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# Rationality on final decisions leads to sequential equilibrium*

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**Abstract.** The purpose of this article is to investigate epistemic conditions that induce a sequential equilibrium outcome in a given extensive form game. If players mutually know that every player maximizes her own expected payoff at any information sets then the outcome yields a sequential equilibrium: This is an extension of the result of Aumann (1995, *Games and Economic Behavior*, 8:6-19) in perfect-information game. In this paper, we suppose that each player has *$\mu$-rationality*, which means that he knows that he maximizes his own payoff according to the belief $\mu$. Furthermore we introduce the notion of $\mu$-consistency in imperfect information game. Our main theorem states that mutual knowledge of $\mu$-rationality and $\mu$-consistency induces the sequential equilibrium outcome in an extensive form game.

**Keywords:** Knowledge, Rationality, Epistemic conditions, Backward induction, Sequential equilibrium.

## 1. Introduction

This paper investigates what epistemic conditions induce a sequential equilibrium, that is, what each player should know in order to achieve the sequential equilibrium in a given game. Though there are many equilibrium solutions in an extensive form game, it is not clear how players achieve these solutions. This paper aims to fill this gap for sequential equilibrium in an extensive form game in imperfect information.

In normal-form game, Aumann and Brandenburger (1995) gives epistemic conditions for leading to Nash equilibrium: Suppose that the players have a common prior, that their payoff functions and their rationality are mutually known, and that their conjectures for the opponents' actions are commonly known. Then the conjectures form Nash equilibrium.

In extensive form game we are bothered by the contradiction between players' rationality and solution concepts. The contradiction is presented by Rosenthal (1981) informally and by Reny (1992) and Ben-Porath (1997) formally. They show that players' rationality at the root in the extensive form game does not always lead to the backward induction outcome by examining the centipede game.

On the other hand Aumann (1995) establishes the theorem that players' rationality at every node in perfect information games can lead to the backward induction outcome.

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* This is a preliminary version and the final form will be published elsewhere.

In this paper we investigate in the same line of Aumann. We extend his result in perfect information game to in imperfect information game as follows:

**Main Theorem.** *The mutual knowledge of $\mu$-rationality leads to a sequential equilibrium in an extensive form game.*

Precisely, if everybody knows that each maximizes his own expected payoff according to the common belief $\mu$ at each information set, then the assignment associated with $\mu$ induces the sequential equilibrium.

This paper is organized as follows: In Section 2 we recall an extensive form game and the sequential equilibrium based on Kreps and Wilson (1982). In addition, we introduce knowledge of players and $\mu$-rationality, and we show some basic lemmas. In section 3 we present the main theorem and give the proof.

## 2. Game and Knowledge

### 2.1. Extensive-form Games

We consider a finite extensive form game. By this we mean a structure $G = \langle (T, \prec), N, (\mathcal{I}_i)_{i\in N}, (A_i)_{i\in N}, (u_i)_{i\in N}\rangle$ consisting of as follows: $T$ is the finite set of nodes that is divided into the set of players' decision nodes $X$ and the set of the terminal nodes $Z$. We assume there is no chance moves for simplicity.[1] $(T, \prec)$ forms a tree with the unique root: The relation $\prec$ is a totally order on the predecessors $P(x)$ of each member $x$ in $T$ and $p(x)$ is the immediate predecessor of $x$. $N$ is a set of finitely many players. For each $i \in N$, $X_i$ is the subset of $X$ that consists of $i$'s decision nodes and thus $X$ is the disjoint union of all the sets of $X_i$'s. We denote by $\iota(x)$ the player making his decision at $x \in X$.

The information that player $i$ possesses is represented by $i$'s *information partition* $\mathcal{I}_i$ on $X_i$ consisting of components $I_i$ called *$i$'s information set.* When a set $I_i \in \mathcal{I}_i$ contains a node $x \in X_i$, we denote it by $I_i(x)$ (or simply by $I(x)$.) Each information set is identified with the set of all the decision nodes among which the player can not distinguish. In addition $\mathcal{I}$ denotes the disjoint union of all $\mathcal{I}_i$'s.

Each player $i$ has a feasible action set $A_i(I)$ at every $I \in \mathcal{I}_i$. Since each of $i$'s information sets is the set of nodes that she can not distinguish, the feasible action sets $A_i(x)$, $A_i(x')$ at $x, x' \in I$ are identified wit each other, which denotes $A_i(I)$. We denote by $A_i$ the set of all profiles of $i$'s feasible-actions; that is, $A_i \equiv \times_{I\in\mathcal{I}_i} A_i(I)$.

In this paper we focus on games with *perfect recall.*[2] An extensive form game $G$ is said to be with perfect recall if the following conditions are satisfied:

1. For any two nodes in a same information set, it is impossible that one node is the predecessor of the other one.
2. For any three nodes $x, x', x'' \in X_i$ with $x' \in I(x'')$ and $x \in P(x')$, there exist $\hat{x} \in I(x) \cap P(x'')$ and $a \in A_i(I(x))$ such that if $a$ respectively reaches $x'$ and $x''$ then it is played at both $x$ and $\hat{x}$.

[1] We restrict our attention into the case that the number of the initial node is just one for simplicity.

[2] Kuhn (1953).

The assumption of perfect recall plays crucial in the main theorem. $i$'s payoff function $u_i : Z \to \mathbf{R}$ is a real-valued von Neumann-Morgenstern utility on the outcomes for all players.

A *local strategy* at $I \in \mathcal{I}_i$ for player $i$ is a probability distribution $b_i^I$ on $A_i(I)$, and $i$'s *behavior strategy* $b_i$ is the profile $(b_i^I)_{I \in \mathcal{I}_i}$. A behavior strategy $b_i$ is called $i$'s *pure strategy* if each component of $b_i$ assigns the probability one to the specific action $a^I \in A_i(I)$ at each information set $I$. In addition, $i$'s mixed strategy is defined to be the probability distribution on $A_i$. By Kuhn's theorem in Kuhn (1953) there is a one to one correspondence between behavior strategies and mixed strategies in a game with perfect recall, and hence we restrict our attention to behavior strategies; hereafter behavior strategies are simply called *strategies* in this paper.

Let $\mathcal{B}_i$ denote the set of all strategies for player $i$ and $\mathcal{B} = \times_{i \in N} \mathcal{B}_i$ the set of all profiles of strategies for the game. Each strategy $b \in \mathcal{B}$ induces the probability distribution $\boldsymbol{P}^b$ on $T$ defined as follows: For $x \in T$,

$$\boldsymbol{P}^b(x) := \prod_{a \in \pi(x)} b(a), \tag{1}$$

where $\pi(x)$ is the set of all actions reaching $x$ from the root. The formula (1) represents the probability to reach $x$ from the root calculated by the strategies on $P(x)$. $i$'s *expected utility* $U_i$ induced from $\boldsymbol{P}$ on $\mathcal{B}$ is defined by

$$U_i(b) := \sum_{z \in Z} \boldsymbol{P}^b(z) u_i(z). \tag{2}$$

## 2.2. Sequential Equilibrium[3]

A *system of beliefs* is the class of probability distributions $\mu$ on each information set $I \in \mathcal{I}$; hence $\sum_{x \in I} \mu(x) = 1$ for each $I \in \mathcal{I}$. Let $\mu(x)$ interpret as a belief assigned by $\iota(x)$ to $x \in I$ if an information set $I$ is reached. Let $\mathcal{M}$ denote the set of beliefs. Each member of $\mathcal{B} \times \mathcal{M}$ is called an *assessment.* Given an assessment $(b, \mu) \in \mathcal{B} \times \mathcal{M}$, we define the *conditional* probability $\boldsymbol{P}^{b,\mu}(\cdot|\, I)$ over $Z$ by

$$\boldsymbol{P}^{b,\mu}(z|\, I) = \begin{cases} 0 & \text{if } x \notin P(z) \cap I \\ \mu(x) \displaystyle\prod_{a \in \pi(x,z)} b(a) & \text{if } x \in P(z) \cap I, \end{cases} \tag{3}$$

where $\pi(x, z)$ is the set of actions which are used to reach $z$ from $x \in I$. This formula represents the probability of player's assessment of reaching each terminal node when she is at an information set $I$. Then we define the conditional expectation $U_i^\mu$ under $i$'s information set $I$ by

$$U_i^\mu(b|\, I) := \sum_{z \in Z} \boldsymbol{P}^{b,\mu}(z|\, I) u_i(z). \tag{4}$$

Let $\mathcal{B}^+$ denote the set of strategies $b \in \mathcal{B}$ such that $b(a) \gneqq 0$ for any $a \in A$, and $\mathcal{M}^+$ the subset of $\mathcal{M}$ which consists of $\mu \in \mathcal{M}$ such that $\mu(x) \gneqq 0$ at each $x \in X$.

[3] Kreps and Wilson (1982).

For given $b \in \mathcal{B}^+$, we say that the belief $\mu$ is *associated* with $b$ if it is defined by the Bayes' rule:

$$\mu(x|\, b) = P^b(x) \Big/ \sum_{\hat{x} \in I} P^b(\hat{x}). \tag{5}$$

We can now define the sequential equilibrium as follows.

**Definition 1.** Let $G$ be an extensive form game. We denote by $\mathcal{SE}(G|\, I)$ the set of all the assessments $(b^*, \mu^*)$ satisfying both the conditions $(\mathbf{C_I})$ and $(\mathbf{SR_I})$ at an information set $I$:

$(\mathrm{C_I})$ An assessment $(b^*, \mu^*)$ is *consistent* at the information set $I$. That is, there exists a sequence $\{(b^n, \mu(\cdot|b^n))\} \subseteqq \mathcal{B}^+ \times \mathcal{M}^+$ such that for all $x \in I$ and all $a \in A_{\iota(I)}(I)$,

$$\lim_{n \to \infty} (b^n(a), \mu(x|\, b^n)) = (b^*(a), \mu^*(x)).$$

$(\mathrm{SR_I})$ An assessment $(b^*, \mu^*)$ is *sequential rational* at the information set I. That is, for the information set $I$ and for any alternative strategy profile $b_i' \in \mathcal{B}_i$,

$$U_i^\mu(b^*|\, I) \geqq U_i^\mu(b_i', b_{-i}^*|\, I),$$

where $i = \iota(I)$ and $b_{-i}^*$ denotes the profile $(b_j^*)_{j \in N \setminus \{i\}}$.

Let $\mathcal{SE}(G)$ denote the intersection of $\mathcal{SE}(G|I)$ over $I \in \mathcal{I}$. We call $(b^*, \mu^*) \in \mathcal{SE}(G)$ a *sequential equilibrium* of the game $G$.

### 2.3. Knowledge Structure on G

Aumann (1995) introduced the partition model of knowledge on extensive form games. He shows that the backward induction outcome is reached by the common knowledge of rationality in perfect information games. We will extend the model of knowledge on perfect information game into that on imperfect information game.

A *knowledge structure* on an extensive form game $G$ is a triple $\langle \Omega, (\Pi_i)_{i \in N}, \mathrm{b} \rangle$ consisting of the following structures and interpretations: $\Omega$ is a non-empty set, each element $\omega$ is called a *state* and a subset $E$ of $\Omega$ is called an *event.* $\Pi_i$ is a mapping of $\omega$ into $2^\omega$ such that the image makes a partition on $\Omega$ consisting of components $\Pi(\omega)$ for $\omega \in \Omega$. b is a function from $\Omega$ to $\mathcal{B}$ and $\mathrm{b}(\omega)$ represents the $|N|$-tuple of the players' strategies at the state $\omega$.

To avoid the confusion we call $\Pi_i$ $i$'s *knowledge partition.* Intuitively a component $\Pi_i(\omega)$ of $i$'s knowledge partition is interpreted as the event consisting of all the states that player $i$ cannot distinguish from $\omega$. $i$'s *knowledge operator* $K_i$ on $2^\Omega$ is defined by

$$K_i E = \{\omega \in \Omega|\; \Pi_i(\omega) \subseteqq E\} \quad \text{for } E \subseteqq \Omega.$$

We will record the properties as follows:[4] For any $E, F \subseteqq \Omega$,

(N) $K_i \Omega = \Omega$;

[4] Bacharach (1985).

(M) If $E \subseteqq F$, then $K_iE \subseteqq K_iF$;
(K) $K_i(E \cap F) = K_iE \cap K_iF$;
(T) $K_iE \subseteqq E$;
(4) $K_iE \subseteqq K_i(K_iE)$;
(5) $\Omega \setminus K_iE \subseteqq K_i(\Omega \setminus K_iE)$.

The *mutual knowledge* operator $K_E$ on $\Omega$ is defined by $K_EF = \bigcap_{i \in N} K_iF$. The event $K_EF$ is interpreted as that 'every player knows $F$.' The *common-knowledge* operator $K_C$ is defined by

$$K_CE := \bigcap_{k=1,2,\ldots} \bigcap_{\{i_1,i_2,\ldots,i_k\} \subseteq N} K_{i_1}K_{i_2}\cdots K_{i_k}E.$$

The event $K_CE$ is interpreted as that 'all players know that all players know that $\cdots$ that all players knows $E$.'

Now, if $\phi$ is a function on $\Omega$ and $v$ is its value then $[\phi = v]$ (or simply $[v]$) denotes the event $\{\omega \in \Omega | \ \phi(\omega) = v\}$. Therefore for any $b_i \in \mathcal{B}_i$, $[b_i]$, denote the set $\{\omega \in \Omega \mid \mathbf{b}_i(\omega) = b_i \ \}$. We assume that

$$[b_i] \subseteqq K_E[b_i] \quad \text{for every } b_i \in \mathcal{B}_i, \tag{6}$$

which is interpreted as that everybody knows every behavior strategy for each player. In view of the assumption (6) we can observe that each strategies of player $i$ is $\Pi_i$-measurable, and thus $K_i[b_i] = [b_i]$ by (T).

*Example 1.* Let $G$ be an extensive form game $G = \langle (T, \prec), N, (\mathcal{I}_i)_{i \in N}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$. Let $\Omega = T \setminus Z$ and $\Pi_i$ the function from $\Omega$ to $2^{\Omega}$ defined by $\Pi_i(\omega) = \mathcal{I}_i(\omega)$. Given $\mathbf{b}_i^I : I \to \mathcal{B}_i(I)$ an arbitrary map, we set the function $\mathbf{b}_i = \sum_{I \in \mathcal{T}_i} \mathbf{b}_i^I$ as the disjoint union of $\mathbf{b}_i^I$ over $i$'s information sets, where $\mathcal{B}_i(I)$ is the sets of feasible behavior strategies at $I$. Define the knowledge operator $K_j$ for player $j$ as follows:

$$K_j[b_i^I] = \begin{cases} I & \text{if } i = j \\ \emptyset & \text{if } i \neq j, \end{cases} \tag{7}$$

for any $b_i^I \in \mathcal{B}_i(I)$. Then for any $b_i \in \mathcal{B}_i$ and $b_i^I \in \mathcal{B}_i(I)$, it can be observed that $[b_i] = \sqcup_{I \in \mathcal{I}_i}[b_i^I] \subseteqq \sqcup K_E[b_i^I] = \sqcup_{I \in \mathcal{I}_i} I$ by (M), where the symbol $\sqcup$ denotes the disjoint union operator.

## 2.4. Rationality and Consistency

The notion of rationality defined here is an extension of that of rationality defined in Aumann (1995). For $\mu \in \mathcal{M}$ we say that player $i$ is *$\mu$-rational* at $I \in \mathcal{I}_i$ if each strategy that $i$ does not know never yield her expected utility value according to $\mu$ at $I \in \mathcal{I}_i$ greater than the actual expected utility value at $I$. If she is rational at any $I \in \mathcal{I}_i$, then we say $i$ to be $\mu$-rational. Formally, the event $\mathcal{R}_i^{\mu}(I)$ that 'player $i$ is $\mu$-rational at $I \in \mathcal{I}_i$' is given by:

$$\mathcal{R}_i^{\mu}(I) := \bigcap_{b_i' \in \mathcal{B}_i} \sim K_i\Big[U_i^{\mu}(b_i', \mathbf{b}_{-i}) | \ I) \gneqq U_i^{\mu}(\mathbf{b} | \ I)\Big], \tag{8}$$

where $\sim$ denotes the complementation. We denote

$$\mathcal{R}_i^\mu = \bigcap_{I \in \mathcal{I}_i} \mathcal{R}_i^\mu(I) \qquad \text{and} \qquad \mathcal{R}^\mu = \bigcap_{i \in N} \mathcal{R}_i^\mu.$$

The former event is interpreted as that player $i$ is $\mu$-rational and the latter as that all players are $\mu$-rational. Furthermore we define the notion of *$\mu$-consistency*. For given $\mu \in \mathcal{M}$, the event of $\mu$-consistency $C^\mu$ is the set of all the states $\omega$ such that there exists a sequence $\{(b^n, \mu(\cdot|\, b^n))\} \subsetneqq \mathcal{B}^+ \times \mathcal{M}^+$ with $\lim_{n\to\infty}(b^n, \mu(\cdot|\, b^n)) = (\mathbf{b}(\omega), \mu)$.

It is well end this section in a remark: Rationality in perfect information game is clearly equivalent to $\mu$-rationality when the belief $\mu$ is the constant function 1. That is, the rationality in Aumann (1995) is the 1-rationality $\mathcal{R}^1$ for all players in our sense. One of the purposes in this paper is to extend the result of Aumann (1995) in the case of $\mu$-rationality.

## 3. The Result

Let $G$ be an extensive form game and $\mu \in \mathcal{M}$. We denote by $SE^\mu(G)$ the event consisting of the states $\omega \in \Omega$ such that the assessment $(\mathbf{b}(\omega), \mu) \in \mathcal{B} \times \mathcal{M}$ constitutes a sequential equilibriums in $G$; that is,

$$SE^\mu(G) = \{\omega \in \Omega \mid (\mathbf{b}(\omega), \mu) \in \mathcal{SE}(G)\ \}.$$

Similarly $SE^\mu(G|\, I)$ is the event consisting of the states $\omega \in \Omega$ such that $(\mathbf{b}(\omega), \mu)$ is a member of $\mathcal{SE}(G|\, I)$ for each information set $I$. In addition, by the *final decisions* of player $i$ we mean the set of all the nodes in $I \in \mathcal{I}_F \cap \mathcal{I}_i$ which does not give the chance to decide again to player $i$. We denote by $\mathcal{I}_F$ the subset of $\mathcal{I}$ consisting of all the information sets in which each player finally decides in the game $G$. $R_F^\mu$ is the event of $\mu$-rationality over $\mathcal{I}_F$, that is, $R_F^\mu = \cap_{h \in \mathcal{I}_F} R_i^\mu(h)$. The main theorem states that if $\mu$-rationality at final decision information sets for each players under $\mu$-consistency for some $\mu \in \mathcal{M}$ is mutually known then the sequential equilibrium is achieved in the given game $G$. We can now state the main theorem formally as follows:

**Theorem 1.** $K_E(R_F^\mu \cap C^\mu) = SE^\mu(G)$.

*Proof.* It suffices to prove that $K_E(R_F^\mu \cap C^\mu) \subsetneqq SE^\mu(G)$. We prove it by induction as follows. It may be assumed that $K_E(R_F^\mu \cap C^\mu) \neq \emptyset$. For each information set $I \in \mathcal{I}_i$, let $S_i(I)$ be the subset of $\mathcal{I}_i$ consisting of $i$'s information sets next after $i$ decides at $I$. We shall shall the two pints: First that for each $i \in N$ and any $h \in \mathcal{I}_F \cap \mathcal{I}_i$, $K_i(R_F^\mu \cap C^\mu) \subsetneqq SE^\mu(G|\, h)$. Let $\mathcal{I}^{\prec}(I)$ denote the set of all the information sets at which $\iota(I)$ decides after $I$. Secondly we show that if $K_i(R_F^\mu \cap C^\mu) \subsetneqq SE^\mu(G|\, h)$ at any $h \in \mathcal{I}^{\prec}(I)$ then $K_i(R_F^\mu \cap C^\mu) \subsetneqq SE^\mu(G|\, I)$.

We shall verify the first point: For each player $i \in N$, it follows that

$$K_E(R_F^\mu \cap C^\mu) \subsetneqq \bigcap_{b_i' \in \mathcal{B}_i} \sim K_i\Big[U_i^\mu(b_i', \mathbf{b}|\, h) \gneqq U_i^\mu(\mathbf{b}|\, h)\Big] \cap C^\mu.$$

We note that for any $\omega \in K_E(R_F^\mu \cap C^\mu)$ and for any $b_i' \in \mathcal{B}_i$,

$$\begin{aligned} \omega \notin K_i\left[U_i^\mu(b_i', \mathbf{b}|\ I) \gneqq U_i^\mu(\mathbf{b}|\ I)\right] \\ \Leftrightarrow \exists\ \xi \in \Pi_i(\omega),\ \xi \notin \left[U_i^\mu(b_i', \mathbf{b}|\ I) \gneqq U_i^\mu(\mathbf{b}|\ I)\right] \\ \Leftrightarrow \exists\ \xi \in \Pi_i(\omega),\ U_i^\mu(\mathbf{b}(\xi)|\ I) \geqq U_i^\mu(b_i', \mathbf{b}(\xi)|\ I). \end{aligned}$$

Furthermore it is observed that $\mathbf{b}_i(\omega) = b_i(\xi)$ for any $\xi \in \Pi_i(\omega)$ because $[b_i] \subseteqq K_E[b_i]$, and thus it can be plainly obtained that for any $\omega \in K_E(R_F^\mu \cap C^\mu)$ and for any $b_i' \in \mathcal{B}_i$,

$$U_i^\mu(\mathbf{b}(\omega)|\ I) \geqq U_i^\mu(b_i', \mathbf{b}(\omega)|\ I).$$

Therefore we have shown that for each $\omega \in K_i(R_F^\mu \cap C^\mu)$, $(\mathbf{b}(\omega), \mu)$ is $\mu$-rational on any $h \in \mathcal{I}_F \cap \mathcal{I}_i$, and it is easily observed to be $\mu$-consistent. Therefore $K_i(R_F^\mu \cap C^\mu) \subseteqq SE^\mu(G|\ h)$.

The following lemma is needed to verify the second point. We denote $\mathbf{P}(x|\ b) := \mathbf{P}^b(x)$ for simplicity.

**Lemma 1.** *For $b \in \mathcal{B}$, each $i \in N$, and $I \in \mathcal{I}_i$ such that $S_i(I) \neq \emptyset$,*

$$U_i^\mu(b|\ I) = \sum_{h \in S_i(I)} \frac{\sum_{\tilde{x} \in h} \mathbf{P}(\tilde{x}|\ b)}{\sum_{\hat{x} \in h} \mathbf{P}(\hat{x}|\ b)} U_i^\mu(b|\ h).$$

*Proof.* For $b \in \mathcal{B}, x \in I$ and $x' \in h \in S_i(I)$, it can be observed that

$$\mu(x|\ b^n) = \frac{\sum_{\tilde{x} \in h} \mathbf{P}(\tilde{x}|\ b^n)\ \mu(x'|\ b^n)}{\sum_{\hat{x} \in h} \mathbf{P}(\hat{x}|\ b^n)\ \prod_{a \in \pi(x, x')} b^n(a)}.$$

Therefore it follows that

$$\begin{aligned} U_i^\mu(b|\ I) &= \lim_{n \to \infty} \sum_{x \in I} \mu(x|\ b^n) \prod_{a \in \pi(x,z)} b^n(a) u_i(z) \\ &= \lim_{n \to \infty} \sum_{x \in I} \frac{\sum_{\tilde{x} \in h} P(\tilde{x}|\ b^n)}{\sum_{\hat{x} \in I} P(\hat{x}|\ b^n)} \mu(x'|\ b^n) \prod_{a \in \pi(x',z)} b(a) u_i(z) \\ &= \lim_{n \to \infty} \sum_{h \in S_i(I)} \frac{\sum_{\tilde{x} \in h} P(\tilde{x}|\ b^n)}{\sum_{\hat{x} \in I} P(\hat{x}|\ b^n)} \sum_{x' \in h} \mu(x'|\ b^n) \prod_{a \in \pi(x',z)} b(a) u_i(z) \\ &= \sum_{h \in S_i(I)} \frac{\sum_{\tilde{x} \in h} P(\tilde{x}|\ b)}{\sum_{\hat{x} \in I} P(\hat{x}|\ b)} U_i^\mu(b|\ h), \end{aligned}$$

in completing the proof of the lemma.

We proceed to the proof of the second point. Assume now that for each $i \in N$ and $h \in S_i(I)$, $K_i(R_F^\mu \cap C^\mu) \subseteqq SE^\mu(G|\ h)$. Suppose to the contrary that there exists $\tilde{b}_i \in B_i$ such that at $\omega \in K_E(R_F^\mu \cap C^\mu)$,

$$U_i^\mu(\tilde{b}_i, \mathbf{b}_{-i}(\omega)|\ I) \gneqq U_i^\mu(\mathbf{b}(\omega)|\ I).$$

It then follows from the above lemma that

$$
\begin{aligned}
U_i^{\mu}(b|\, I) &= \sum_{h\in I} \frac{\sum_{\tilde{x}\in h} P(\tilde{x}|\, \mathbf{b}(\omega))}{\sum_{\hat{x}\in I} P(\hat{x}|\, \mathbf{b}(\omega))} U_i^{\mu}(\mathbf{b}(\omega)|\, h) \\
&\geq \sum_{h\in S_i(I)\setminus\{h'\}} \frac{\sum_{\tilde{x}\in h} P(\tilde{x}|\, \mathbf{b}(\omega))}{\sum_{\hat{x}\in I} P(\hat{x}|\, \mathbf{b}(\omega))} U_i^{\mu}(\mathbf{b}(\omega)|\, h) \\
&\quad + \frac{\sum_{\tilde{x}\in h} P(\tilde{x}|\, \mathbf{b}(\omega))}{\sum_{\hat{x}\in I} P(\hat{x}|\, \mathbf{b}(\omega))} U_i^{\mu}(\bar{b}_i, \mathbf{b}_{-i}(\omega)|\, h') \\
&= \lim_{n\to\infty} \sum_{h\in S_i(I)\setminus\{h'\}} \frac{\sum_{\tilde{x}\in h} P(\tilde{x}|\, b^n)}{\sum_{\hat{x}\in I} P(\hat{x}|\, b^n)} U_i^{\mu(\cdot|\, b^n)}(b^n|\, h) \\
&\quad + \frac{\sum_{\tilde{x}\in h} P(x'|\, b^n)}{\sum_{\hat{x}\in I} P(\hat{x}|\, b^n)} U_i^{\mu(\cdot|b^n)}(\bar{b}_i^n, b_{-i}^n|\, h') \\
&= U_i^{\mu}(\bar{b}_i, \mathbf{b}_{-i}(\omega)|\, I),
\end{aligned}
$$

in contradiction because player $i$ is sequential rational at $h \in S_i(I)$. This completes the proof of our theorem.

## 4. Concluding Remarks

This paper examines what epistemic conditions about players' rationality can lead to the outcomes induced by a sequential equilibrium. Originally Aumann (1995) shows that if players act on the rational behavior in a perfect-information game then they can obtain the outcomes by backward induction solution. In this article we extend this result to that about sequential equilibrium. Though he requires common knowledge of rationality for all players, we require here only the mutual knowledge of it. Therefore it is sufficient only to know rationality of each player. Furthermore our theorem insists that rationality is sufficient only at the information sets in final decision for each player.

Some related works lead to the different result from Aumann's (e.g. Reny (1992), Ben-Porath (1997)). In Aumann (1995) and our research is required rationality on every information set, however they suppose only players' beliefs at the beginning of a game. Since players have the Bayesian rationality in their studies, players can revise their own beliefs about their opponents' behaviors or their present nodes through moving plays. These are the different views in examining the extensive form games. While Aumann regards rationality of players as an representation of the equilibrium, Reny and Ben-Porath capture it as playability in a given game. We would like to examine the relationship between the two views in the further research.

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