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Generic Inefficiency of Equilibria with Incomplete Markets

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In the incomplete markets model with real assets and one good we show Pareto inefficiency of equilibrium allocations generically in all agents' utility functions and endowments. All we assume on their utility functions is monotonicity. No convexity is needed on them. We use Thom Transversality Theorem in 1-jet space to generically characterize the set of allocations fulfilling first order necessary conditions for Pareto optimum. We also show that it is not utility maximization behavior of each agent but his budget constraint that yields inefficiency of equilibrium allocations with incomplete markets.

1. Introduction

It is well known that with incomplete asset markets, the equilibrium allocations are generally Pareto inefficient. "For a long time the Pareto inefficiency of equilibrium allocations when markets are incomplete was regarded as a Folk Theorem. The intuition was simply that barring unusual initial circumstances, such as starting at a Pareto optimum, with an insufficient number of markets to trade on, agents' marginal rates of substitution will not be equalized by trade. Formal proofs are however recent ... and establishing the result requires techniques from differential topology. ...these techniques were introduced in the 1970s for studying qualitative properties of the standard general equilibrium model — they have turned out to be crucial for studying the model with incomplete markets since many of its properties are only generic." (Magill M. and M.Quinzii ' Theory of Incomplete Markets Volume 1 ' p.136. 1996.)

In the above mentioned book, Magill and Quinzii show that in one good -two period exchange economy model the equilibrium allocation with incomplete markets is generically Pareto inefficient with respect to the agents' initial endowments. Their claim is, however, restrictive in two points. One is that they assume strict quasi-concavity on an agent's utility function. The other is that they demonstrate inequality of the present-valued (normalized marginal rates of substitution) vectors of *all* agents, which is too demanding for the proof of invalidity of Pareto optimum.

We also consider one good-two period exchange economy model like Magill and Quinzii's. But differently from their approach we take not only agents' endowments but also their utility functions as economy parameters. Moreover, we require only monotonicity to the agents' utility functions. In the analysis first order necessary conditions for Pareto optimum play a crucial role in twofold sense. First of all, given the utility function space, the particular form of the conditions enables us to use Thom Transversality Theorem (product functional form) in 1-jet space.

Secondly to show invalidity of the conditions implies impossibility theorem of Pareto optimum. By using transversality theorem recursively (this is the approach developed in Nagata (1998, 2000), we obtain the result that for almost all agents' monotone utility functions and endowments, the equilibrium allocations with incomplete markets are Pareto inefficient. In the course of investigation, we also obtain as a by-product the fact about the cause of inefficiency of the equilibrium allocations. That is, it is not optimization behavior of each agent but his budget constraint on assets that yields inefficiency of equilibrium allocations with incomplete markets.

2. The Model

The model to be considered here is the simplest two period exchange economy under uncertainty. First and second period is each specified by t=0 and 1 and one of S states of nature (s=1, ..., S) occurs at date 1. For simplicity, we call date t=0, state s=0 so that in total there are S + 1 states. The economy consists of I consumers(i=1, ..., I) and a single consumption good. Thus the commodity space is R^{S+1}.

The characteristics of each agent i consist of three ingredients, that is, his consumption set X_i, his utility function u^i and his initial endowment ω^i. We make assumptions on those ingredients as follows. For each i (i=1, ..., I),

Assumption 1. X_i is $R^{S+1}{}_{++}$.

Assumption 2. u^i satisfies

(1) $u^i \in C^r(R^{S+1}{}_{++}, R)$ for sufficiently large $r > 0$

(2) $Du^i(x) \in R^{S+1}{}_{++}$ for each $x \in R^{S+1}{}_{++}$.

Assumption 3. $\omega^i \in R^{S+1}{}_{++}$.

For simplicity , let $u=(u^1, ..., u^I)$ and $\omega=(\omega^1, ..., \omega^I)$ in the following.

There are J real assets (j=1, ..., J) in the economy. Our interest is in the case of incomplete asset markets so that $J < S$. Each asset j can be purchased for the price q_j at date 0 and delivers a return $v^j=(v^j_1,...,v^j_S)$ across the states at date 1 where each v^j_s indicates a certain amount of good. We see each v^j (j=1, ..., J) as a column vector and combine them to form $S \times J$ matrix of returns $v= [v^1,...,v^J]$. Let $\mathscr{E}(u , \omega ; v)$ denote the economy composed of u, ω and v.

We investigate inefficiency of equilibria with incomplete markets from a generic viewpoint with respect to u and ω. Thus the asset structure v is fixed on which we may assume that rank v = J without loss of generality.

Since u and ω are only parameters that specify the economy we consider the space that consists of permissible u and ω. Let U be the set of functions satisfying Assumption 2 and let $\mathscr{U}$ be I-product of U, that is, U^I. Then the space of permissible u and ω is $\mathscr{U} \times R^{(S+1)I}{}_{++}$ which is called the economy space. To $R^{(S+1)I}{}_{++}$ a standard Euclidean topology is given whereas $C^r(R^{S+1}{}_{++}, R)$ is endowed with the Whitney C^r topology so that $C^r(R^{S+1}{}_{++}, R)^I$ is considered to be a product topological space.

Given the asset structure v, each agent has a chance to purchase some amounts of J assets and adjust his income stream so that he can optimize his intertemporal consumptions. Let $z^i=(z^i_1,...,z^i_J) \in R^J$ denote the number of units of the J assets purchased by agent i. z^i is called a portfolio of agent i. Then the problem he has to solve is as follows.

$$\mathrm{Max}_{x^i,z^i} \quad u^i(\mathbf{x}^i)$$
$$\text{s.t. } x^i_0=\omega^i_0 - \mathbf{q}\mathbf{z}^i, \quad \mathbf{z}^i \in R^J \qquad \text{..........} (*)$$
$$\mathbf{x}^i_1=\boldsymbol{\omega}^i_1 + \mathrm{v}\mathbf{z}^i$$

where $\mathbf{x}^i_1$ and $\boldsymbol{\omega}^i_1$ denote (x^i_1, ..., x^i_S) and (ω^i_1, ..., ω^i_S) respectively. Note that there is only one good in the economy so that the price of the good is normalized to be one (the good at date 0 is interpreted as a numeraire).

Now we define the equilibrium for the economy $\mathscr{E}$(u , ω ; v).

Definition 1. An asset market equilibrium for $\mathscr{E}$(u , ω ; v) is a tuple ($(\mathbf{x}^i, \mathbf{z}^i)_i$, $\mathbf{q}$) such that

(i) $(\mathbf{x}^i, \mathbf{z}^i)$ is a solution of the problem ($*$). i=1, ..., I.

(ii) $\Sigma_{i=1} \mathbf{x}^i = \Sigma_{i=1} \boldsymbol{\omega}^i$

(iii) $\Sigma_{i=1} \mathbf{z}^i = 0$

Next we consider efficiency of allocations. Given u and ω, a Pareto optimal allocation is defined as follows.

Definition 2. An allocation $x=(\bar{\mathbf{x}}^1, ..., \bar{\mathbf{x}}^I) \in R^{(S+1)I}_{++}$ is a Pareto optimum if

(i) $\Sigma_{i=1} \bar{\mathbf{x}}^i = \Sigma_{i=1} \boldsymbol{\omega}^i$

(ii) there does not exist $x=(\mathbf{x}^1, ..., \mathbf{x}^I) \in R^{(S+1)I}_{++}$ such that $\Sigma_{i=1} \mathbf{x}^i = \Sigma_{i=1} \boldsymbol{\omega}^i$ and $u^i(\mathbf{x}^i) \geqq u^i(\bar{\mathbf{x}}^i)$, i=1, ..., I with a strict inequality for at least one i.

Lastly we consider a particular feasibility for the economy $\mathscr{E}$(u , ω ; v) .

Definition 3. An allocation $x=(\mathbf{x}^1,...,\mathbf{x}^I) \in R^{(S+1)I}_{++}$ is v-feasible if

(i) $\Sigma_{i=1} \mathbf{x}^i = \Sigma_{i=1} \boldsymbol{\omega}^i$

(ii) $\mathbf{x}^i_1 \in \langle \mathrm{v} \rangle + \boldsymbol{\omega}^i_1$, i=1, ..., I

where $\langle \mathrm{v} \rangle$ indicates a vector subspace spanned by the columns of v.

Let F_v (ω) denote the set of v-feasible allocations with respect to ω . If a tuple ($(\mathbf{x}^i, \mathbf{z}^i)_i$, $\mathbf{q}$) is an asset market equilibrium for $\mathscr{E}$(u , ω ; v) , then obviously $(\mathbf{x}^i)_i$ is an element of F_v(ω). F_v(ω) will play a very critical role in the analysis.

3. Main Result

Our analysis proceeds according to the following scenario. First we consider the set of allocations which satisfy the first order necessary condition for a Pareto optimum. To characterize the set we can use Thom Transversality Theorem in 1-jet space and obtain the result that the set constitutes a definite dimensional manifold in $R^{(S+1)I}_{++}$ generically in u. Then we turn to F_v (ω). It can be easily seen that F_v (ω) is a manifold in $R^{(S+1)I}_{++}$ for every ω. To examine transversality between F_v (ω) and the set previously mentioned we can also use a transversality theorem (another form) again and conclude that for almost all ω the intersection of those sets is empty, which implies the consequence to be shown.

Now let us consider the first order necessary conditions of Pareto optimal allocations.

Proposition 1. Given u and ω fulfilling assumption 2 and 3 and let an allocation $\bar{x}=(\bar{x}^1, \dots, \bar{x}^I) \in R^{(S+1)I}_{++}$ be Pareto optimal. Then the following equation holds.

$$Du^1(\bar{x}^1)/\Sigma_{i=1} Du^1_i(\bar{x}^1) = \dots = Du^I(\bar{x}^I)/\Sigma_{i=1} Du^I_i(\bar{x}^I)$$

Proof. Since a Pareto optimal allocation can be characterized as a solution of the next maximization problem,

$$\begin{aligned} &\text{Max} \quad u^1(x^1) \\ &\text{s.t.} \quad u^i(x^i) \geq \bar{u}^i \quad i=2, \dots, I \\ &\qquad \Sigma_{i=1} x^i = \Sigma_{i=1} \omega^i \end{aligned}$$

the consequence is easily deduced by Lagrange multiplier method. Q.E.D.

Let the set of allocations which satisfy the equation in the proposition be A(u). Note that an element of A(u) is not required to fulfill feasibility (i.e. condition (i) of definition 2).

We are going to apply Thom Transversality Theorem to characterize A(u) generically in u. To this end, some preparations are needed.

Since the structure of A(u) requires us to consider all agents' utility functions simultaneously, we need a product functional version of original Thom Transversality Theorem.

Let $C^r(N,M)^k$ be k-product of $C^r(N,M)$ and $J^r(N,M)^k$ be k-product of r-jet space $J^r(N,M)$ where N and M are differential manifolds. k-product function of $f_1 \times \dots \times f_k \in C^r(N,M)^k$ and its r-jet extension $j^r_k(f_1 \times \dots \times f_k)$ are defined as follows.

$$f_1 \times \dots \times f_k : N^k \rightarrow M^k \text{ by } f_1 \times \dots \times f_k (x_1, \dots, x_k) = (f_1 (x_1), \dots, f_k(x_k))$$

$$j^r_k(f_1 \times \dots \times f_k) : N^k \rightarrow J^r(N,M)^k \text{ by } j^r_k(f_1 \times \dots \times f_k) (x_1, \dots, x_k) = (j^r f_1 (x_1), \dots, j^r f_k(x_k))$$

Theorem 1. Thom Transversality Theorem (product functional form)

Let Q be a submanifold of $J^r(N,M)^k$. Then

$$\mathscr{T} = \{ f_1 \times \dots \times f_k \in C^r(N,M)^k \mid j^r_k(f_1 \times \dots \times f_k) \text{ is transversal with respect to } Q \}$$

is a residual subset of $C^r(N,M)^k$ in the Whitney C^r product topology. So $\mathscr{T}$ is dense in $C^r(N,M)^k$.

Proof. See Theorem 7.3. of Nagata (2001).

Since our product function space is not $C^r(N,M)^k$ itself but a subset (i.e. $\mathscr{U} \subset C^r(R^{S+1}_{++}, R)^I$) we need to check if the above theorem is applicable to the subset.

Proposition 2. $\mathscr{U}$ is an open subset of $C^r(R^{S+1}_{++}, R)^I$.

Proof. It is sufficient to show that U is an open subset of $C^r(R^{S+1}_{++}, R)$ in the Whitney C^1 topology. Consider the following set V.

$$V = \{ (a, b, c_1, \dots, c_{S+1}) \in J^1(R^{S+1}_{++}, R) \mid c_i > 0, i=1, \dots, S+1 \}$$

Then obviously $U = \{ f \in C^r(R^{S+1}_{++}, R) \mid J^1 f (R^{S+1}_{++}) \subset V \}$. Noting that $J^1(R^{S+1}_{++}, R)$ is substantially equal to $R^{S+1}_{++} \times R \times R^{S+1}$, it is obvious that V is open in $J^1(R^{S+1}_{++}, R)$,which implies that U is an open subset of $C^r(R^{S+1}_{++}, R)$.

Q.E.D.

In the sequel we are solely concerned with 1-jet space $J^1(R^{S+1}_{++}, R)$ which is naturally identified with $R^{S+1}_{++} \times R \times R^{S+1}$. Let $J^1_+(R^{S+1}_{++}, R)$ denote $R^{S+1}_{++} \times R \times R^{S+1}_{++}$ which is obviously an open submanifold of $J^1(R^{S+1}_{++}, R)$. Then we define the map

$\phi : J^1{}_+(R^{S+1}{}_{++}, R) \rightarrow R^{S+1}{}_{++} \times R \times \triangle^S{}_+$

by ϕ $(\mathbf{a}, b, c_1, \ldots, c_{S+1}) = (\mathbf{a}, b, c_1/\Sigma_{i=1} c_i, \ldots, c_{S+1}/\Sigma_{i=1} c_i)$ where $\triangle^S{}_+$ indicates strictly positive S-simplex. I-product function of ϕ (denoted by $\Phi : J^1{}_+(R^{S+1}{}_{++}, R)^I \rightarrow (R^{S+1}{}_{++} \times R \times \triangle^S{}_+)^I$) is defined in the same way as $f_1 \times \ldots \times f_k$, i.e. $\Phi(y^1, \ldots, y^I) = (\phi(y^1), \ldots, \phi(y^I))$. Considering Theorem 1, we obtain the following proposition.

Proposition 3. Let W be a submanifold of $(R^{S+1}{}_{++} \times R \times \triangle^S{}_+)^I$. Then

$$\mathscr{T} = \{ u \in \mathscr{U} \mid \Phi \cdot j^1{}_I u \text{ is transversal with respect to } W \}$$

is dense in $\mathscr{U}$.

Proof. See appendix.

Then we have a consequence regarding A(u).

Proposition 4. There exists a dense subset $\mathscr{U}^*$ of $\mathscr{U}$ such that for any u in $\mathscr{U}^*$, A(u) constitutes a $S + I$ dimensional submanifold in $R^{(S+1)I}{}_{++}$.

Proof. Let W be the set $\{(\mathbf{a}^i, b^i, \mathbf{c}^i)_i \in (R^{S+1}{}_{++} \times R \times \triangle^S{}_+)^I \mid \mathbf{c}^1 = \mathbf{c}^2 = \ldots = \mathbf{c}^I \}$. It is obvious that W is a $(S + 1)I + I + S$ dimensional submanifold of $(R^{S+1}{}_{++} \times R \times \triangle^S{}_+)^I$. From proposition 2 there exists a dense subset $\mathscr{U}^*$ of $\mathscr{U}$ such that for any u in $\mathscr{U}^*$, $\Phi \cdot j^1{}_I u$ is transversal with respect to W . Thus for those u, $\Phi \cdot j^1{}_I u^{-1}(W)$ constitutes a $S + I$ dimensional submanifold of $R^{(S+1)I}{}_{++}$. Since $\Phi \cdot j^1{}_I u^{-1}(W) = A(u)$, the proof is completed.

Q.E.D.

Now we turn to $F_v(\omega)$. A particular property of $F_v(\omega)$ is obtainable with an additional assumption on the number of agents, assets and states.

Assumption 4. $(J + 1)I > S + 1$.

Proposition 5. Under assumption 4, $F_v(\omega)$ constitutes a $(J + 1)I - (S + 1)$ dimensional submanifold of $R^{(S+1)I}{}_{++}$ for each $\omega \in R^{(S+1)I}{}_{++}$.

Proof. Let X^i be the set $\{ (x_0, x_1, \ldots, x_S) \in R^{(S+1)}{}_{++} \mid (x_1, \ldots, x_S) \in \langle v \rangle + \omega^i{}_1 \}$. It can be easily seen that X^i is a $J + 1$ dimensional submanifold of $R^{(S+1)}{}_{++}$ since rank v = J. Define a map $g : \Pi_i X^i \rightarrow R^{(S+1)}$ by $g(\mathbf{x}^1, \ldots, \mathbf{x}^I) = \Sigma_i \mathbf{x}^i - \Sigma_i \boldsymbol{\omega}^i$. Since $\dim \Pi_i X^i = (J + 1)I >$ $\dim R^{(S+1)} = S + 1$ by assumption 4, $Dg(\mathbf{x}^1, \ldots, \mathbf{x}^I)$ is surjective at each $(\mathbf{x}^1, \ldots, \mathbf{x}^I) \in \Pi_i X^i$ so that g is a submersion. Since 0 is a regular value of g, $g^{-1}(0) = F_v(\omega)$ is a $(J + 1)I - (S + 1)$ dimensional submanifold of $R^{(S+1)I}{}_{++}$.

Q.E.D.

Finally we investigate transversality between A(u) and $F_v(\omega)$. To this end, we fix an A(u) for any given u of $\mathscr{U}^*$ and check if $F_v(\omega)$ is transversal to A(u) generically in ω. We take some device to do so. First pick an arbitrary ω out of $R^{(S+1)I}{}_{++}$ and fix it. Then define the map ϕ : $F_v(\omega) \times N_1{}^+(\omega) \rightarrow R^{(S+1)I}{}_{++}$ by $\phi(x, y) = x + y$ where $N_1{}^+(\omega) = \{ y \in R^{(S+1)I}{}_{++} \mid \| y - \omega \| < 1 \}$.

Proposition 6. For almost all $y \in N_1{}^+(\omega)$, $\phi(\cdot, y) : F_v(\omega) \rightarrow R^{(S+1)I}{}_{++}$ is transversal with respect to A(u).

Proof. It is obvious from the structure of ϕ that ϕ is a submersion. Thus ϕ is transversal with respect to any submanifold of $R^{(S+1)I}{}_{++}$, especially A(u). Here we apply the transversality theorem (Guillemin and Pollack (1974) p.68) to ϕ and obtain the result.

Q.E.D.

Proposition 7. For each $y \in N_1{}^+(\omega)$, $\phi(F_v(\omega), y) = F_v(\omega + y)$.

Proof. Since $F_v(\omega)$ is the set of $x=(x^1,\ldots,x^I) \in R^{(S+1)I}{}_{++}$ that satisfies (i) $\Sigma_{i=1} x^i = \Sigma_{i=1} \omega^i$, (ii) $x^i_1 \in \langle v \rangle + \omega^i_1$, i=1, ..., I, it is easily seen that the image of $F_v(\omega)$ by ϕ is $F_v(\omega + y)$.

Q.E.D.

Now we are in a position to state our main claim.

Theorem. Under assumptions 1 ~ 4, for almost all u and ω, each equilibrium allocation of the economy $\mathscr{E}(u, \omega ; v)$ is Pareto inefficient.

Proof. Let u be an element of $\mathscr{U}^*$. For any given $\omega \in R^{(S+1)I}{}_{++}$, define $N_1{}^+(\omega)$. Let y be an element of $N_1{}^+(\omega)$ such that $\phi(\cdot, y)$ is transversal with respect to A(u). Note that from proposition 6 such an y is an element of a dense set of $N_1{}^+(\omega)$. Now suppose that $F_v(\omega + y) \cap A(u) \neq \emptyset$. Since $\phi(\cdot, y)$ is transversal with respect to A(u), it follows from proposition 7 that at any $x \in F_v(\omega + y) \cap A(u)$, $T_z F_v(\omega) + T_x A(u) = R^{(S+1)I}$ where T indicates a tangent space and $z = \phi(\cdot, y)^{-1}(x)$. But considering proposition 4, proposition 5 and assumption 4,

$$\begin{aligned} \dim F_v(\omega) + \dim A(u) &= (J+1)I - (S+1) + S + I \\ &\leqq SI + I - 1 \\ &< (S+1)I \\ &= \dim R^{(S+1)I}, \end{aligned}$$

which is a contradiction. Thus $F_v(\omega + y) \cap A(u) = \emptyset$, that is to say, there exist no Pareto optimal allocations fulfilling v-feasibility. Since each equilibrium allocation of $\mathscr{E}(u, \omega ; v)$ is obviously v-feasible, it is Pareto inefficient. Noting that ω is arbitrarily taken from $R^{(S+1)I}{}_{++}$ and u and y are respectively arbitrary elements of the dense sets, our claim follows.

Q.E.D.

4. Concluding Remarks

We have investigated inefficiency of equilibrium allocations with incomplete markets from the generic viewpoint with respect to all agents' utility functions and endowments in one good-two period exchange economy model. One of our findings is that as long as all agents' utility functions are strictly monotone and endowments are strictly positive, for almost all utility functions and endowments each equilibrium allocation with incomplete markets is, if any, Pareto

Magill and Quinzii have also shown inefficiency of equilibrium allocations with incomplete markets. In their argument, however, the agents ' endowments are only considered as economy parameters given their utility functions satisfying monotonicity and some convexity (Magill and Quinzii (1996)). Our result substantially extends theirs in that (i) all agents' utility functions are included in the genericity analysis and (ii) no convexity is needed on the functions.

Another finding is that it is not utility maximization behavior of each agent but his budget constraint (v-feasibility) that yields Pareto inefficiency of equilibrium allocations with incomplete markets. This is easily seen from the fact that $F_v(\omega) \cap A(u) = \emptyset$ generically in u and ω, which is shown in the proof of the theorem . To put it in another way, as soon as the agents participate in the incomplete asset markets, they are kept away from Pareto optimal allocations regardless of their optimization behavior. In this connection, we may add another remark that is more suggestive. It is about likelihood of inefficiency of equilibrium allocations. When J=S, then the incomplete markets model turns into complete markets model so that every equilibrium allocation is Pareto optimal by the first fundamental theorem of welfare economics. Therefore, even if the asset markets are incomplete, likelihood of inefficiency of equilibrium allocations is intuitively expected to be dependent on how much J differs from S. It is intuitively inferred that equilibrium allocations with incomplete markets are more likely to be Pareto optimal when J is close to S than when J is way below S. But it is revealed from our argument that such an intuition is wrong. The reason is this. It is always true that $F_v(\omega) \pitchfork A(u)$ generically in u and ω. The number J has an effect on dim $F_v(\omega)$, that is, the more the number grows, the bigger dim $F_v(\omega)$ is. However, as long as $J < S$, dim $F_v(\omega)+$ dim $A(u) < (S+1)I$, which necessarily yields that $F_v(\omega) \cap A(u) = \emptyset$.

Appendix.

To prove Proposition 2, we give two lemmas.

Lemma 1. Let X, Y and Z be C^r manifolds with W a submanifold of Z and $f: X \to Y$ be C^r mapping. Let $g: Y \to Z$ be a submersion. Then $f \pitchfork g^{-1}(W)$ if and only if $g \cdot f \pitchfork W$.

Proof. See Nagata (2001) , lemma 7.3, p.124.

Lemma 2. A smooth map $f: R^{ln}{}_{++} \to \Delta^{(l-1)n}{}_{+}$ defined by

$$f(\mathbf{x}^1,\dots,\mathbf{x}^n) = (\mathbf{x}^1/\Sigma_{i=1} x^1_i, \dots, \mathbf{x}^n/\Sigma_{i=1} x^n_i)$$

is a submersion.

Proof. Considering the structure of f, it is sufficient to show that a map $h: R^l{}_{++} \to \Delta^{l-1}{}_{+}$ defined by $h(\mathbf{x}) = \mathbf{x}/\Sigma_{i=1} x_i$ is a submersion. To this end, we show that at any $\mathbf{x}$ ($\in R^l{}_{++}$) $Dh(\mathbf{x}): R^l \to R^{l-1}$ is surjective. It is easily seen that i-th column vector of $Dh(\mathbf{x})$ is

$$(x_1/(\Sigma_{i=1} x_i)^2, \dots, x_{i-1}/(\Sigma_{i=1} x_i)^2, \Sigma_{j\neq i} x_j/(\Sigma_{i=1} x_i)^2, x_{i+1}/(\Sigma_{i=1} x_i)^2, \dots, x_l/(\Sigma_{i=1} x_i)^2)^t$$

where t designates the transpose. Thus we see that first $l - 1$ column vectors of $Dh(\mathbf{x})$ are linearly independent. Indeed, set $\Sigma_{i=1}^{l-1} \lambda_i (x_1/(\Sigma_{i=1} x_i)^2, \dots, x_{i-1}/(\Sigma_{i=1} x_i)^2, \Sigma_{j\neq i} x_j/(\Sigma_{i=1} x_i)^2,$

$x_{i+1}/(\Sigma_{i=1} x_i)^2, \ldots, x_l/(\Sigma_{i=1} x_i)^2)' = 0$ where each λ_i is a scalar ($i=1, \ldots, l-1$). Then its l-th equation yields $\Sigma_{i=1}^{l-1} \lambda_i = 0$, which in turn makes each k-th equation turn into $\lambda_k/(\Sigma_{i=1} x_i)=0$ ($k=1, \ldots, l-1$). It follows that $\lambda_1 = \lambda_2 = \ldots = \lambda_{l-1} = 0$. So $Dh(\mathbf{x})$ is surjective.

Q.E.D.

Proof of Proposition 2.

Note that when u is an element of $\mathscr{U}$, then the range of $j^l{}_l u$ is not $J^l(R^{s+1}_{++}, R)^l$ but $J^l{}_+(R^{s+1}_{++}, R)^l$. Take $R^{(s+1)l}_{++}$, $J^l{}_+(R^{s+1}_{++}, R)^l$ and $(R^{s+1}_{++} \times R \times \triangle^s{}_+)^l$ for X, Y and Z in Lemma 1. Then if we show that Φ is a submersion, the proof is completed by Lemma 1 and Theorem 1. But by the structure of Φ, to show its submersiveness we have only to show that the map $\zeta : R^{(s+1)l}_{++} \to \triangle^{sl}_+$ defined by

$$\zeta\,(c^1, c^2, \ldots, c^l) = (c^1/\Sigma_{i=1} c^1_i,\ c^2/\Sigma_{i=1} c^2_i, \ldots, c^l/\Sigma_{i=1} c^l_i)$$

is a submersion. By applying Lemma 2 we see that this map is indeed a submersion.

Q.E.D.

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