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Mean-square stability of numerical schemes for stochastic differential systems

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Abstract

Stochastic differential equations (SDEs) represent physical phenomena dominated by stochastic processes. As for deterministic ordinary differential equations (ODEs), various numerical schemes are proposed for SDEs. We have proposed the *mean-square* stability of numerical schemes for a scalar SDE, that is, the numerical stability with respect to the mean-square norm. However we studied it for only scalar SDEs because of difficulty and complexity in SDE systems. In the present note we will consider a 2-dimensional linear system with one multiplicative noise and try to analyze them.

1 Introduction

We have proposed the numerical mean-square stability (MS-stability) for a scalar stochastic differential equation (SDE) with one multiplicative noise [7]. However we studied it for only scalar SDEs. Komori and Mitsui [4, 5] analyzed numerical MS-stability for a 2-dimensional SDE with special case, that is, simultaneously diagonalizable case. In this note we will try to analyze numerical MS-stability of the Euler-Maruyama scheme for general 2-dimensional SDE systems.

Consider the SDE of Ito-type given by

$$\mathrm{d}X(t) = f(t, X)\mathrm{d}t + g(t, X)\mathrm{d}W(t) \tag{1}$$

with $f(0,t) = g(0,t) = 0$ so that the steady state $X(t) = 0$ is the equilibrium solution. The Euler-Maruyama scheme for the discrete approximate solution $\{\bar{X}_n\}$ is

$$\bar{X}_{n+1} = \bar{X}_n + f(t_n, \bar{X}_n)h + g(t_n, \bar{X}_n)\Delta W_n$$

where h and ΔW_n stand for the step-size and the increment of the Wiener process, respectively. Then we can give the definition of the MS-stability.

Definition 1 *Steady solution* $X(t) \equiv 0$ *is* asymptotically stable *in mean-square if*

$$\forall\varepsilon > 0,\ \exists\delta > 0; \qquad \mathbf{E}\left(\|X(t)\|^2\right) < \varepsilon \quad \text{for all} \quad t \geq 0 \quad \text{and } \|X_0\| < \delta$$

and

$$\exists\delta_0; \qquad \lim_{t\to\infty} \mathbf{E}\left(\|X(t)\|^2\right) = 0 \quad \text{for all} \quad \|X_0\| < \delta_0$$

Here the norm $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^2$.

We will consider three types of linear SDE systems, and try to analyze them. In the next section we describe the results of MS-stability for three types of the SDE system. Section 3 shows the results of numerical MS-stability of the Euler-Maruyama scheme corresponding to results in Section 2. In Section 4 we will show the numerical experiments confirming our stability analysis in Section 3. Finally we will describe our conclusion and future aspects.

2 MS-stability

We will restrict the SDE (1) to an Ito-type 2-dimensional linear SDE system with one multiplicative noise, which has the form

$$\begin{cases} \mathrm{d}\boldsymbol{X}(t) &= \boldsymbol{D}\boldsymbol{X}(t)\mathrm{d}t + \boldsymbol{B}\boldsymbol{X}(t)\mathrm{d}W(t), \\ \boldsymbol{X}(0) &= \mathbf{1}. \end{cases} \tag{2}$$

Here the real constant matrices $\boldsymbol{D}$ and $\boldsymbol{B}$ are given by

$$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_2 & \alpha_2 \end{bmatrix}.$$

Komori and Mitsui [4, 5] analyzed MS-stability for SDE system (2) with $\beta_1 = 0$ and $\beta_2 = 0$ (simultaneously diagonalizable case). We will consider more general SDE system, namely $\beta_1 \neq 0$ and $\beta_2 \neq 0$. First we will introduce the conventional and the logarithmic norms of matrices for stability analysis of the SDE system (2).

Definition 2 *Corresponding to the vector norms l^1, l^2 and l^∞ in $\mathbb{R}^n$, we define the subordinate matrix norms of square $n \times n$ matrix $A = (a_{ij})$ by*

$$\|A\|_1 = \max_j \left\{\sum_{i=1}^n |a_{ij}|\right\}, \qquad \|A\|_\infty = \max_i \left\{\sum_{j=1}^n |a_{ij}|\right\},$$
$$\|A\|_2 = \left\{\textit{maximum eigenvalue of } A^T A\right\}^{1/2}.$$

Definition 3 *Logarithmic matrix norm $\mu[A]$ (see [1, 6]) is defined by*

$$\mu[A] = \lim_{h \to 0+} (\|I + hA\| - 1)/h$$

where I is the unit matrix and $h \in \mathbb{R}$.

For the matrix norms $\|\cdot\|_1$, $\|\cdot\|_\infty$ and $\|\cdot\|_2$, the following identities are well known to evaluate the logarithmic norms.

$$\mu_1[A] = \max_j \left\{a_{jj} + \sum_{i \neq j} |a_{ij}|\right\}, \qquad \mu_\infty[A] = \max_i \left\{a_{ii} + \sum_{j \neq i} |a_{ij}|\right\},$$
$$\mu_2[A] = \text{maximum eigenvalue of } (A + A^T)/2.$$

Let $\boldsymbol{P}(t) = \mathbf{E}(\boldsymbol{X}(t)\boldsymbol{X}(t)^T)$ be the 2×2 matrix-valued second moment of the solution of (2). Then $\boldsymbol{P}(t)$ obeys the initial value problem of the following *matrix* ordinary differential equation (ODE)

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = D\boldsymbol{P} + \boldsymbol{P}D^T + B\boldsymbol{P}B^T \quad (t > 0), \tag{3}$$

with $\boldsymbol{P}(0) = \boldsymbol{X}_0\boldsymbol{X}_0^T$. Due to the symmetry of the matrix $\boldsymbol{P}$ we have its governing ODEs of 3-dimension

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \mathcal{M}Y \tag{4}$$

where

$$\begin{gathered} Y(t) = (Y^1(t), Y^2(t), Y^3(t)), \quad Y^1(t) = \mathbf{E}(X^1(t))^2, \\ Y^2(t) = \mathbf{E}(X^2(t))^2, \quad Y^3(t) = \mathbf{E}(X^1(t)X^2(t)). \end{gathered}$$

We can readily obtain the following lemma owing to the logarithmic matrix norm μ.

Lemma 1 *The linear test system with the unit initial value is asymptotically MS-stable w.r.t. logarithmic norm μ iff*

$$\mu(\mathcal{M}) < 0$$

We will study MS-stability for the following three types of the test system. Drift matrix $\boldsymbol{D}$ in (2) is fixed with real numbers $\lambda_1 < \lambda_2 < 0$ and diffusion matrices $\boldsymbol{B}$ are either

$$\text{Type I:} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad \text{Type II:} \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}, \quad \text{or} \quad \text{Type III}: \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}.$$

Here real numbers α and β are non-negative.

Theorem 1 *In Type I the matrix in (4) is given by*

$$\mathcal{M} = \begin{bmatrix} 2\lambda_1 + \alpha^2 & 0 & 0 \\ 0 & 2\lambda_2 + \alpha^2 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 + \alpha^2 \end{bmatrix}.$$

Henceforth the stability criterion w.r.t. μ_2, μ_∞ and μ_1 yields

$$\max\{2\lambda_1 + \alpha^2, 2\lambda_2 + \alpha^2\} < 0. \tag{5}$$

We employed the following identity to derive (5).

$$\lambda_1 + \lambda_2 + \alpha^2 = \frac{2\lambda_1 + \alpha^2 + 2\lambda_2 + \alpha^2}{2} \tag{6}$$

Type II has the following

Theorem 2 *The coefficient matrix in Type II is given by*

$$\mathcal{M} = \begin{bmatrix} 2\lambda_1 & \beta^2 & 0 \\ \beta^2 & 2\lambda_2 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 + \beta^2 \end{bmatrix},$$

which implies the stability criterion w.r.t. μ_∞ *and* μ_1 *as*

$$\max\{2\lambda_1 + \beta^2, 2\lambda_2 + \beta^2\} < 0.$$

Again we employed (6).

Note that the condition represented by μ_∞ is a sufficient condition for the convergence to the zero solution. We will show this through the following example.

Example 1 The combination with

$$\boldsymbol{D} = \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

yields

$$\mathcal{M} = \begin{bmatrix} -200 & 4 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & -97 \end{bmatrix},$$

whose logarithmic norms are

$$\mu_\infty(\mathcal{M}) = 2 > 0 \quad \text{but} \quad \mu_2(\mathcal{M}) = -101 + \sqrt{9817} < 0.$$

Finally we will study Type III as the composition of Types I and II. We conclude with the theorem.

Theorem 3 *Type III has the coefficient matrix given by*

$$\mathcal{M} = \begin{bmatrix} 2\lambda_1 + \alpha^2 & \beta^2 & 2\alpha\beta \\ \beta^2 & 2\lambda_2 + \alpha^2 & 2\alpha\beta \\ \alpha\beta & \alpha\beta & \lambda_1 + \lambda_2 + \alpha^2 + \beta^2 \end{bmatrix},$$

which brings the stability condition w.r.t. μ_∞ *as*

$$\max\{2\lambda_1 + (|\alpha| + |\beta|)^2, 2\lambda_2 + (|\alpha| + |\beta|)^2\} < 0$$

Note that the stability criterion for Type III is given only in μ_∞.

3 MS-stability of Euler-Maruyama scheme

We now ask what conditions must be imposed in order that the numerical solution $\{\bar{X}_n\}$ of (2) generated by a numerical scheme satisfies

$$\bar{Y}_n = \mathbf{E}|\bar{X}_n|^2 \to 0 \qquad \text{as } n \to \infty. \tag{7}$$

When we apply a numerical scheme to (2) and calculate the components of the second moment of $\bar{X}_n$, we obtain a one-step difference equation of the form

$$\bar{Y}_{n+1} = \overline{\mathcal{M}}\bar{Y}_n \tag{8}$$

where

$$\bar{Y}_n = (\bar{Y}_n^1, \bar{Y}_n^2, \bar{Y}_n^3), \quad \bar{Y}_n^1 = \mathbf{E}(\bar{X}_n^1)^2,$$
$$\bar{Y}_n^2 = \mathbf{E}(\bar{X}_n^2)^2, \quad \bar{Y}_n^3 = \mathbf{E}(\bar{X}_n^1\bar{X}_n^2).$$

We shall call $\overline{\mathcal{M}}$ the *stability matrix* of the scheme. Note that $\bar{Y}_n \to 0$ as $n \to \infty$ *if*

$$||\overline{\mathcal{M}}|| < 1. \tag{9}$$

Definition 4 *The numerical scheme is said to be MS-stable w.r.t.* $||\cdot||$ *if it has* $\overline{\mathcal{M}}$ *satisfying* $||\overline{\mathcal{M}}|| < 1$.

We will calculate the stability matrices $\overline{\mathcal{M}}$ and MS-stability conditions w.r.t. $||\cdot||$ of the Euler-Maruyama scheme for Type I, II and III. Let $r(x)$ be $1 + x$ in the following theorems.

Theorem 4 *For Type I we obtain*

$$\overline{\mathcal{M}} = \begin{bmatrix} r^2(\lambda_1 h) + \alpha^2 h & 0 & 0 \\ 0 & r^2(\lambda_2 h) + \alpha^2 h & 0 \\ 0 & 0 & r(\lambda_1 h)r(\lambda_2 h) + \alpha^2 h \end{bmatrix},$$

which yields the stability condition w.r.t. $||\cdot||_2$, $||\cdot||_\infty$ *and* $||\cdot||_1$ *as*

$$\max\{(1 + \lambda_1 h)^2 + \alpha^2 h, (1 + \lambda_2 h)^2 + \alpha^2 h\} < 1. \tag{10}$$

The inequality

$$r(\lambda_1 h)r(\lambda_2 h) + \alpha^2 h \leq \frac{r^2(\lambda_1 h) + r^2(\lambda_2 h) + 2\alpha^2 h}{2} \tag{11}$$

is utilized to derive the above result. When we observe the left-hand side in the MS-stability condition (10), we conclude to check the numerical MS-stability whether the pair $(\bar{h}, k) = (\lambda h, \alpha^2/\lambda)$ satisfying $|R(\bar{h}, k)| < 1$ for every λ_1 and λ_2. Namely we should check $(\bar{h}_1, k_1) = (\lambda_1 h, \alpha^2/\lambda_1)$, $(\bar{h}_2, k_2) = (\lambda_2 h, \alpha^2/\lambda_2) \in \mathcal{R}_{\mathrm{EM}}$. Here $\mathcal{R}_{\mathrm{EM}}$ is the MS-stability region of the Euler-Maruyama scheme in scalar case. We will show the region in Fig. 1.

Next we will focus on Type II. We will calculate the $\overline{\mathcal{M}}$ and stability condition as same as Type I.

Theorem 5 *Type II has the stability matrix given by*

$$\overline{\mathcal{M}} = \begin{bmatrix} r^2(\lambda_1 h) & \beta^2 h & 0 \\ \beta^2 h & r^2(\lambda_2 h) & 0 \\ 0 & 0 & r(\lambda_1 h) r(\lambda_2 h) + \beta^2 h \end{bmatrix}, \tag{12}$$

which brings the stability condition w.r.t. $\|\cdot\|_\infty$ *and* $\|\cdot\|_1$ *as*

$$\max\{(1+\lambda_1 h)^2 + |\beta^2 h|, (1+\lambda_2 h)^2 + |\beta^2 h|\} < 1.$$

We result in stability function of the Euler-Maruyama scheme (scalar case), namely $R(\bar{h}, k)$ again applicable by $\bar{h} = \lambda h$, $k = \beta^2/\lambda$ like as Type I.

Finally we try to analyze Type III.

Theorem 6 *For Type III we have*

$$\overline{\mathcal{M}} = \begin{bmatrix} r^2(\lambda_1 h) + \alpha^2 h & \beta^2 h & 2\alpha\beta h \\ \beta^2 h & r^2(\lambda_2 h) + \alpha^2 h & 2\alpha\beta h \\ \alpha\beta h & \alpha\beta h & r(\lambda_1 h) r(\lambda_2 h) + (\alpha^2 + \beta^2) h \end{bmatrix},$$

which implies the stability condition w.r.t. $\|\cdot\|_\infty$ *as*

$$\max\{(1+\lambda_1 h)^2 + (|\alpha| + |\beta|)^2 h, (1+\lambda_2 h)^2 + (|\alpha| + |\beta|)^2 h\} < 1.$$

Like as Type I and II, we conclude that stability function of the Euler-Maruyama scheme (scalar case) $R(\bar{h}, k)$ again applicable with $\bar{h} = \lambda h,\ k = (|\alpha| + |\beta|)^2/\lambda$.

4 Numerical experiments

In this section we will show the confirmation for our MS-stability of the Euler-Maruyama scheme through numerical experiments. We will describe four examples corresponding to Type I, II, and III (2 examples) as follows.

Example 2 *(Type I)*

$$\mathrm{d}\boldsymbol{X} = \begin{bmatrix} -200 & 0 \\ 0 & -100 \end{bmatrix} \boldsymbol{X}\mathrm{d}t + \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \boldsymbol{X}\mathrm{d}W(t) \tag{13}$$

$h = 0.005,\ (\bar{h}, k) = (-1, -0.5), (-0.5, -1)$: *stable*
$h = 0.01,\ (\bar{h}, k) = (-2, -0.5), (-1, -1)$: *unstable*
$h = 0.02,\ (\bar{h}, k) = (-4, -0.5), (-2, -1)$: *unstable*
$h = 0.05,\ (\bar{h}, k) = (-10, -0.5), (-5, -1)$: *unstable*

Example 3 *(Type II)*

$$\mathrm{d}\boldsymbol{X} = \begin{bmatrix} -200 & 0 \\ 0 & -100 \end{bmatrix} \boldsymbol{X}\mathrm{d}t + \begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix} \boldsymbol{X}\mathrm{d}W(t)$$

$h = 0.005,\ (\bar{h}, k) = (-1, -0.5), (-0.5, -1)$: *stable*
$h = 0.01,\ (\bar{h}, k) = (-2, -0.5), (-1, -1)$: *unstable*
$h = 0.02,\ (\bar{h}, k) = (-4, -0.5), (-2, -1)$: *unstable*
$h = 0.05,\ (\bar{h}, k) = (-10, -0.5), (-5, -1)$: *unstable*

Example 4 *(Type III)*

$$d\boldsymbol{X} = \begin{bmatrix} -200 & 0 \\ 0 & -100 \end{bmatrix} \boldsymbol{X} dt + \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix} \boldsymbol{X} dW(t)$$

$h = 0.005,\ (\bar{h}, k) = (-1, -0.625), (-0.5, -1.25)$: *stable*
$h = 0.01,\ (\bar{h}, k) = (-2, -0.625), (-1, -1.25)$: *unstable*

Example 5 *(Type III)*

$$d\boldsymbol{X} = \begin{bmatrix} -200 & 0 \\ 0 & -100 \end{bmatrix} \boldsymbol{X} dt + \begin{bmatrix} 5 & 10 \\ 10 & 5 \end{bmatrix} \boldsymbol{X} dW(t)$$

$h = 0.005,\ (\bar{h}, k) = (-1, -0.625), (-0.5, -1.25)$: *stable*
$h = 0.01,\ (\bar{h}, k) = (-2, -0.625), (-1, -1.25)$: *unstable*

We took the initial value $\boldsymbol{X}(0) = (1, 1)$ and $10,000$ samples. We will show the results of Example 2 to Fig. 2, Example 3 to Fig. 3, Example 4 to Fig. 4 and Example 5 to Fig. 5.

5 Conclusions and Future aspects

We extended numerical MS-stability for a scalar SDE with one multiplicative noise to it for a 2-dimensional SDE system with one multiplicative noise. We will analyze MS-stability for general pair of the matrices $\boldsymbol{D}$ and $\boldsymbol{B}$, and more dimensional case. And we will investigate the relation of the MS-stability conditions in matrix norms, for example, between $\|\cdot\|_\infty$ and $\|\cdot\|_2$.

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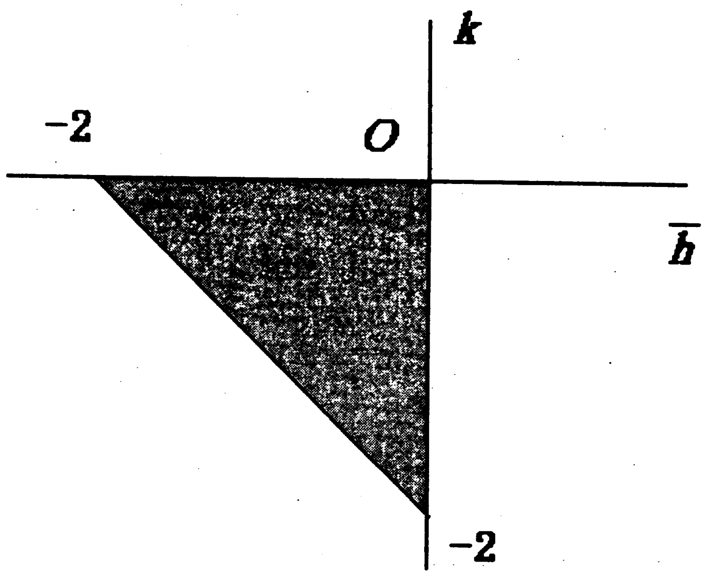

Figure 1: MS-stability region of Euler-Maruyama scheme

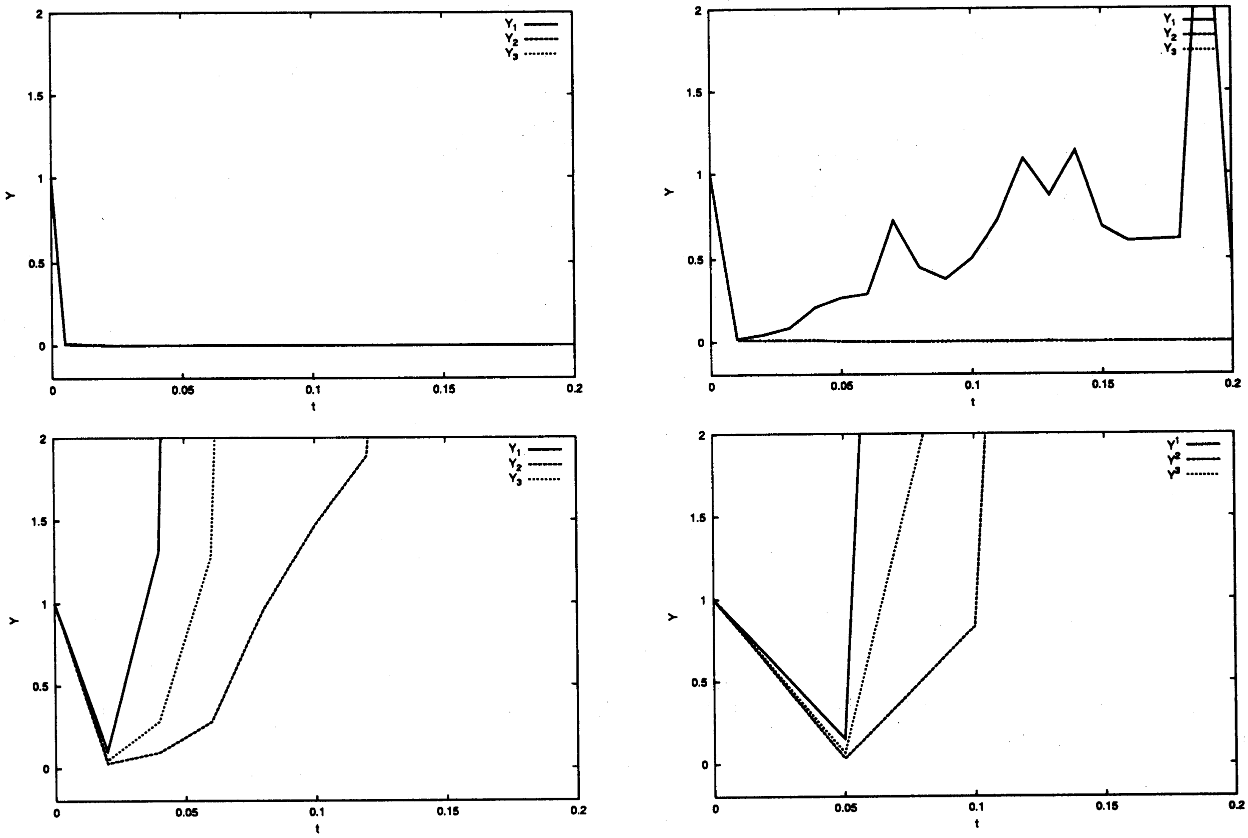

Figure 2: Example 1 (upper left:$h = 0.005$, upper right:$h = 0.01$, lower left:$h = 0.02$, lower right:$h = 0.05$)

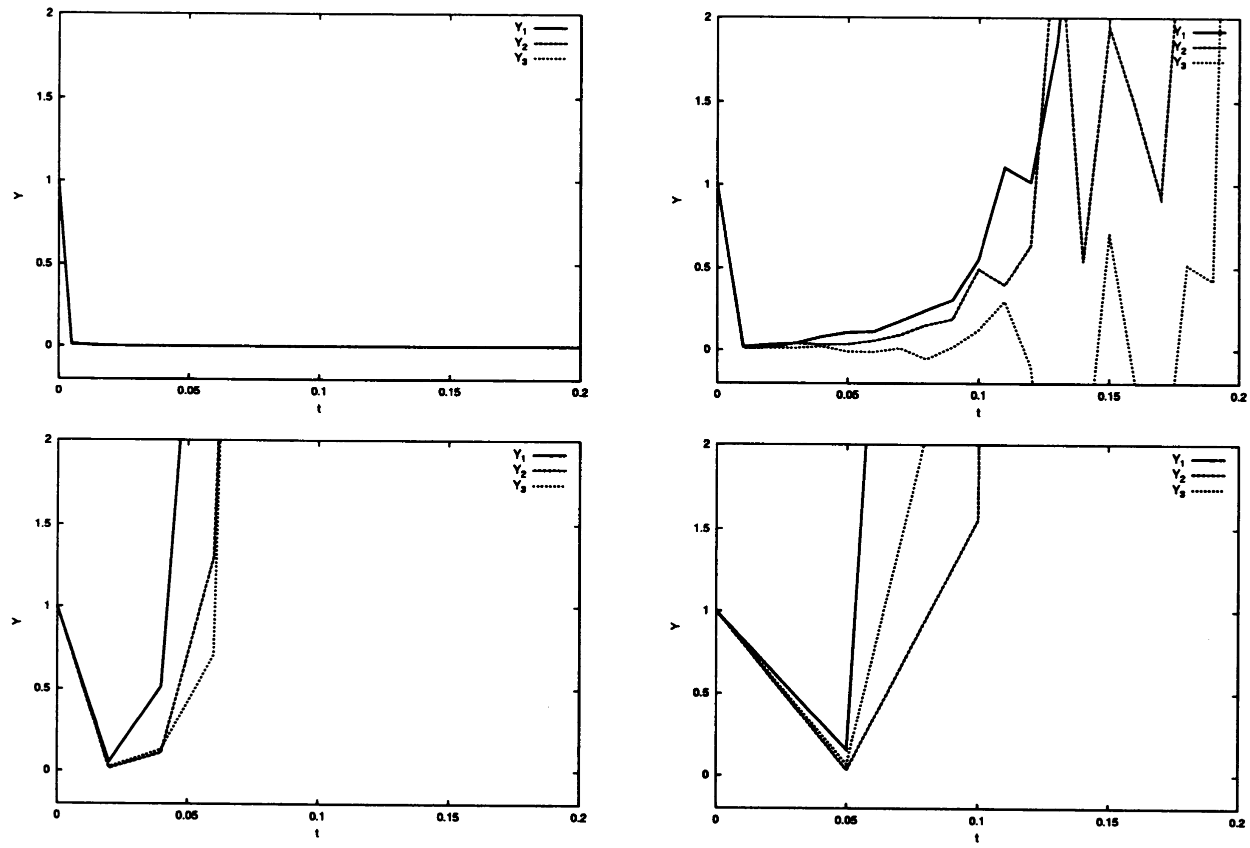

Figure 3: Example 2 (upper left:$h = 0.005$, upper right:$h = 0.01$, lower left:$h = 0.02$, lower right:$h = 0.05$)

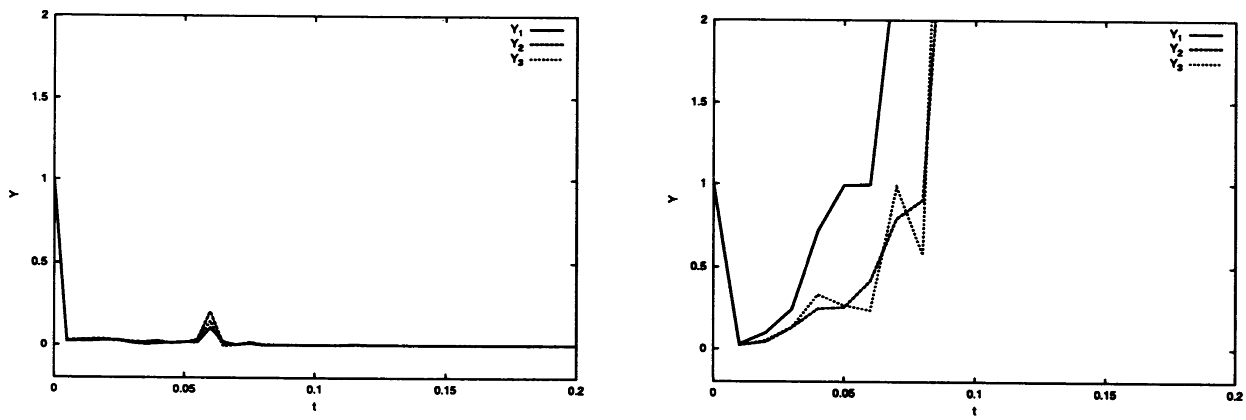

Figure 4: Example 3 (left:$h = 0.005$, right:$h = 0.01$)

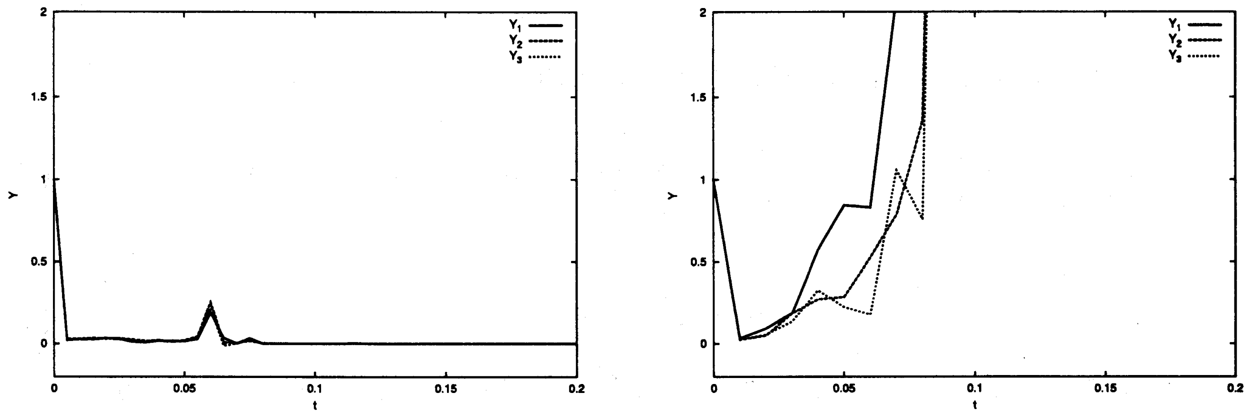

Figure 5: Example 4 (left:$h = 0.005$, right:$h = 0.01$)