Stability analysis of numerical methods for delay integro-differential equations

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Abstract

Stability of $\theta$-methods for delay integro-differential equations (DIDEs) is studied on the basis of the linear equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau) + \kappa \int_{t-\tau}^{t} u(\sigma)d\sigma,$$

where $\lambda$, $\mu$, $\kappa$ are complex numbers and $\tau$ is a constant delay. It is shown that every $A$-stable $\theta$-method possesses a similar stability property to $P$-stability, i.e., the method preserves the delay-independent stability of the exact solution under the condition that $\tau/h$ is an integer, where $h$ is a step-size. It is also shown that the method does not possess the same property if $\tau/h$ is not an integer. As a result, any $\theta$-method cannot possess a similar stability property to $GP$-stability with respect to DIDEs.

1. Introduction

We study stability of (2-stage) $\theta$-methods for delay integro-differential equations (DIDEs) on the basis of the linear equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau) + \kappa \int_{t-\tau}^{t} u(\sigma)d\sigma,$$  \hspace{1cm} (1.1)

where $\lambda$, $\mu$, $\kappa$ are complex numbers and $\tau$ is a constant delay. When $\kappa = 0$, the equation (1.1) coincides with the test equation

$$\frac{du}{dt} = \lambda u(t) + \mu u(t - \tau),$$  \hspace{1cm} (1.2)

which was proposed by Barwell [1] to examine stability of numerical methods for delay differential equations (DDEs). As described in [1], if $\lambda$, $\mu$ satisfy

$$|\mu| < -\text{Re} \lambda,$$  \hspace{1cm} (1.3)

the zero solution of (1.2) is asymptotically stable for any $\tau \geq 0$. This asymptotic property is called delay-independent stability, and analogous stability properties of numerical methods are considered on the basis of the condition (1.3). For example,
a numerical method for DDEs is said to be $P$-stable if every numerical solution to (1.2) tends to zero whenever $\lambda$, $\mu$ satisfy (1.3) and $\tau/h$ is an integer, where $h$ is the step-size. A numerical method is said to be $GP$-stable if the same holds for any constant step-size.

In the last two decades, various studies were carried out concerning stability properties of numerical methods for DDEs (see, e.g., [12]). In particular, an earliest study by Watanabe and Roth [10] has revealed that every $A$-stable $\theta$-method is $GP$-stable. To the contrary, little is known about stability properties of numerical methods for DIDEs. It is quite recent that we studied delay-independent stability of linear DIDEs [7], and even stability of $\theta$-methods for (1.1) remains to be investigated.

By Theorem 2 of [7], the zero solution of (1.1) is asymptotically stable for any $\tau \geq 0$ if and only if $\lambda$, $\mu$, $\kappa$ satisfy

$$\lambda + \mu + \kappa \tau \neq 0 \quad \text{for any } \tau \geq 0,$$

$$z^2 - z \lambda - \kappa = 0, \ z \in \mathbb{C}, \ z \neq 0 \quad \Rightarrow \quad \text{Re } z < 0,$$

$$\left| \frac{\mu z - \kappa}{z^2 - z \lambda - \kappa} \right| < 1 \quad \text{for any } \text{Re } z = 0 \text{ with } z \neq 0. \quad (1.6)$$

Moreover, the conditions (1.5), (1.6) are rewritten as

$$\text{Re } \lambda < 0 \quad \text{and} \quad \text{Re } \lambda \Re(\lambda \kappa) + (\text{Im } \kappa)^2 < 0 \text{ or } \kappa = 0,$$

$$\text{Im}[(\lambda + \mu)\overline{\kappa}] = 0 \quad \text{and} \quad \left| \mu \right|^2 < (\text{Re } \lambda)^2 + 2 \text{Re } \kappa \quad \text{or} \quad \text{Im } \lambda = 0, \ |\mu|^2 = (\text{Re } \lambda)^2 + 2 \text{Re } \kappa \quad (1.7)$$

respectively (Sect. 3 in [7]). When $\lambda$, $\mu$, $\kappa$ are all real and $\kappa \neq 0$, these conditions are reduced to the simple condition

$$\lambda < 0, \ \kappa < 0, \ |\mu|^2 \leq \lambda^2 + 2 \kappa. \quad (1.9)$$

We study stability properties of $\theta$-methods by comparing the region determined by these conditions with a kind of stability regions of the methods.

**Fig. 1** Delay-independent v.s. delay-dependent stability regions
It should be noted that a considerable number of papers [2, 3, 4, 6, 9] are devoted to stability analysis of $\theta$-methods for DDEs, which does not seem strange from a practical viewpoint. Some important instances of stiff DDEs are obtained from the space-descritization of partial functional differential equations (see, e.g., [13]). The $\theta$-methods have practicality in such a situation.

2. Stability regions of $\theta$-methods

Consider delay integro-differential equations (DIDEs) with a constant delay,
\[
\frac{du}{dt} = f\left(t, u(t), u(t-\tau), \int_{t-\tau}^{t} g(t, \sigma, u(\sigma))d\sigma \right). 
\]
(2.1)
For a given step-size $h > 0$, let $m$ be the smallest integer greater than or equal to $\tau/h$. Then, the delay $\tau$ is represented in the form
\[
\tau = (m - \delta)h, \quad 0 \leq \delta < 1, 
\]
(2.2)
and the relation
\[
t_{n} - \tau = t_{n-m} + \delta h 
\]
(2.3)
holds for the step points $t_{n} = t_{0} + nh$, $n \in Z$.

By approximating the delayed argument and the integrand in (2.1) with linear interpolation, we can adapt a $\theta$-method to (2.1) as follows:
\[
u_{n+1} = u_{n} + h(1-\theta)f(t_{n}, u_{n}, v_{n}, G_{n}) + h\theta f(t_{n+1}, u_{n+1}, v_{n+1}, G_{n+1}),
\]
(2.4)
where, $0 \leq \theta \leq 1$, $u_{n}$ is an approximate value of $u(t_{n})$, and
\[
u_{n} = (1-\delta)u_{n-m} + \delta u_{n-m+1},
\]
(2.5)
\[G_{n} = \frac{h(1-\delta)^{2}}{2}g(t_{n}, t_{n-m}, u_{n-m}) + \frac{h(2-\delta^{2})}{2}g(t_{n}, t_{n-m+1}, u_{n-m+1}) + h \sum_{k=2}^{m-1}g(t_{n}, t_{n-m+k}, u_{n-m+k}) + \frac{h}{2}g(t_{n}, t_{n}, u_{n}).
\]
(2.6)

As a result, the integral term of (2.1) is approximated with the trapezoidal rule. When $\theta = 1/2$ and $\delta = 0$, the formula (2.4)-(2.6) determines a method that belongs to a class of Runge-Kutta methods discussed in [7]. But, when $\theta \neq 1/2$, it gives another type of numerical method.

In the case of the test equation (1.1), the formula (2.4)-(2.6) is reduced to
\[
u_{n+1} = u_{n} + (1-\theta)\alpha u_{n} + \theta\alpha u_{n+1} + \beta\left[(1-\delta)(1-\theta)u_{n-m} + (\delta + \theta - 2\delta\theta)u_{n-m+1} + \delta\theta u_{n-m+2}\right] + \gamma\left[\frac{(1-\delta)^{2}(1-\theta)}{2}u_{n-m} + \frac{(2-\delta^{2})(1-\theta) + (1-\theta)^{2}\theta}{2}u_{n-m+1}
\]
\[+ \frac{2 - \delta^{2}\theta}{2}u_{n-m+2} + \sum_{k=3}^{m-1}u_{n-m+k} + \frac{1+\theta}{2}u_{n} + \frac{\theta}{2}u_{n+1}\right],
\]
(2.7)
\[ \alpha = h \lambda, \quad \beta = h \mu, \quad \gamma = h^2 \kappa. \]  

The characteristic equation of (2.7) is written as

\[
z^{m+1} - z^m - (1 - \theta)\alpha z^m - \theta\alpha z^{m+1} \\
- \beta \left[ (1 - \delta)(1 - \theta) + (\delta + \theta - 2\delta \theta)z + \delta \theta z^2 \right] \\
- \gamma \left[ \frac{(1 - \delta)^2(1 - \theta)}{2} + \frac{(2 - \delta^2)(1 - \theta) + (1 - \delta)^2 \theta}{2} z \\
+ \frac{2 - \delta^2 \theta}{2} z^2 + \sum_{k=3}^{m-1} z^k + \frac{1 + \theta}{2} z^m + \frac{\theta}{2} z^{m+1} \right] = 0. \tag{2.9}
\]

Using (2.9) we define the sets \( S_{\theta,m}^{(\delta)} \) and \( S_{\theta}^{(\delta)} \) for \( 0 \leq \delta < 1 \) by

\[
S_{\theta,m}^{(\delta)} = \{ (\alpha, \beta, \gamma) \in \mathbb{C}^3 : \text{all the roots of (2.9) satisfy } |z| < 1 \}, \tag{2.10}
\]

\[
S_{\theta}^{(\delta)} = \bigcap_{m \geq 1} S_{\theta,m}^{(\delta)}. \tag{2.11}
\]

The set \( S_{\theta}^{(\delta)} \) is an analogue of the \( \delta \)-stability region of the \( \theta \)-method [4].

When \( z = 1 \), the left hand side of (2.9) is equal to \(-[\alpha + \beta + (m - \delta)\gamma]\). Hence, for any \( m \geq 1 \), \( z = 1 \) is not a root of (2.9) if and only if

\[ (C_0) \quad \alpha + \beta + (m - \delta)\gamma \neq 0 \text{ for any } m \geq 1. \]

Substituting \( \sum_{k=3}^{m-1} z^k = (z^3 - z^m)/(1 - z) \) into (2.9) and multiplying \((1 - z)\), we get

\[
z^m q(z) - p(z) = 0, \tag{2.12}
\]

\[
q(z) = q_0 z^2 + q_1 z + q_2, \tag{2.13}
\]

\[
p(z) = p_0 z^3 + p_1 z^2 + p_2 z + p_3, \tag{2.14}
\]

where

\[
q_0 = \theta \alpha + \frac{\theta}{2} \gamma - 1, \quad q_1 = (1 - 2\theta)\alpha + \frac{\gamma}{2} + 2,
\]

\[
q_2 = - (1 - \theta)\alpha + \frac{1 - \theta}{2} \gamma - 1,
\]

\[
p_0 = - \delta \theta \beta + \frac{\delta^2 \theta}{2} \gamma, \quad p_1 = (3\delta \theta - \delta - \theta)\beta + \frac{-3\delta^2 \theta + \delta^2 + 2\delta \theta + \theta}{2} \gamma,
\]

\[
p_2 = (-3\delta \theta + 2\delta + 2\theta - 1)\beta + \frac{3\delta^2 \theta - 2\delta^2 - 4\delta \theta - 2\delta^2 + 2\delta + 1}{2} \gamma,
\]

\[
p_3 = (\delta \theta - \delta - \theta + 1)\beta + \frac{-\delta^2 \theta + \delta^2 + 2\delta \theta - 2\delta - \theta + 1}{2} \gamma.
\]

Moreover, we set

\[
r(z) = p(z)/q(z), \tag{2.15}
\]

and consider the following conditions.
(a) $q(z) \neq 0$ for any $|z| \geq 1$.

(b) $|r(z)| < 1$ for any $|z| = 1$ with $z \neq 1$.

These are regarded as conditions for $\alpha$, $\beta$, $\gamma$. We also write

c) $(\alpha, \beta, \gamma) \in S_{\theta}^{(\delta)}$.

Under this notation, we can characterize $S_{\theta}^{(\delta)}$ as follows.

**Theorem 2.1** The following implications hold:

$$(C_0) \text{ and (a) and (b)} \implies (c) \implies (\bar{a}) \text{ and (b)}.$$  

If, in addition,

$$(C_1) \quad p(z), q(z) \text{ have no common zero on } |z| = 1,$$

then (c) implies (a).

**Proof.** Assume $(C_0)$, (a) and (b). We first show that $\hat{r}(z) = r(z)/z$ satisfies $|\hat{r}(z)| < 1$ for any $|z| \geq 1$ with $z \neq 1$.

The linear fractional transformation

$$z = \frac{w + 1}{w - 1} \quad (2.16)$$

maps $\text{Re} w > 0$ conformally onto $|z| > 1$, with $w = \infty$ corresponding to $z = 1$. The function $\hat{R}(w) = \hat{r}[(w + 1)/(w - 1)]$ is represented in the form

$$\hat{R}(w) = \frac{\hat{P}(w)}{\hat{Q}(w)} = \gamma w^2 + (-2\beta + 2\delta\gamma - \gamma)w + 2(1 - 2\delta)\beta - 2\delta(1 - \delta)\gamma \times \left[w - (1 - 2\theta)\right], \quad (2.17)$$

$$\hat{Q}(w) = (w + 1)\left[\gamma w^2 + [2\alpha - (1 - 2\theta)\gamma]w - 2(1 - 2\theta)\alpha - 4\right]. \quad (2.18)$$

Then, it follows from (a) that $\hat{R}(w)$ is a bounded, holomorphic function in $\text{Re} w > 0$. Hence, by the Phragmén-Lindelöf theorem (see, e.g., [8], p. 168), it follows from (b) that $|\hat{R}(w)| < 1$ for any $\text{Re} w > 0$, which implies that $|\hat{r}(z)| < 1$ for any $|z| \geq 1$ with $z \neq 1$.

If $|z| \geq 1$ and $z \neq 1$, then

$$z^m q(z) - p(z) = q(z)z\left[z^{m-1} - \hat{r}(z)\right] \neq 0,$$
which, together with \((C_0)\), implies \((c)\).

Assume \((c)\). If \(q(z_0) = 0\) for some \(|z_0| > 1\), then there exists \(\epsilon > 0\) such that 
\[C(z_0, \epsilon) \subset \{|z| > 1\}\] and \(q(z) \neq 0\) on \(C(z_0, \epsilon)\), where
\[C(z_0, \epsilon) = \{z \in \mathcal{C} : |z - z_0| = \epsilon\}.
By Rouché's theorem, the polynomial \(z^m q(z) - p(z)\) has a root in the interior of \(C(z_0, \epsilon)\) for sufficiently large \(m\), which contradicts \((c)\). Therefore, \((\hat{a})\) holds.

Moreover, if \(|r(z_0)| > 1\) for some \(|z_0| = 1\), then there exists \(\epsilon > 0\) such that \(\epsilon > 0\) such that \(C(z_0, \epsilon) \subset \{|z| > 1\}\) and \(q(z) \neq 0\) on \(C(z_0, \epsilon)\), where \(C(z_0, \epsilon) = \{z \in \mathcal{C} : |z - (1 + \epsilon)z_0| < \epsilon\}\). Hence,
\[\rho = \max_{z \in \mathcal{V}_\epsilon} |\psi(z)| < 1,
and \(1 \in \mathcal{C} \setminus B(0, \rho)\), where \(B(0, \rho) = \{z \in \mathcal{C} : |z| \leq \rho\}\). On the other hand, we have
\[\mathcal{C} \setminus B(0, \rho) \subset \bigcup_{m \geq 1} \{z^m \psi(z) : z \in \mathcal{V}_\epsilon\},
(2.20)
by Proposition 7 of [11]. Since \(|z| > 1\) for any \(z \in \mathcal{V}_\epsilon\), it follows from (2.20) that \(z^m = r(z)\) holds for some \(m \geq 1\) and \(|z| > 1\), which contradicts \((c)\). Therefore, \((\hat{b})\) holds.

It is easy to see that \((\hat{a})\) and \((\hat{b})\) imply \((a)\) under the condition \((C_1)\).

\[\square\]

3. Stability regions in the case \(\delta = 0\)

We consider the case \(\delta = 0\). Since \(q(1) = \gamma\), \(z = 1\) satisfies \(q(z) = 0\) if and only if \(\gamma = 0\). We assume that \(\gamma \neq 0\) for a while, and rewrite the conditions \((a)\), \((\hat{a})\), \((b)\), \((\hat{b})\) by making use of the linear fractional transformation (2.16).

The function \(R(w) = r[(w+1)/(w-1)]\) is represented in the form
\[
R(w) = P(w)/Q(w),
(3.1)
\]
\[
P(w) = (\gamma w - 2\beta)[w - (1 - 2\theta)],
(3.2)
\]
\[
Q(w) = \gamma w^2 + [2\alpha - (1 - 2\theta)\gamma]w - 2(1 - 2\theta)\alpha - 4.
(3.3)
\]
Hence, \((a)\), \((\hat{a})\), \((b)\), \((\hat{b})\) are equivalent to
\[(A)\] \(Q(w) \neq 0\) for any \(\text{Re} w \geq 0\),
\[(\hat{A})\] \(Q(w) \neq 0\) for any \(\text{Re} w > 0\),
\[(B)\] \(|R(w)| < 1\) for any \(\text{Re} w = 0\),
\[(\hat{B})\] \(|R(w)| \leq 1\) for any \(\text{Re} w = 0\),
\[(\hat{B})\] \(|R(w)| \leq 1\) for any \(\text{Re} w = 0\),
respectively.

When $\alpha$, $\gamma$ are real, (A), (A) are reduced to

\[
\gamma\left[2\alpha - (1 - 2\theta)\gamma\right] > 0, \quad \gamma\left[-4 - 2(1 - 2\theta)\alpha\right] > 0,
\]

\[
\gamma\left[2\alpha - (1 - 2\theta)\gamma\right] \geq 0, \quad \gamma\left[-4 - 2(1 - 2\theta)\alpha\right] \geq 0,
\]

respectively. In addition, putting $w = iy$, $y \in \mathbb{R}$, we have

\[
|Q(w)|^2 - |P(w)|^2 = 4\text{Im}\{(\alpha + \beta)\overline{\gamma}\}y^3 + 4(\alpha^2 - \beta^2 + 2\text{Re}\gamma)y^2
\]

\[
+ \left\{16\text{Im}\alpha + 4(1 - 2\theta)^2 \text{Im}\{(\alpha + \beta)\overline{\gamma}\}\right\}y
\]

\[
+ 4 + 2(1 - 2\theta)\alpha\right| y^2 - |2(1 - 2\theta)\beta|^2.
\]

When $\alpha$, $\beta$, $\gamma$ are real, it is reduced to

\[
|Q(w)|^2 - |P(w)|^2 = 4(\alpha^2 - \beta^2 + 2\gamma)y^2 + 4\eta,
\]

\[
\eta = \left[(1 - 2\theta)(\alpha + \beta) + 2\right]\left[(1 - 2\theta)(\alpha - \beta) + 2\right].
\]

Hence, in this case, (B), (B) are equivalent to

\[
\beta^2 \leq \alpha^2 + 2\gamma, \quad \eta > 0,
\]

\[
\beta^2 \leq \alpha^2 + 2\gamma, \quad \eta \geq 0,
\]

respectively.

\[\text{Fig. 2 } \gamma\text{-section of } S^{(0)}_\theta \cap \mathbb{R}^3 (0 \leq \theta < 1/2)\]
Let $\alpha < 0$ and $\gamma < 0$. The conditions (3.4), (3.9) are reduced to

$$\alpha > -\frac{2}{1 - 2\theta}, \quad \gamma > \frac{2\alpha}{1 - 2\theta}, \quad \beta^2 \leq \alpha^2 + 2\gamma, \quad |\beta| < \alpha + \frac{2}{1 - 2\theta},$$

(3.11)

when $0 \leq \theta < 1/2$ (Fig. 2), and

$$\beta^2 \leq \alpha^2 + 2\gamma,$$

(3.12)

when $1/2 \leq \theta \leq 1$. If $\alpha(0), \beta$ satisfy $\beta^2 \leq \alpha^2 + 2\gamma$ for $\gamma < 0$, then $\alpha + \beta < 0$, and (C0) holds. Hence, by Theorem 2.1, these determine the region

$$S_{\theta}^{(0)} \cap \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha < 0, \gamma < 0\},$$

(3.13)

except for ambiguity of the boundary.

We now denote by $\Omega$ the set of all the triplicate $(\lambda, \mu, \kappa)$ for which the zero solution of (1.1) is asymptotically stable for any $\tau \geq 0$, i.e.,

$$\Omega = \{(\lambda, \mu, \kappa) \in \mathbb{C}^3 : (1.4), (1.5), (1.6) are satisfied\}.$$  

(3.14)

It is easy to see that

$$(\lambda, \mu, \kappa) \in \Omega \implies (h\lambda, h\mu, h^2\kappa) \in \Omega \text{ for any } h > 0.$$  

(3.15)

The following theorem shows that $A$-stable $\theta$-methods possess a similar stability property to $P$-stability with respect to DIDEs.

**Theorem 3.2** If $1/2 \leq \theta \leq 1$, then $\Omega \subset S_{\theta}^{(0)}$.

**Proof.** The inclusion $\Omega \cap \{\gamma = 0\} \subset S_{\theta}^{(0)}$ follows from the known result as in the case of DDEs (see, e.g., Theorem 2.6 in [6]). We consider the case $\gamma \neq 0$.

Let $(\alpha, \beta, \gamma) \in \Omega$. The condition (C0) follows from (1.4). Moreover, it follows from (3.6) and $\text{Im}[(\alpha + \beta)\overline{\gamma}] = 0$ that for $w = iy$, $y \in \mathbb{R}$,

$$|Q(w)|^2 - |P(w)|^2 = \eta_0 y^2 + 2\eta_1 y + \eta_2,$$

(3.16)

$$\eta_0 = 4 \left( |\alpha|^2 - |\beta|^2 + 2 \text{Re} \gamma \right), \quad \eta_1 = 8 \text{Im} \alpha,$$

$$\eta_2 = |2(1 - 2\theta)\alpha + 4|^2 - |2(1 - 2\theta)\beta|^2.$$

Since

$$\eta_2 = 16 + 16(1 - 2\theta) \text{Re} \alpha + 4(1 - 2\theta)^2 \left( |\alpha|^2 - |\beta|^2 \right) \geq 16,$$

(3.17)

$$\eta_1^2 - \eta_0 \eta_2 \leq 64(\text{Im} \alpha)^2 - 64 \left( |\alpha|^2 - |\beta|^2 + 2 \text{Re} \gamma \right)$$

$$= -64 \left[ (\text{Re} \alpha)^2 + 2 \text{Re} \gamma - |\beta|^2 \right],$$

(3.18)

we have

$$|Q(w)| \geq |P(w)| \text{ for any } \text{Re} \; w = 0.$$  

(3.19)
which implies (B).

When $\theta = 1/2$, it holds that

$$Q(w) = \gamma w^2 + 2\alpha w - 4 = -w^2 \left[ (2/w)^2 - \alpha (2/w) - \gamma \right]. \quad (3.20)$$

Hence, (A) for $\theta = 1/2$ follows from (1.5).

The condition (A) for $\theta = 1/2$, together with (3.19), implies (A) for $1/2 < \theta \leq 1$. In fact, if $Q(w) = 0$ has a solution with $\text{Re} \ w \geq 0$ for some $1/2 < \theta \leq 1$, then it follows from (A) for $\theta = 1/2$ that there exists $1/2 < \theta_0 \leq \theta$ such that $Q(w) = 0$ for $\theta = \theta_0$ has a solution with $\text{Re} \ w = 0$. But this is impossible by (3.19). \qed

4. Stability regions in the case $\delta \neq 0$

The same result as in Theorem 3.2 does not hold in the case $\delta \neq 0$. As a result, any $\theta$-method cannot possess a similar stability property to GP-stability with respect to DIDEs.

**Theorem 4.3** If $0 < \delta < 1$, there exists $(\alpha, \beta, \gamma) \in \Omega$ which does not belong to $S^{(6)}_{\theta}$.  

**Proof.** The function $R(w) = r[(w+1)/(w-1)]$ can be written as

$$R(w) = \tilde{P}(w)/\tilde{Q}(w), \quad (4.1)$$

$$\tilde{P}(w) = \left[ \gamma w^2 + (-2\beta + 2\delta \gamma - \gamma)w + 2(1-2\delta)\beta - 2\delta(1-\delta)\gamma \right] \times \left[ w - (1 - 2\theta) \right], \quad (4.2)$$

$$\tilde{Q}(w) = (w-1) \left\{ \gamma w^2 + \left[ 2\alpha - (1 - 2\theta)\gamma \right] w - 2(1-2\theta)\alpha - 4 \right\}. \quad (4.3)$$

When $\alpha$, $\beta$, $\gamma$ are real, we have for $w = iy$, $y \in \mathbb{R}$,

$$|\tilde{Q}(w)|^2 - |\tilde{P}(w)|^2 = 4(y^2 + 1) \left[ (\alpha^2 - \beta^2 + 2\gamma)y^2 + \eta \right]$$

$$+ 4\delta(1-\delta)(2\beta - \delta \gamma) \left[ 2\beta + (1-\delta)\gamma \right] y^2 + (1-2\theta)^2, \quad (4.4)$$

$$\eta = \left[ (1 - 2\theta)(\alpha + \beta) + 2 \right] \left[ (1 - 2\theta)(\alpha - \beta) + 2 \right]. \quad (4.5)$$

When $\alpha = -\sqrt{-2\gamma}$ and $\beta = 0$, (4.4) is a quadratic function of $y$ and the coefficient of $y^2$ is given by

$$4 \left[ -(1 - 2\theta)\sqrt{-2\gamma} + 2 \right]^2 - 4\delta^2(1-\delta)^2\gamma^2. \quad (4.6)$$

If $0 < \delta < 1$ and $-\gamma$ is sufficiently large, the value (4.6) is negative. This implies that (h) does not hold near $(\alpha, \beta) = (-\sqrt{-2\gamma}, 0)$, a point on the hyperbola $\beta^2 = \alpha^2 + 2\gamma$, if $-\gamma$ is sufficiently large. Therefore, by Theorem 2.1, there are points in $\Omega$ which do not belong to $S^{(6)}_{\theta}$. \qed
In some cases, the region $S_\theta^{(\delta)} \cap \mathbb{R}^3$ is determined on the basis of Theorem 2.1. Let $1/2 \leq \theta \leq 1$, and assume that $\alpha < 0$ and $\gamma < 0$. Then, (a), which does not depend on $\delta$, is satisfied, and $(C_0)$ holds if $\beta^2 \leq \alpha^2 + 2\gamma$. Moreover, (b) is rewritten as

$$(\text{B}) \quad |\tilde{Q}(w)| > |\tilde{P}(w)| \quad \text{for any} \ Re \ w = 0.$$
The condition \((\tilde{B})\) holds if and only if \(a \geq 0, \ c > 0\), which is equivalent to
\[
\beta^2 \leq \alpha^2 + 2\gamma, \quad \alpha < \frac{\gamma}{4} + 2,
\]
(4.14)
when \(\delta = 1/2\).

References


