p *is given.*

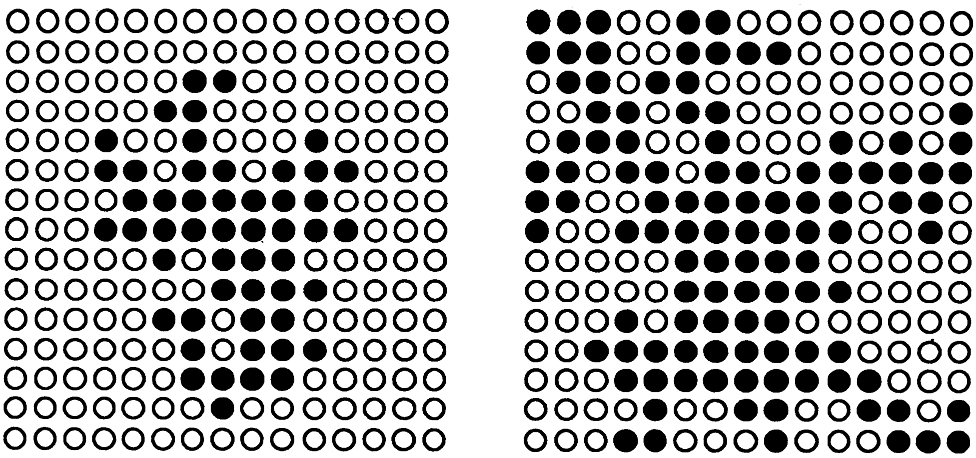
The following theorem is the reason why the percolation theory has grown into the biggest interests in the study of statistical physics and probability theory.

Theorem 1 *There uniquely exists* $p^c \in (0,1)$ *such that*

$$\theta(p) \begin{cases} = 0 & (p < p^c) \\ > 0 & (p > p^c) \end{cases} . \tag{2}$$

(Proof) *See Grimmett (1999), pp.15-19.*

Theorem 1 means that the cluster will be finite with probability 1 if $p < p^c$, but it will become infinite with positive probability if $p > p^c$ (Fig.2.2). The critical value p^c is called the **critical probability**. The above forest fire model (Fig.2.1) is called (2-dimensional) **site percolation**, and in the case $p^c = 0.592745\cdots$ is known.



(i) A Finite Cluster: $p < p^c$ (ii) An Infinite Cluster: $p > p^c$

Fig.2.2: The phase transition in percolation

Apparently, $\theta(p)$ is a non-decreasing function of the influence rate p, and both of $\theta(0) = 0$ and $\theta(1) = 1$ are satisfied. It is believed that $\theta(p)$ behaves as indicated in Fig.2.3, which shows that a "phase" of the model changes as p increases.

Stauffer and Penna (1998) applied the percolation theory to stock markets in order to explain fat tails of stock return distributions. They emphasized a herding effect in stock markets, that is, traders' tendency to follow market trend without looking at any economic facts. Traders are supposed to be distributed on a square lattice with density p, and they go to a neighborhood bank or broker for investment advice, so that the people who go to the same bank will share the same advice, and they form a percolation cluster of traders sharing the same expectation. Each cluster randomly decides to buy (with probability $\alpha/2$), to sell (with probability $\alpha/2$), or not to trade (with probability $1-\alpha$) during the current time period. Let $C_+^1(t)$, $C_+^2(t)$, $C_+^3(t)$, $\cdots$ be buying clusters, and $C_-^1(t)$, $C_-^2(t)$, $C_-^3(t)$, $\cdots$ selling clusters at period t (Fig.2.4). Then the return $\Delta S/S$ of the stock S at t is propotional to excess demand:

$$\frac{\Delta S_t}{S_t} = \rho\left(\sum_j \left|C_+^j(t)\right| - \sum_j \left|C_-^j(t)\right|\right), \tag{3}$$

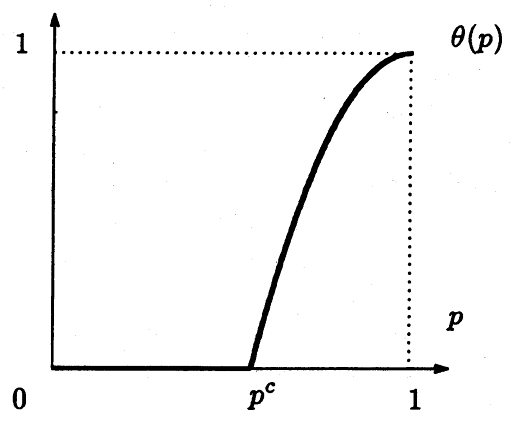


Fig.2.3: The percolation probability

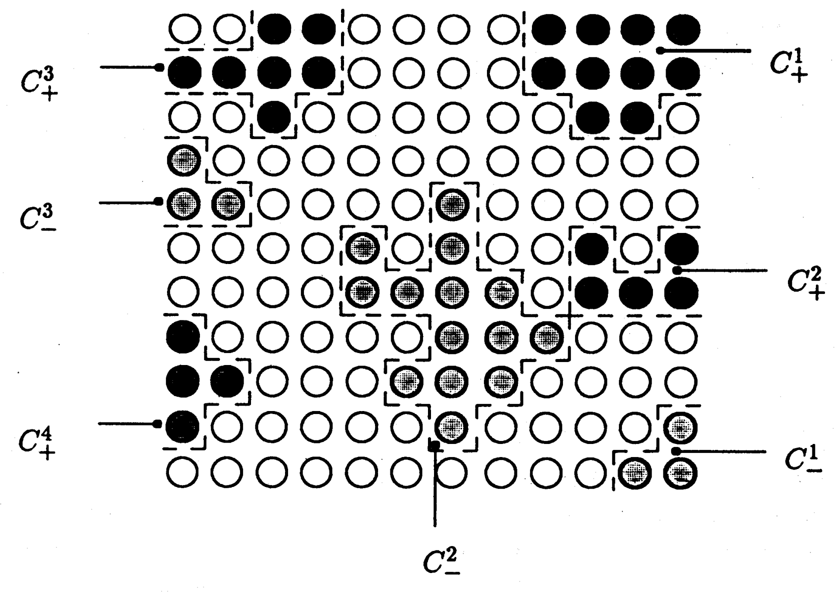


Fig.2.4: The Stauffer-Penna model

where $\Delta S_t = S_{t+1} - S_t$. Stauffer and Penna investigated how the model behaves when the activity parameter α increases from small values towards unity, and showed by computer simulations that the generated distributions have fat tails when $0 < \alpha \ll 1$, but that they resemble Gaussian when $\alpha \sim 1$.

Their model succeeded in generating fat-tail distributions, however, their assumption of the herding effect by which traders form percolation clusters seems somewhat unrealistic. In fact, they admitted that their model's assumption may be valid only for, for example, markets in 19th century because in modern markets "everybody on Earth shares the same information within minutes". Adding to this, their assumption admits only constant influence rate p. This means that their model cannot have the property of phase transition that is the most fascinating aspect of percolation theory. The model of this paper, however, is based on another assumption, that is, bounded sight of traders. This assumption not only seems more plausible than that of Stauffer-Penna, but can accept variable influence rates. In other words, the model of this paper has the phase transition property.

3 The Model

In this section, we will give the model of this paper. The model is specified for simplicity of computer simulations. The results will be used to show the model does indeed have fat tails.

Suppose N traders, and trader $i\,(=1,\cdots,N)$ holds M_i kinds of stocks, $B_i^1,\cdots,B_i^{M_i}$. For example, when $M_1 = \cdots = M_N = 2$ and

$$B_1^1 = B_2^2,\ B_2^1 = B_3^2,\ \cdots\ ,\ B_{N-1}^1 = B_N^2, \quad \textit{and} \quad B_N^1 = B_1^2,$$

the market has N kinds of stocks, and the structure of the market can be represented by a 1-dimensional torus S^1 (Fig.3.1). Each trader is supposed to look at only the stocks he holds and a stock price index S, such as NIKKEI 225 or S&P500. In other words, he has no interest in (or does not notice) selling or buying of stocks he does not hold. For simplicity, the contents of each trader's portfolio are assumed to be unchanged through time.

At every period t, traders behave as follows:

(Step 1) At the beginning of period t, a randomly chosen trader i becomes bull (with probability 1/2) or bear (with probability 1/2). If he becomes bull, he sends a buying order and tries to increase his holdings $B_i^1,\cdots,B_i^{M_i}$. If he becomes bear, on the other hand, he tries to sell his holdings. The reason why trader i wants to buy or sell his stocks may be rational (e.g. he has received positive or negative news on fundamentals), irrational (e.g. fads, trends, or sunspots), or exogenious (e.g. he needs money for marriage).

(Step 2) By looking at trader i's buying (or selling) order, neighborhood traders of i, i.e. traders sharing stocks with trader i, can reason that trader i becomes bull (or bear).

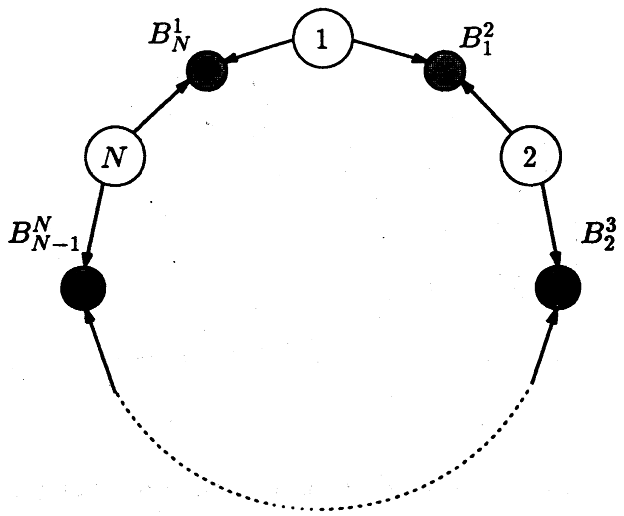


Fig.3.1: The S^1 network structure of the market

If they see that trader i is trying to buy the stocks, each of them becomes bull with probability p^u, or remains neutral with probability $1 - p^u$. Traders influenced by trader i's bull order also send buying orders to the market.

If they see trader i's selling, on the other hand, each of them becomes bear with probability p^d, or remains neutral with probability $1 - p^d$. A trader who becomes bear also tries to sell his holdings.

In other words, the influence rate of a bull order is given by p^u, and that of a bear order is p^d. The influence rates, p^u and p^d, are assumed to be functions of the stock price index S: that is, $p^u = p^u(S_t)$ is a decreasing function of the stock price index S_t at period t, and $p^d = p^d(S_t)$ is an increasing function of S_t (Fig.3.2).

(Step 3) The return $\Delta S_t / S_t$ of the stock price index S_t is assumed to be propotional to the number of bull (or bear) traders at every t: that is,

$$\frac{\Delta S_t}{S_t} = \rho \times sgn(C_t) \times |C_t|, \tag{4}$$

where $\Delta S_t = S_{t+1} - S_t$, ρ is a positive constant, $|C_t|$ is the size of the cluster of bull (or bear) traders C_t at t, and

$$sgn(C_t) = \begin{cases} +1 & (if\ C_t\ is\ a\ bull\ cluster) \\ -1 & (if\ C_t\ is\ a\ bear\ cluster) \end{cases} . \tag{5}$$

Equation (4) defines a stochastic process $S = (S_t)_{t=1,2,3,\cdots}$ of the price index. Note that both of the price and the quantity of the stocks are ignored in this model.

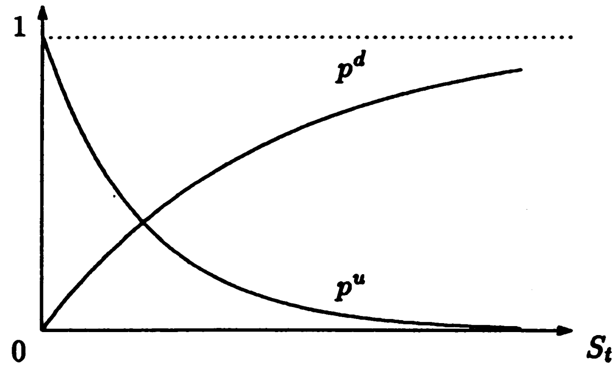


Fig.3.2: The influence rates of bull and bear order

For simplicity of numerical simulations, a 2-dimentional square lattice network is investigated in this paper. There are $N \times N$ traders in the market, and each trader (i, j), $1 \leq i, j \leq N-1$, holds 4 kinds of stocks, $B_{(i,j)}^{(i+1,j)}$, $B_{(i,j)}^{(i-1,j)}$, $B_{(i,j)}^{(i,j+1)}$ and $B_{(i,j)}^{(i,j-1)}$. Trader (i, N) is assumed to hold $B_{(i,N)}^{(i,1)}$ instead of $B_{(i,N)}^{(i,N+1)}$, and trader (N, j) holds $B_{(N,j)}^{(1,j)}$ instead of $B_{(N,j)}^{(N+1,j)}$. A network structure of the market is given by $B_{(i,j)}^{(k,l)} = B_{(k,l)}^{(i,j)}$: that is, each trader shares one of his four holdings with only one of his four neighborhood traders. Traders on an edge of the network share their stocks with traders on the opposite edge (for example, trader (i, N) shares stock $B_{(i,N)}^{(i.1)}$ with trader $(i, 1)$). This market is modeled by site percolation on a 2-dimensional torus S^2 (Fig.3.3).

Influence rates of a bull and bear order, p^u and p^d, are specified by

$$p^u(S) = e^{-\alpha S} \quad and \quad p^d(S) = 1 - e^{-\delta S} . \tag{6}$$

Since it is known that the cluster size distribution converges as $N \to \infty$, this specified model is determined by three parameters, α, δ, and ρ, if N^2 is large enough. Fig.3.4 is a sample path of S in the case of $\alpha = 1.4$, $\delta = 0.44$, $\rho = 0.008$, and $N = 21$. Fig.3.5 is the log of the p.d.f. for the returns of the data generated by the simulation. The fat tails are clearly shown in the figure.

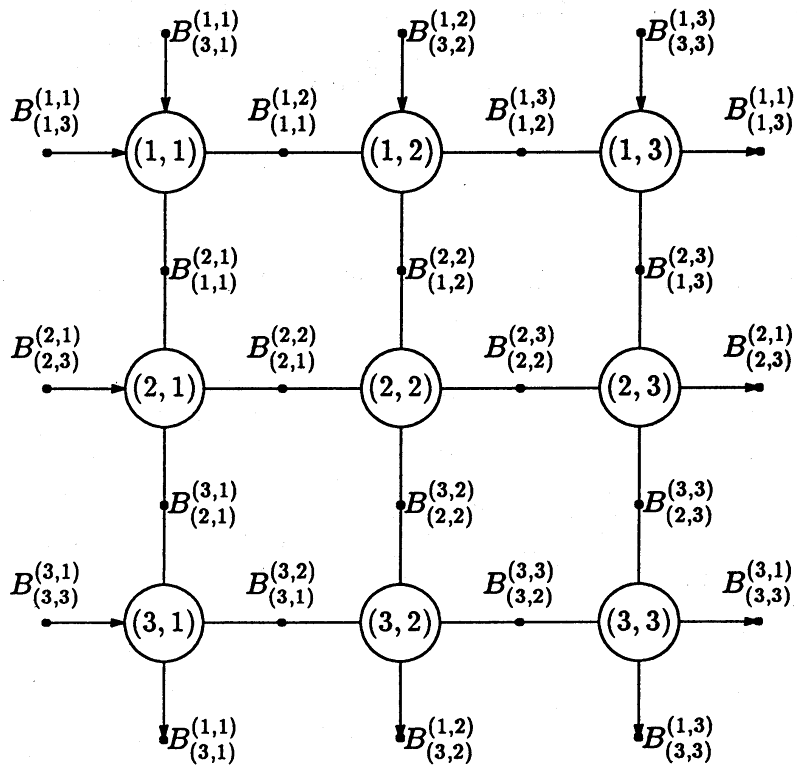


Fig.3.3: The S^2 network structure of the market (N = 3)

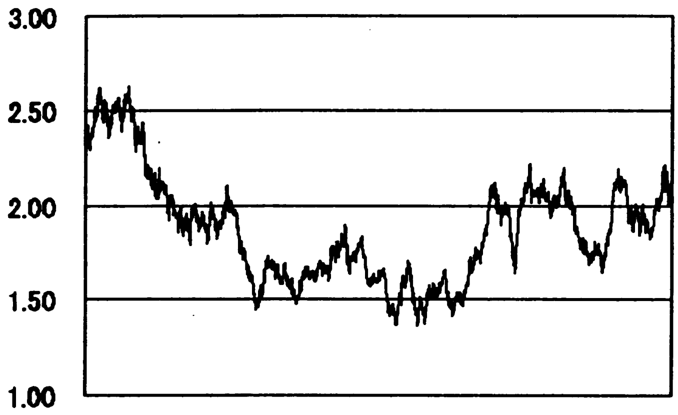


Fig.3.4: A sample path of S in the case of $\alpha = 1.4$, $\delta = 0.44$, $\rho = 0.008$, $N = 21$

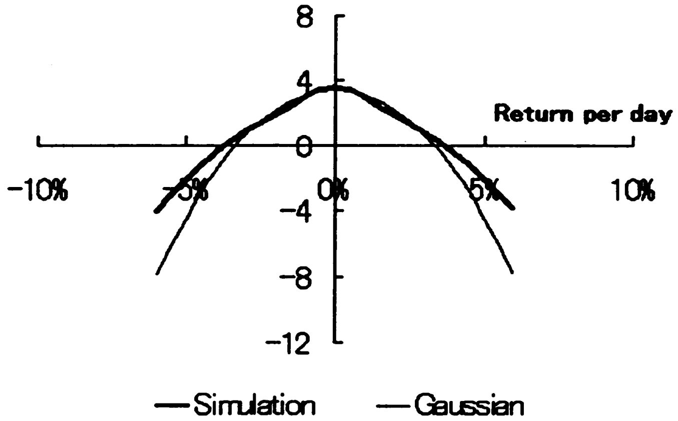


Fig.3.5: The log of the p.d.f. for the returns of S in Fig.3.4

4 The Floor and Ceiling for NIKKEI 225

In this section, we will define the floor and ceiling for the price index process $S = (S_t)_{t \in \mathbf{N}}$ given in the previous section, and calculate them from empirical data of NIKKEI 225. We define the floor S_* and ceiling S^* for S as follows:

Definition 2 *Let p^c be the critical probability of a 2-dimentional site percolation, that is, $p^c = 0.592745 \cdots$.*

The floor S_* and ceiling S^* for the price index process $S = (S_t)_{t \in \mathbf{N}}$ given by equation (4) are prices satisfying

$$p^u(S_*) = p^c \ , \ p^d(S^*) = p^c \ . \tag{7}$$

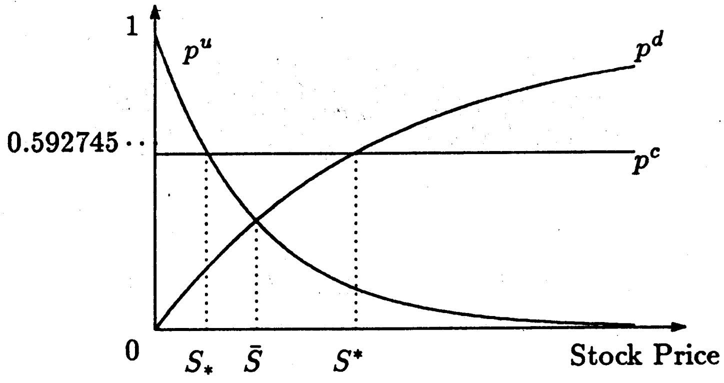


Fig.4.1: The floor and ceiling for S

Definition 2 ($p^u(S_*) = p^d(S^*) = p^c$) means that the probability of large price changes, such as the falling over 20%which the U.S. stock market experienced on October 1987, becomes positive if the value of S becomes lower than the floor S_* or higher than the ceiling S^*: that is, S will drastically rise when $S < S_*$, or catastrophically fall when $S > S^*$ with positive probability.

In the case of the specified model (6) in the previous section, S_* and S^* are explicitly solved as follows:

$$S_* = -\frac{1}{\alpha}\log p^c \ , \ S^* = -\frac{1}{\delta}\log(1-p^c) \ . \tag{8}$$

Therefore, we can calculate the floor and ceiling if we know the values of the parameters α and δ. We will decide values of α and δ by repeating numerical simulation until we find those which fit the empirical data well. The following conjecture is useful to reduce the hardship of searching them.

[Conjecture] *Let $\bar{S}$ be a price which satisfies $p^u(\bar{S}) = p^d(\bar{S})$ (see Fig.4.1). Then, the following will hold:*

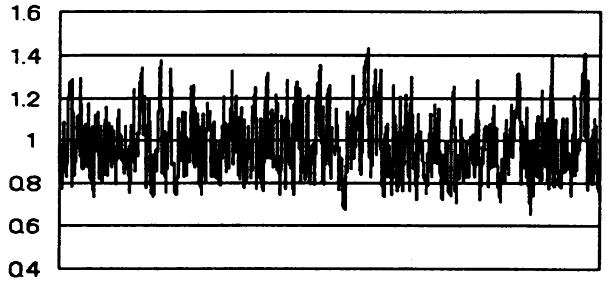
$$\lim_{T\to\infty}\left[\frac{1}{T}\sum_{t=1}^{T} S_t\right] = \bar{S} \quad . \tag{9}$$

Fig.4.2 shows two sample paths characterized by parameters satisfying $\bar{S} = 1$ (see also Fig.4.3). Both of them seem to fluctuate around $S = 1$.

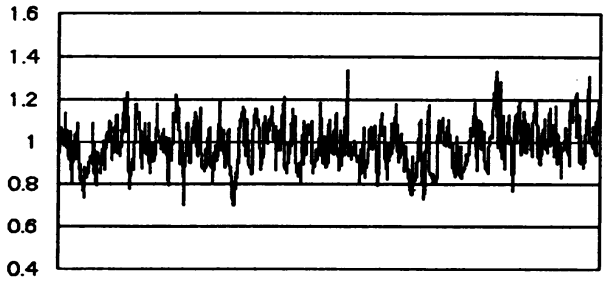
Because it is known that the mean of NIKKEI 225 from January 1991 to September 2001 is ¥18, 265, we can represent δ as a function of α by solving $p^u(\bar{S}) = p^d(\bar{S})$, that is,

$$\delta(\alpha \mid \bar{S}) = -\frac{1}{\bar{S}}\log(1-e^{-\alpha\bar{S}}) \ , \tag{10}$$

and substituting $\bar{S} = 1.8265$. By the function $\delta = \delta(\alpha)$, we can reduce the number of the parameters.



(i) $\alpha = 0.6, \delta = 0.7959, \rho = 0.5, N = 441$



(ii) $\alpha = 0.7, \delta = 0.6863, \rho = 0.5, N = 441$

Fig.4.2: Two examples satisfying $\bar{S} = 1$

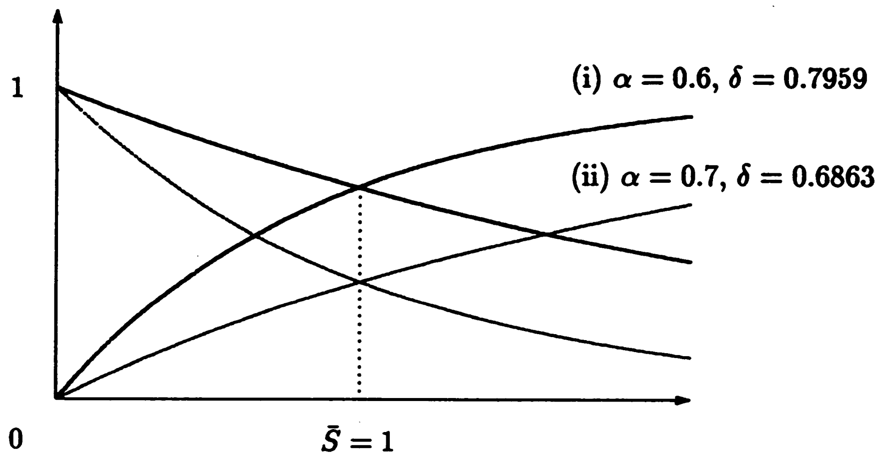


Fig.4.3: p^u and p^d in the case of Fig.4.2 (i) and (ii)

Fig.4.4 shows a sample path in the case of $\alpha = 1.6$, $\delta = 0.030$, and $\rho = 0.015$, and Fig.4.5 is the log of the p.d.f. for the absolute values of its returns. The mean and standard deviation of the sample path in Fig.4.4 are 18,333 and 3,032, while those of NIKKEI 225 from January 1991 to September 2001 are ¥18,265 and ¥3,205.

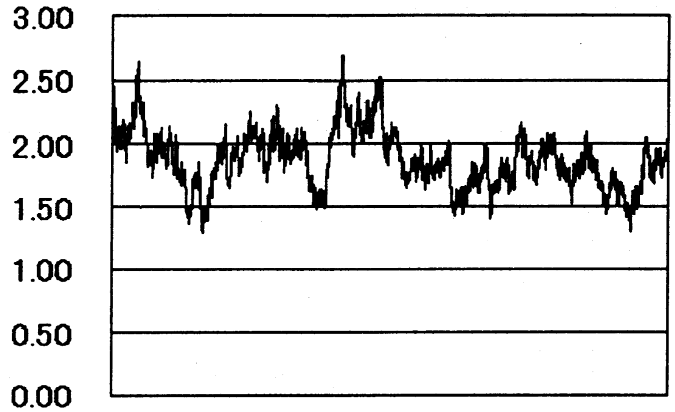


Fig.4.4: A sample path in the case of $\alpha = 1.6$, $\delta = 0.030$, and $\rho = 0.015$

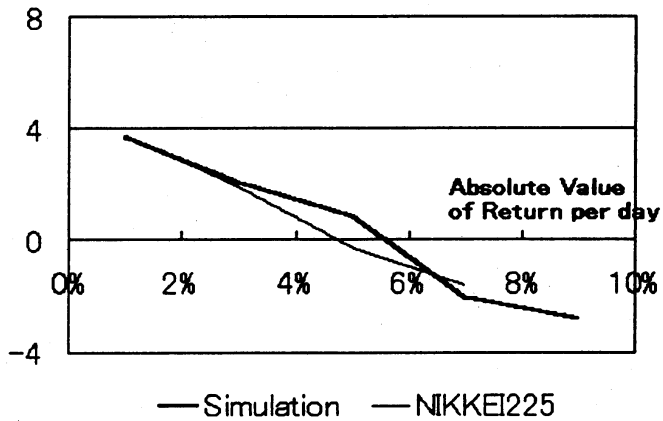


Fig.4.5: The log of the p.d.f. in the case of $\alpha = 1.6$, $\delta = 0.030$, and $\rho = 0.015$

In this case, the model seems to fit the empirical data well. The floor and ceiling calculated from the above parameters are, however, respectively ¥3,269 and ¥296,672, and they do not seem to be realistic. One possible explanation of such under- or overestimation is that the data we used is of too long periods of time. It is unlikely that the parameters, which depend on a variety of economic conditions, remain constant for more than ten years.

Therefore, we should use empirical data of shorter periods to derive more precise values of the parameters. Fig.4.6 is NIKKEI 225 from July to December in 2001, and the parameters fitting these data are $\alpha = 0.7$, $\delta = 0.58$, and $\rho = 0.0009$. In this case, the floor and ceiling for NIKKEI 225 become respectively ¥7,471 and ¥15,489, which seem more realistic than the previous values.

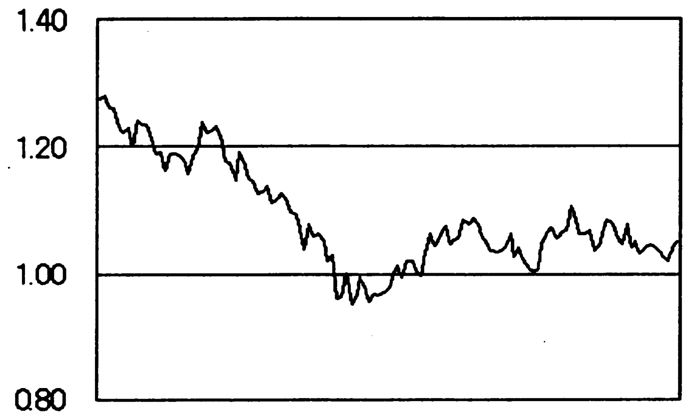


Fig.4.6: NIKKEI 225 from Jul.2001 to Dec.2001

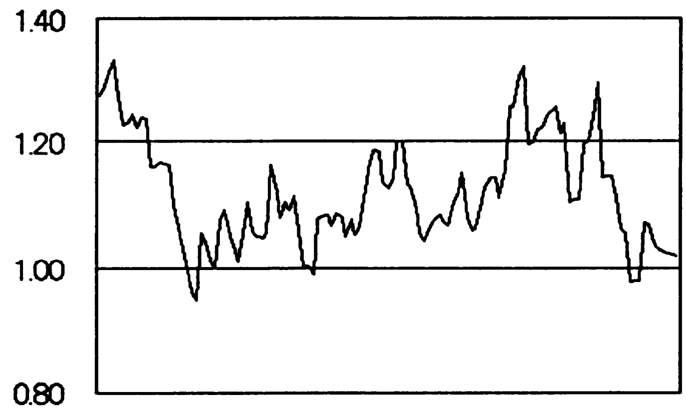


Fig.4.7: A sample path in the case of $\alpha = 0.7$, $\delta = 0.58$, and $\rho = 0.0009$

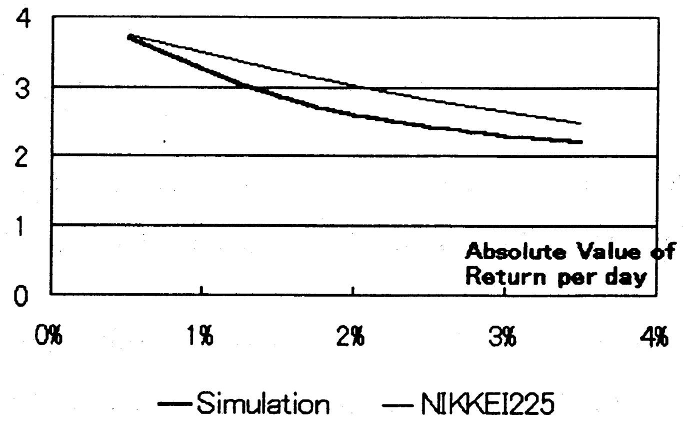


Fig.4.8: The log of the p.d.f. in the case of $\alpha = 0.7$, $\delta = 0.58$, and $\rho = 0.0009$

5 Concluding Remarks

In this paper, we modeled the stock market as a network of locally interacting traders. Numerical simulations showed that the model had fat-tail distributions of returns, which are actually observed in many markets. As application of the model, we defined the floor and ceiling price for a stock price index as critical values in the percolation process. Moreover, we calculated them for NIKKEI 225 using the empirical data.

Traders in the market were assumed to have bounded sight. This assumption is essential in the model, and it implies that, even through trading, each trader's private information may not become public but remain local. Usual models, such as Milgrom-Stokey (1982), assume that agents have unlimited sight and rationality. Each agent can reason the others' private information through trading, so that any information cannot remain unrevealed. In these models, sinse every trader knows all information in the market, and therefore each of them decides his activity independently of the other traders, distributions of market fluctuations will become normal because of the Central Limit Theorem. In the setting of this paper, however, private information is revealed not to all but only to part of traders in the market through trading. In other words, a trader's selling or buying affects his neighborhood traders' decision, and this local interaction makes it possible for the model of this paper to escape from the realm of the Central Limit Theorem.

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