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A Unified Approach for Global Games, Purification, and Quantal Response Equilibria*

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Abstract

This paper offers a model which enables us to understand the uniqueness of equilibrium in global games, purification of mixed strategy equilibria, and the quantal response equilibrium in a unified way.

1 Introduction and the Unified Framework

This paper offers a model which enables us to understand the uniqueness of equilibrium in global games, purification of mixed strategy equilibria, and the quantal response equilibrium in a unified way. These concepts are presented independently in the literature but the structure of these idea has a common feature. This is what we intend to demonstrate in this paper.

We consider a class of Bayesian games with I players. Each player has a finite action set A_i. An action profile is denoted by $a = (a_1, ..., a_I) \in A \equiv A_1 \times \cdots \times A_I$. As is in the standard framework, each player i chooses an action after observing her private type t_i chosen from a set of private types T_i, and the players have a common prior probability measure P over the set of type profiles $T_1 \times \cdots \times T_I \equiv T$. When a type profile t, is realized, player i's payoff to action profile a is $u_i(a, t)$. Each player is assumed to be a Bayesian expected utility maximizer, thus after observing a private type t_i player i chooses an action that maximizes the conditional expected utility. We allow for mixed actions, and denote by the set of probability measures on A_i and A by $\triangle(A_i)$ and $\triangle(A)$, respectively.

*This is more or less a research plan and the results reported in this paper are to be elaborated.

We further make two assumptions on utility functions. First we assume that utility function is additively separable as follows.

Assumption 1 *For each player i, $u_i(a,t) = g_i(a) + \varepsilon_i(a_i, t_i)$ for all $a \in A$ and $t \in T$, for some functions $g_i : A \to \mathbb{R}$, and $\varepsilon_i : A_i \times T_i \to \mathbb{R}$.*

Notice that the function g_i depends on the action profile only, thus there is no uncertainty about the part of payoffs given by g_i. The function ε_i only depends on player $i's$ action and type. An interpretation of this assumption is that the function g_i describes the "fundamentals" of the game and ε represents the uncertain or "noise" part of the utility.

The second assumption is about the role of private information.

Assumption 2 *For each player i, $t_i \neq t_i'$ implies that $\varepsilon_i(a_i, t_i) \neq \varepsilon_i(a_i, t_i')$ for some $a_i \in A_i$.*

So each private type t_i is associated with a unique vector of payoffs $(\varepsilon_i(a_i,t_i))_{a_i \in A_i} \in \mathbb{R}^{\#A_i}$. Thus, under the assumption that the payoff functions are common knowledge among the players, if a player could observe payoffs to each action profile, he needs not to observe t_i, that is, the vector $(\varepsilon_i(a_i,t_i))_{a_i \in A_i}$ contains all the private information he could learn.

So under this assumption, it is convenient to suppress the reference to private type t_i, and assume that the vector of player's own payoffs $(\varepsilon_i(a_i,t_i))_{a_i \in A_i}$ is observable to each player. That is, each player i observes the realization of random vector $t \longmapsto (\varepsilon_i(a_i,t_i)) \in \mathbb{R}^{\#A_i} \equiv \Omega_i$, where t is drawn from the sample space (state space) T according to the probability measure P. With a slight abuse of notation, we denote the random vector by $\tilde{\varepsilon}_i$, a realization of $\tilde{\varepsilon}_i$ by $\varepsilon_i \in \mathbb{R}^{\#A_i}$. For a realized payoff ε_i, Denote the element corresponding to action a_i by $\varepsilon_i(a_i)$.

To sum up, player i observes the realization of the random vector $\tilde{\varepsilon}_i$ in $\Omega \equiv \Pi_{i=1}^{I}\Omega_i$, where $\Omega_i = \mathbb{R}^{\#A_i}$ as above, then choose a possibly random action. The joint distribution of $\tilde{\varepsilon}_i$, $i = 1,...,I$, which we denote by P abusing notation, is common knowledge. A game is thus characterized by $(g, \tilde{\varepsilon})$ where $g = (g_i)_{i=1}^{I}$ and $\tilde{\varepsilon} = (\tilde{\varepsilon}_i)_{i=1}^{I}$.

A strategy of player i is a measurable function $\sigma_i : \Omega_i \to \Delta(A_i)$. The expected utility to a strategy profile $\sigma = (\cdots, \sigma_i, \cdots)$ is given by $\int \sum_{a \in A} [g_i(a) + \varepsilon_i(a_i)]\, \sigma(a)(\omega)\, P(d\omega)$, where $\sigma(a|\omega) \equiv \Pi_{i=1}^{I}\sigma_i(a_i|\omega_i)$ is the probability of action profile a is chosen given ω. We simply write $E[g_i(\sigma) + \varepsilon_i(\sigma_i)]$ for this expression. The Bayesian Nash equilibrium of the game $(g, \tilde{\varepsilon})$ is then defined as follows.

Definition 1 *A strategy profile $\sigma^* = (\sigma_1^*, ..., \sigma_I^*)$ is a Bayesian Nash Equilibrium (BNE) of $(g, \tilde{\varepsilon})$ if $\sigma_i^* \in \arg\max_{\sigma_i} E\left[g_i\left(\sigma_i, \sigma_{-i}^*\right) + \varepsilon_i(\sigma_i)\right]$ for each $i = 1, ..., I$.*

A BNE σ^* induces an equilibrium action distribution $\mu \in \Delta(A)$, $\mu(a) = \Pr[\sigma^*(\omega) = a]$. Let $\mathcal{E}_g(\tilde{\varepsilon})$ be the set of all equilibrium action distributions induced by BNE equilibria of game $(g, \tilde{\varepsilon})$. Note that $\mathcal{E}_g(\tilde{0})$ is the set of mixed strategy equilibrium distribution of game g as a complete information strategic form game, where $\tilde{0}$ is the constant random vector yielding 0 with probability one.

2 Examples

We shall give a couple of examples to show that our framework given in the previous section is capable to discuss some well discussed example in the global game literature (e.g., Carlsson and van Damme (1993), Morris and Shin (2000)).

2.1 Investment game with private signal

The first is so called the investment game. Set $I = 2$ and suppose each player has two actions {invest (α), not invest (β)}. Payoffs are given by:

	α_2	β_2
α_1	θ, θ	$\theta - 1, 0$
β_1	$0, \theta - 1$	$0, 0$

but it is assumed that players do not observe θ directly. Player i observes an independent private signal x_i with mean θ, then choose his action. It is thus interpreted that each player observes only a noisy signal of the fundamental value θ.

Assume that $x_i \longmapsto E[\theta|x_i]$ is one to one, which holds under the maintained assumptions in the aforementioned papers. Let $\varepsilon_i(\alpha_i) = E[\theta|x_i]$, $\varepsilon_i(\beta_i) = 0$. Then the investment game can be reduced to the game $(g, \tilde{\varepsilon})$ as follows; when ε is realized,

$g:$

	α_2	β_2
α_1	$0, 0$	$-1, 0$
β_1	$0, -1$	$0, 0$

, $\varepsilon:$

	α_2	β_2
α_1	$\varepsilon_1(\alpha_1), \varepsilon_2(\alpha_2)$	$\varepsilon_1(\alpha_1), 0$
β_1	$0, \varepsilon_2(\alpha_2)$	$0, 0$

,

thus

$g + \varepsilon:$

	α_2	β_2
α_1	$\varepsilon_1(\alpha_1), \varepsilon_2(\alpha_2)$	$\varepsilon_1(\alpha_1) - 1, 0$
β_1	$0, \varepsilon_2(\alpha_2) - 1$	$0, 0$

.

Then it can be shown that the set of BNE action distributions of the investment game is exactly $E_g(\tilde{\varepsilon})$.

2.2 Investment with public signal

This example is also discussed in Morris and Shin (2000). The setting is the same as the investment game, except that each player i observes two signals y and x_i. Both are normally distributed around θ, and they are *uncorrelated.* It is common knowledge that y is publicly observed and each x_i is privately observed.

To transform this game to our framework, let $\varepsilon_i(\alpha_i) = E[\theta|y, x_i], \varepsilon_i(\beta_i) = 0$. It turns out that under normal distribution, any BNE of the investment game is a function of ε. (Morris - Shin). But probably this is a general result for models with normal type distribution. Again BNE action distributions of the investment game with public signal is $E_g(\tilde{\varepsilon})$.

3 Purification and Quantal Response

In this section we argue how our model captures the ideas of the quantal response equilibrium and purification. We will focus on 2×2 symmetric coordination games of the following type, but it can be generalized: the function g is given by the following table:

Game g

	α_2	β_2
α_1	x, x	v, u
β_1	u, v	y, y

We shall further assume for simplicity that $x > u$, $y > v$, and $u = v = 0$. Thus the class of games we consider is: when ε is realized, payoffs are given by the following table:

	α_2	β_2
α_1	$x + \varepsilon_1(\alpha_1), x + \varepsilon_2(\alpha_2)$	$v + \varepsilon_1(\alpha_1), u + \varepsilon_2(\beta_2)$
β_1	$u + \varepsilon_1(\beta_1), v + \varepsilon_2(\alpha_2)$	$y + \varepsilon_1(\beta_1), y + \varepsilon_2(\beta_2)$

Assume further that $x = y$, and ε's are independently distributed and have continuous distributions. Then irrespective of the opponent's strategy, each player has a unique best action after observing ε, with probability one. Notice that

- α_1 is the best action iff $\Pr(\alpha_2)\, x + \varepsilon_1(\alpha_1) > \Pr(\beta_2)\, x + \varepsilon_1(\beta_1)$ occurs
- Probability of α_1 is chosen $= \Pr[\Pr(\alpha_2)\, x + \varepsilon_1(\alpha_1) > \Pr(\beta_2)\, x + \varepsilon_1(\beta_1)]$

Thus μ is a BNE action distribution iff μ has the form $\mu(a) = \mu_1(a_1) \times \mu_2(a_2)$ and $\mu_i(a_i)$ is exactly the probability that a_i is the best action, given μ_j. Thus the equilibrium action distribution constitutes a quantal response equilibrium as in McKelvey and Palfray (1995).

Note further that if ε is symmetric around 0, $\mu_i(\alpha_i) = \mu_i(\beta_i) = \frac{1}{2}$ is a BNE distribution in which players use (interim) pure strategies, which is indeed the idea of purification.

4 Uniqueness of equilibrium

In our setting, the uniqueness result typical in the global game literature obtains and it has a simple and intuitive representation and interpretation. We shall again deal with the 2×2 game discussed in the previous section, but again the argument can be generalized.

Assume that ε's are jointly normally distributed, but they may be correlated. Note that it still has the unique best action property as in the previous section. Further assume:

- unbiasedness: $E(\varepsilon_i(\alpha_i)) = E(\varepsilon_i(\beta_i)) = 0$
- symmetric variance: $Var(\varepsilon_i(\alpha_i)) = Var(\varepsilon_i(\beta_i)) = \rho^2$ $(\rho \geq 0)$
- uncorrelated payoff noise: $Var(\varepsilon_i(\alpha_i), \varepsilon_i(\beta_i)) = 0$
- informational correlation:
  - $Var(\varepsilon_i(\alpha_i), \varepsilon_j(\alpha_j)) = Var(\varepsilon_i(\beta_i), \varepsilon_j(\beta_j)) = \phi\rho^2$, $(|\phi| < 1)$
  - $Var(\varepsilon_i(\alpha_i), \varepsilon_j(\beta_j)) = 0$
- $\varepsilon = 0$ iff $(\rho, \phi) = (0, 0)$.

So in this parametrization, parameter ρ measures noisiness of the information, and parameter ϕ measures the degree of information about the other"s payoffs; for instance, if ϕ is small, there is less information about the other's payoffs.

Let $\gamma(\rho, \phi) \equiv \frac{1}{\rho}\sqrt{\frac{1-\phi}{1+\phi}}$, and let r^* be defined by $\Phi\left(\frac{\gamma(\rho,\phi)}{\sqrt{2}} r^*\right) = \frac{r^*+x-u}{x-u+y-v}$,where Φ is the cumulative probability distribution of the standard normal distribution.

Proposition 1 *If* $\gamma(\rho, \phi) \leq \frac{2\sqrt{\pi}}{x-u+y-v}$, *then every BNE* σ^* *has the property:* $\sigma_i(\alpha_i|\varepsilon_i) = 1$ *if* $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i) > r^*$ *and* $\sigma_i(\alpha_i|\varepsilon_i) = 0$ *if* $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i) < r^*$ *for* $i \in \{1, 2\}$.

This result says that if payoff noise is sufficiently large (ρ is large) and/or the information of the other's payoffs is sufficiently good (ϕ is close to one), there is an essentially unique equilibrium.

There are two effects that drive the result. The first is the information linkage (or, "infection") effect. Think of small ρ, and ϕ close to one.

1. If the realization of $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i)$ is very large, α_i dominates β_i irrespective of the other's strategy.

2. If ϕ is close to one, a large difference $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i)$ implies large difference $\varepsilon_j(\alpha_j) - \varepsilon_j(\beta_j)$, and hence player i must infer that player j has α_j as a dominant action.

3. If player i believes that player j plays α_j with at least probability, say, p_1, α_i will be a unique best response even if $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i)$ is not as large.

4. Then player i must believe that player j plays α_j with at least $p_2 > p_1$. So α_i will be a best response even for smaller $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i)$.

5. This argument iterates, and eventually determines the action for all realization of $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i)$.

The second effect is the large noise effect. Think of $\phi = 0$, thus observing one's type give no information about the opponent's type. Note that the iteration argument above does not work, thus there is no informational linkage effect in this case. But still the uniqueness obtains because of the following:

1. If ρ is very large, with probability close to one, players have dominant actions.

2. Thus ex ante, player i must assign probability close to $\frac{1}{2}$ to each action α_j and β_j.

3. If so, even if player i does not have a dominant action, he should choose a risk dominant action.

We shall sketch the idea of the proof below, and a formal proof can be found in Ui (2001b). Consider "strategy r": play α_i if $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i) > r$. Let $G(r_j, r_i) =$ (EU of playing α_i) - (EU of playing β_i) for player i, when player j is playing strategy r_j and player i has observed $r_i = \varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i)$. Under the conditions of the uniqueness result, $G(r_j, r_i)$ is decreasing in r_j and increasing r_i, and $G(r^*, r^*) = 0$. We know $G(r_j, r) > 0$ for any r_j if r is large. Let $\bar{r}^1$ be the smallest of such r. Note that $G(\bar{r}^1, \bar{r}^1) > 0$, so there is a unique $\bar{r}^2 < \bar{r}^1$ with $G(\bar{r}^1, \bar{r}^2) = 0$. Then player i is sure that j will choose α if $\varepsilon_j(\alpha_j) - \varepsilon_j(\beta_j) > \bar{r}^1$, so playing β_i after observing $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i) > \bar{r}^2$ is dominated. The argument iterates until $G(r^*, r^*) = 0$ is reached.

5 Informational Robustness of Equilibria.

Let $N(\delta)$ be the closed ball of radius $\delta \geq 0$ about 0 in Ω. If $\varepsilon \in N(\delta)$ is realized, the game (g, ε) is "close to g" Fix a non increasing sequence

$\delta_n \downarrow 0$. Equilibrium $\mu \in E_g(0)$ is $\{\delta_n\}$-robust if for any sequence ε^n with $\Pr[\varepsilon^n \in N(\delta_n)] \to 1$, there is $\mu^n \in E_g(\varepsilon^n)$ such that $\mu^n \to \mu$.

Equilibrium $\mu \in \mathcal{E}_g(0)$ is robust if it is $\{\delta_n\}$-robust for any $\delta_n \downarrow 0$. When $\delta_n = 0$, for all n, $\{\delta_n\}$-robustness is identical by definition to the robustness concept proposed by Kajii - Morris (1997), and further elaborated in Ui (2001a). It is interesting to find out the exact relationship.

Now consider the class of noise is restricted to the normal distribution as in the previous section. Note that the equilibrium cut off value $r^* < 0$ is bounded away from 0. Thus if noise is close to zero with high probability, $\varepsilon_i(\alpha_i) - \varepsilon_i(\beta_i) > r^*$ with probability close to one. Then in the induced equilibrium, (α_1, α_2) is played with probability close to one. So the uniqueness result implies that there is no equilibrium that assigns high probability to (β_1, β_2). When noise is not normal, uniqueness result does not hold in general and thus the argument above does not go through. But it is known that (α_1, α_2) is robust, whereas (β_1, β_2) is not. We speculate thus there is a clean argument in this restrictive class, without referring to the uniqueness, to establish the robustness.

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