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The Massless Nelson Model Without Infrared Cutoff in a Non-Fock Representation

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Abstract

Some results on the Nelson model are reported including: (i) an algebraic definition; (ii) field equations ; (iii) existence of a ground state in the massless case in a non-Fock representation.

Keywords: Nelson model, massless quantum field, Fock space, infrared problem, ground state, field equation

1 Introduction

The Nelson model, which was introduced in [10], describes a system of N quantum particles coupled to a quantum scalar field on the d-dimensional space $\mathbb{R}^d$ ($d, N \in \mathbb{N}$). If the quantum field is massive (resp. massless), then the model is called massive (resp. massless). Nelson showed that, in the massive case, the model is ultravioletly renormalizable, i.e., the model without ultraviolet cutoff (high-energy cutoff) can be constructed within the Hilbert space in which the unperturbed Hamiltonian is defined. Recently Ammari constructed a scattering theory for the renormalized Nelson model [1].

From the view-point of the radiation theory of atoms interacting with the quantized radiation field, it is more important and interesting to consider the massless Nelson model.

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In this note we report on some results of the Nelson model including the massless case. For more details, see [2].

Among problems concerning the massless Nelson model, the problem of existence of a ground state is particularly important. It should be remarked, however, that, generally speaking, there is some subtlety on the existence of ground states of massless quantum field models. This is due to possible infrared divergences, which are related to the phenomenon where the total energy of bosons emitted at low energy is finite, but the number of such bosons (*soft bosons*) blows up. Indeed, in a class of models which describe interactions of particles and massless quantum fields, it is proved or suggested that they have no ground states if no infrared cutoff is made in the interactions, althogh it may depend on the strength of the parameters contained in the Hamiltonians [3, 4, 6, 13]. On the other hand, the Pauli-Fierz model in non-relativistic quantum electrodynamics without infrared cutoff has a ground state [5, 8].

It has been shown that the massless Nelson model *with infrared cutoff (low-energy cutoff)* has a ground state [7, 14]. A natural question is then if the model *without infrared cutoff* has a ground state or not. As for this problem, Lörinczi, Minlos and Spohn [9] proved that, in the case $d = 3$ and $N = 1$, the massless Nelson model without infrared cutoff has no ground state within the Fock space where the time-zero fields are given by the usual Fock representation of the canonical commutation relations (CCR) indexed by $\mathcal{S}_{\text{real}}(\mathbb{R}^d)$, the space of real-valued, rapidly decreasing C^∞-functions on $\mathbb{R}^d$.

The purpose of this note is to point out that, if we consider the massless Nelson model in a non-Fock representation of the CCR for time-zero fields, then it has a ground state even in the case where no infrared cutoff is made. This new representation of the massless Nelson model is inequivalent to the Fock one if no infrared cutoff is made.

2 Algebraic characterization and field equations of the Nelson model

2.1 The standard Nelson model

We first review the Nelson model in the standard form [10]. We call it the standard Nelson model (SNM).

The coordinate of the configuration space $\mathbb{R}^{dN}$ of the N particles is denoted $q = (q_1, \cdots, q_N) \in \mathbb{R}^{dN}$ with $q_j := (q_{j1}, \cdots, q_{jd}) \in \mathbb{R}^d$ $(j = 1, \cdots, N)$. The Hamiltonian of the particle system is then given by the Schrödinger operator

$$H_{\text{p}} := -\sum_{j=1}^{N} \frac{1}{2m_j} \Delta_{q_j} + V, \tag{2.1}$$

acting on $L^2(\mathbb{R}^{dN})$, where $m_j > 0$ is the mass of the j-th particle and Δ_{q_j} is the d-dimensional generalized Laplacian in the variable q_j.

For a linear operator T, we denote its domain by $D(T)$.

We assume the following:

(H.1) The operator H_{p} is self-adjoint on its natural domain $D(H_{\text{p}}) = \cap_{j=1}^{N} D(\Delta_{q_j}) \cap D(V)$ and bounded from below.

The Hilbert space for state vectors of the quantum scalar field is given by

$$\mathcal{F}_{\mathrm{b}} := \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} L^2(\mathbb{R}^d), \tag{2.2}$$

the Boson Fock space over $L^2(\mathbb{R}^d)$, where $\otimes_{\mathrm{s}}^{n} L^2(\mathbb{R}^d)$ is the symmetric tensor product of $L^2(\mathbb{R}^d)$ $(\otimes_{\mathrm{s}}^{0} L^2(\mathbb{R}^d) := \mathbb{C})$.

Let ω be a nonnegative Borel measurable function on $\mathbb{R}^d$ such that $0 < \omega(k) < \infty$ for almost everywehre (a.e.) $k \in \mathbb{R}^d$ with respect to the d-dimensional Lebesgue measure and

$$\omega(k) = \omega(-k) \quad \text{a.e.} k. \tag{2.3}$$

For a.e. $k \in \mathbb{R}^d$, $\omega(k)$ physically means the energy of one free boson with momentum k. The function ω defines a nonnegative self-adjoint multiplication operator on $L^2(\mathbb{R}^d)$ which is injective.

Remark 2.1 A physical example for ω is given by

$$\omega_m(k) := \sqrt{k^2 + m^2}, \quad k \in \mathbb{R}^d, \tag{2.4}$$

with $m \geq 0$ a constant denoting the mass of one boson. We are interested in the massless case $m = 0$: $\omega_0(k) = |k|$

The free Hamiltonian of the quantum scalar field is defined by

$$H_{\mathrm{b}} := d\Gamma(\omega), \tag{2.5}$$

the second quantization of ω [11, §X.7].

We denote the annihilation operators on $\mathcal{F}_{\mathrm{b}}$ by $a(f) = \int_{\mathbb{R}^d} a(k) f(k)^* dk$ $(f \in L^2(\mathbb{R}^d))$ [11, §X.7]. The symmetric operator

$$\Phi_{\mathrm{S}}(f) := \frac{1}{\sqrt{2}}(a(f)^* + a(f)), \tag{2.6}$$

called the Segal field operator, is essentially self-adjoint [11, §X.7]. We denote its closure by the same symbol.

Let $\mathcal{S}'_{\mathrm{real}}(\mathbb{R}^d)$ be the set of real tempered distributions on $\mathbb{R}^d$ and, for each $s \in \mathbb{R}$,

$$H_{\omega}^{s} := \{f \in \mathcal{S}'_{\mathrm{real}}(\mathbb{R}^d) | \omega^s \hat{f} \in L^2(\mathbb{R}^d)\}, \tag{2.7}$$

where

$$\hat{f}(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ikx} dx \tag{2.8}$$

is the Fourier transform of f.

For $f \in H_{\omega}^{-1/2}$ and $g \in H_{\omega}^{1/2}$, we can define

$$\phi_{\mathrm{F}}(f) := \Phi_{\mathrm{S}}\left(\frac{\hat{f}}{\sqrt{\omega}}\right), \quad \pi_{\mathrm{F}}(g) := \Phi_{\mathrm{S}}\left(i\sqrt{\omega}\hat{g}\right). \tag{2.9}$$

Let

$$\mathcal{F}_0 := \{\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}} | \psi^{(n)} = 0 \text{ for all but finitely many } n\text{'s}\}, \tag{2.10}$$

called the subspace of finite-particle vectors. Then, for all $f \in H_\omega^{-1/2}$ and $g \in H_\omega^{1/2}$, $\phi_{\mathrm{F}}(f)$ and $\pi_{\mathrm{F}}(g)$ leave $\mathcal{F}_0$ invariant satisfying the CCR

$$[\phi_{\mathrm{F}}(f), \pi_{\mathrm{F}}(g)] = i \int_{\mathbb{R}^d} \hat{f}(k)^* \hat{g}(k) dk, \tag{2.11}$$

$$[\phi_{\mathrm{F}}(f), \phi_{\mathrm{F}}(f')] = 0, \quad [\pi_{\mathrm{F}}(g), \pi_{\mathrm{F}}(g')] = 0, \ f, f' \in H_\omega^{-1/2}, g, g' \in H_\omega^{1/2}, \tag{2.12}$$

on $\mathcal{F}_0$. Namely, $\{\phi_{\mathrm{F}}(f), \pi_{\mathrm{F}}(g) | f \in H_\omega^{-1/2}, g \in H_\omega^{1/2}\}$ gives a representation of the CCR. This representation is called the Fock representation of the CCR. The time-zero fields in the SNM are given by $\phi_{\mathrm{F}}(f)$ and $\pi_{\mathrm{F}}(g)$.

The Hilbert space for the total system of the particles and the quantum scalar field in the SNM is

$$\mathcal{H} := L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}_{\mathrm{b}}. \tag{2.13}$$

As usual, we freely use the natural identification of $\mathcal{H}$ with $\int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{F}_{\mathrm{b}} dq$, the constant fibre direct integral with base space $(\mathbb{R}^{dN}, dq)$ and fibre $\mathcal{F}_{\mathrm{b}}$ [12, §XIII.16].

In what follows, for notational simplicity, a decomposable operator $A = \int_{\mathbb{R}^{dN}}^{\oplus} A(q) dq$ on $\mathcal{H}$ with fibre $A(q)$ (which is an operator on $\mathcal{F}_{\mathrm{b}}$ for each $q \in \mathbb{R}^{dN}$) is denoted $A(q)$ also.

To describe an interaction between the particles and the quantum scalar field, we fix distributions $\rho_j \in \mathcal{S}'_{\mathrm{real}}(\mathbb{R}^d), j = 1, \cdots, N$, which satisfy the following:

(H.2) For $j = 1, \cdots, N$,

$$\rho_j \in H_\omega^s, \quad s = -1/2, \ -1. \tag{2.14}$$

The interaction of the particles and the quantum field in the SNM is given by the operator

$$H_I^{\mathrm{F}} := \sum_{j=1}^{N} \Phi_{\mathrm{S}} \left(e^{-ikq_j} \frac{\hat{\rho}_j^*}{\sqrt{\omega}} \right) \tag{2.15}$$

acting in $\mathcal{H}$. Formally we have $H_I^{\mathrm{F}} = \sum_{j=1}^N \int_{\mathbb{R}^d} \phi_{\mathrm{F}}(q_j - x) \rho_j(x) dx$, where

$$\phi_{\mathrm{F}}(x) := \int_{\mathbb{R}^d} \frac{1}{\sqrt{2(2\pi)^d \omega(k)}} \{a(k)^* e^{-ikx} + a(k) e^{ikx}\} dk.$$

The total Hamiltonian of the SNM is defined by

$$H_{\mathrm{SNM}} := H_0 + \lambda H_I^{\mathrm{F}}, \tag{2.16}$$

where

$$H_0 := H_{\mathrm{p}} + H_{\mathrm{b}} \tag{2.17}$$

and $\lambda \in \mathbb{R} \setminus \{0\}$ denotes the coupling constant of the model.

Proposition 2.1 *Assume (H.1) and (H.2). Then H_{SNM} is self-adjoint with $D(H_{\mathrm{SNM}}) = D(H_0)$ and bounded from below. Moreover, H_{SNM} is essentially self-adjoint on each core of H_0*

In summary, the SNM is characterized in terms of the Hamiltonian H_{SNM} and the time-zero-fields $\{\phi_{\mathrm{F}}(f), \pi_{\mathrm{F}}(g) | f \in H_\omega^{-1/2}, g \in H_\omega^{1/2}\}$.

2.2 An abstract definition of the Nelson model

As is seen above, the SNM uses the Fock representation of the CCR to give its time-zero fields and its Hamiltonain. We want to define the Nelson model in a way independent of the choice of representations of the CCR for time-zero fields. A natural manner for this is to use commutation relations fulfilled by observables, i.e., to find a possible Lie algebraic structure. We shall take commutation relations in a weak sense.

We denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{X}}$ respectively. But, if there is no danger of confusion, then we simply write them as $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$.

Definition 2.2 Let A and B be densely defined linear operators on a Hilbert space $\mathcal{X}$ and $\mathcal{D}$ be a subspace of $\mathcal{X}$ such that $\mathcal{D} \subset D(A) \cap D(B) \cap D(A^*) \cap D(B^*)$. Then we define a quadratic form $[A, B]_{\mathrm{w}}^{\mathcal{D}}$ by

$$[A, B]_{\mathrm{w}}^{\mathcal{D}}(\psi, \phi) := \langle A^*\psi, B\phi \rangle - \langle B^*\psi, A\phi \rangle, \quad \psi, \phi \in \mathcal{D}.$$

("w" means "weak".)

Remark 2.2 If A and B are bounded on $\mathcal{X}$ with $D(A) = D(B) = \mathcal{X}$, then, for all dense subspaces $\mathcal{D}$ of $\mathcal{X}$, $[A, B]_{\mathrm{w}}^{\mathcal{D}}(\psi, \phi) = \langle \psi, [A, B]\phi \rangle$ for all $\psi, \phi \in \mathcal{D}$, where $[A, B] := AB - BA$ (the usual commutator).

Let $\mathcal{F}$ be a Hilbert space and

$$\mathcal{K} := L^2(\mathbb{R}^{dN}) \otimes \mathcal{F} = \int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{F} dq. \tag{2.18}$$

Definition 2.3 Assume (H.1) and (H.2). Let s_0 and s_1 be real constants. A Nelson model is a set $\mathbb{M}_{\text{Nelson}} := \{\mathcal{K}, \mathcal{D}, L_\lambda, \{\phi(f), \pi(g) | f \in H_\omega^{s_0}, g \in H_\omega^{s_1}\}\}$ having the following properties:

(N.1) $\mathcal{D}$ is a dense subspace of $\mathcal{K}$.

(N.2) L_λ is a symmetric operator on $\mathcal{K}$ and the operator

$$H_{\text{NM}} := H_{\text{p}} + L_\lambda \tag{2.19}$$

is self-adjoint ($D(H_{\text{NM}}) := D(H_{\text{p}}) \cap D(L_\lambda)$). We call H_{NM} the Hamiltonian of the Nelson model $\mathbb{M}_{\text{Nelson}}$.

(N.3) The function ω is such that

$$\mathcal{S}_{\text{real}}(\mathbb{R}^d) \subset H_\omega^{s_0} \cap H_\omega^{s_0+2} \cap H_\omega^{s_1} \cap H_\omega^1 \tag{2.20}$$

and, for all $j = 1, \cdots, N, \mu = 1, \cdots, d$,

$$D_\mu \rho_j \in H_\omega^{s_0}, \tag{2.21}$$

where D_μ $(\mu = 1, \cdots, d)$ is the generalized partial differential operator in the μ-th variable x_μ in $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$.

(N.4) $\{\mathcal{K}, \mathcal{D}, \{\phi(f), \pi(f) | f \in \mathcal{S}_{\text{real}}(\mathbb{R}^d)\}\}$ is a representation of the CCR indexed by $\mathcal{S}_{\text{real}}(\mathbb{R}^d)$. Namely, for all $f, g \in \mathcal{S}_{\text{real}}(\mathbb{R}^d)$, $\phi(f)$ and $\pi(f)$ are self-adjoint operators on $\mathcal{K}$ with $\mathcal{D} \subset D(\phi(f)\pi(g)) \cap D(\pi(f)\phi(g)) \cap D(\phi(f)\phi(g)) \cap D(\pi(f)\pi(g))$ satisfying the CCR

$$[\phi(f), \pi(g)] = i \int_{\mathbb{R}^d} f(x)g(x)dx, \tag{2.22}$$

$$[\phi(f), \phi(g)] = 0, \quad [\pi(f), \pi(g)] = 0, \tag{2.23}$$

on $\mathcal{D}$ and the linearity:

$$\phi(af + bg) = a\phi(f) + b\phi(g), \quad \pi(af + bg) = a\pi(f) + b\pi(g), \quad a, b \in \mathbb{R},$$

on $\mathcal{D}$.

(N.5) Let $X = q_{j\mu}, p_{j\mu}, Y = \phi(f), \pi(f), j = 1, \cdots, N, \mu = 1, \cdots, d, \ f \in \mathcal{S}_{\text{real}}(\mathbb{R}^d)$. There exists a dense subspace $\mathcal{E}$ of $\mathcal{K}$ such that

$$\mathcal{E} \subset D(X) \cap D(Y) \cap D(L_\lambda) \cap D(H_{\text{p}}) \tag{2.24}$$

and the following relations hold in the sense of quadratic form on $\mathcal{E}$:

$$\begin{aligned}
&[L_\lambda, q_{j\mu}]_{\text{w}}^{\mathcal{E}} = 0, \quad [L_\lambda, p_{j\mu}]_{\text{w}}^{\mathcal{E}} = i\lambda\phi(D_\mu \rho_j(q_j - \cdot)), \\
&[L_\lambda, \phi(f)]_{\text{w}}^{\mathcal{E}} = -i\pi(f), \\
&[L_\lambda, \pi(f)]_{\text{w}}^{\mathcal{E}} = i\phi(\omega(-i\nabla)^2 f) + i\lambda \sum_{j=1}^{N} (\rho_j * f)(q_j), \\
&[X, Y]_{\text{w}}^{\mathcal{E}} = 0, \quad [H_{\text{p}}, Y]_{\text{w}}^{\mathcal{E}} = 0,
\end{aligned}$$

where $\nabla = (D_1, \cdots, D_d)$ and $\rho_j * f$ is the convolution of ρ_j and f: $(\rho_j * f)(x) := \int_{\mathbb{R}^d} \rho_j(x - y)f(y)dy$.

Remark 2.3 (i) We have

$$(\omega(-i\nabla)^2 f)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \omega(k)^2 \hat{f}(k) e^{ikx} dk, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Hence (N.3) implies that

$$\omega(-i\nabla)^2 f \in H_\omega^{s_0},$$

so that $\phi(\omega(-i\nabla)^2 f)$ is defined.

(ii) It follows from (H.2) and (N.3) that, for all $f \in \mathcal{S}_{\text{real}}(\mathbb{R}^d)$, $\rho_j * f$ is a bounded continuous function on $\mathbb{R}^d$, so that $\rho_j * f(q_j)$ is a bounded self-adjoint multiplication operator with $D(\rho_j * f(q_j)) = \mathcal{K}$.

The SNM is indeed a Nelson model in the above sense. We introduce

$$\mathcal{F}_{\omega,\text{fin}} := \mathcal{L}\{\Omega_0, \ a(f_1)^* \cdots a(f_n)^* \Omega_0 | n \in \mathbb{N}, f_j \in D(\omega), j = 1, \cdots, n\}, \tag{2.25}$$

where $\Omega_0 := \{1, 0, 0, \cdots\} \in \mathcal{F}_{\text{b}}$ is the Fock vacuum and $\mathcal{L}\{\cdots\}$ means the subspace algebraically spanned by all the vectors in the set $\{\cdots\}$, and

$$\mathcal{D}_0 := C_0^\infty(\mathbb{R}^{dN}) \otimes_{\text{alg}} \mathcal{F}_{\omega,\text{fin}}, \tag{2.26}$$

where $\otimes_{\text{alg}}$ denotes algebraic tensor product.

Proposition 2.4 *Assume (H.1) and (H.2). Suppose that*

$$V \in L^2_{\mathrm{loc}}(\mathbb{R}^{dN}) := \left\{ u : \mathbb{R}^{dN} \to \mathbb{C} \,\middle|\, \int_{|q|\le R} |u(q)|^2 dq < \infty, \forall R > 0 \right\}, \tag{2.27}$$

$$\mathcal{S}_{\mathrm{real}}(\mathbb{R}^d) \subset H^{3/2}_\omega \cap H^{-1/2}_\omega, \tag{2.28}$$

and, for $j = 1, \cdots, N, \mu = 1, \cdots, d$,

$$D_\mu \rho_j \in H^{-1/2}_\omega. \tag{2.29}$$

Let

$$L^{\mathrm{SNM}}_\lambda := H_{\mathrm{b}} + \lambda H^{\mathrm{F}}_I. \tag{2.30}$$

Then

$$\mathbb{M}_{\mathrm{SNM}} := \{\mathcal{H}, \mathcal{D}_0, L^{\mathrm{SNM}}_\lambda, \{\phi_{\mathrm{F}}(f), \pi_{\mathrm{F}}(g) | f \in H^{-1/2}_\omega, g \in H^{1/2}_\omega\}\}$$

is a Nelson model.

2.3 Field equations

We derive field equations for the Nelson model $\mathbb{M}_{\mathrm{Nelson}}$ in a weak sense.

Definition 2.5 Let $\mathcal{X}$ be a Hilbert space and $A(t)$ ($t \in \mathbb{R}$) be a linear operator on $\mathcal{X}$. Suppose that there exists a dense subspace $\mathcal{D}$ of $\mathcal{X}$ such that $\mathcal{D} \subset D(A(t))$ for all $t \in \mathbb{R}$. We say that $A(t)$ is weakly differentiable on $\mathcal{D}$ if, for all $\psi, \phi \in \mathcal{D}$, the function $\langle \psi, A(t)\phi \rangle$ is differentiable in $t \in \mathbb{R}$. In that case we define a quadratic form w-$dA(t)/dt|_{\mathcal{D}}$ on $\mathcal{D}$ by

$$\text{w-}\left.\frac{dA(t)}{dt}\right|_{\mathcal{D}}(\psi, \phi) := \frac{d}{dt} \langle \psi, A(t)\phi \rangle, \quad \psi, \phi \in \mathcal{D}.$$

Definition 2.6 Let $\mathcal{X}$ be a Hilbert space and H be a self-adjoint operator on $\mathcal{H}$. Let A be a delnsely defined linear operator on $\mathcal{X}$. We say that A is in the set A_H if it satisfies the following (i) and (ii):

(i) There exists a dense subspace $D_A \subset D(H)$ such that, for all $s \in \mathbb{R}$, $e^{isH} D_A \subset D(A) \cap D(A^*)$.

(ii) For all $\psi \in D_A$, the $\mathcal{X}$-valued functions: $s \to Ae^{isH}\psi$ and $s \to A^* e^{isH}\psi$ are strongly continuous on $\mathbb{R}$.

For $A \in \mathsf{A}_H$, we set

$$D_{A,H} := \mathcal{L}\{e^{isH}\psi | \psi \in D_A, s \in \mathbb{R}\}. \tag{2.31}$$

The canonical Heisenberg operators of the Neson model $\mathbb{M}_{\mathrm{Nelson}}$ (Definition 2.3) are defined as follows:

$$q_{j\mu}(t) := e^{itH_{\mathrm{NM}}} q_{j\mu} e^{-itH_{\mathrm{NM}}}, \quad p_{j\mu}(t) := e^{itH_{\mathrm{NM}}} p_{j\mu} e^{-itH_{\mathrm{NM}}}, \tag{2.32}$$

$$\phi(t, f) := e^{itH_{\mathrm{NM}}} \phi(f) e^{-itH_{\mathrm{NM}}}, \quad \pi(t, f) := e^{itH_{\mathrm{NM}}} \pi(t, f) e^{-itH_{\mathrm{NM}}}, \tag{2.33}$$

$$j = 1, \cdots, N, \ \mu = 1, \cdots, d, \ f \in \mathcal{S}_{\mathrm{real}}(\mathbb{R}^d), \ t \in \mathbb{R}.$$

For a self-adjoint operator T, $Q(T)$ denotes the form domain of the quadratic form associated with T.

Theorem 2.7 *Consider the Nelson model* $\mathbb{M}_{\rm Nelson}$ *(Definition 2.3). Assume (H.1), (H.2) and the following (A.1)–(A.5):*

(A.1) *(2.27) holds and* $H_{\rm p}$ *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}^{dN})$.

(A.2) *For all* $j = 1, \cdots, N$, $\mu = 1, \cdots, d$, *the distributional partial derivative* $D_{j\mu}V$ *in the variable* $q_{j\mu}$ *is a Borel measurable function on* $\mathbb{R}^{dN}$ *which is a.e. finite with respect to the Lebesgue measure. Moreover,* $C_0^\infty(\mathbb{R}^{dN})$ *is a form core of the self-adjoint multiplication operator* $D_{j\mu}V$ *and* $D(H_{\rm p}) \subset \cap_{j=1}^N \cap_{\mu=1}^d Q(D_{j\mu}V)$.

(A.3) *In addition to (2.24),*

$$\mathcal{E} \subset D(H_{\rm p}) \cap \left\{\cap_{j,l=1}^N \cap_{\mu,\nu=1}^d [D(p_{j\mu}q_{l\nu}) \cap D(q_{l\nu}p_{j\mu})]\right\} \cap \left[\cap_{j=1}^N \cap_{\mu=1}^d Q(D_{j\mu}V)\right].$$

(A.4) *For all* $s \in \mathbb{R}$, $e^{isH_{\rm NM}}$ *leaves* $\mathcal{E}$ *invariant:* $e^{isH_{\rm NM}}\mathcal{E} \subset \mathcal{E}$

(A.5) *For all* $j = 1, \cdots, N$, $\mu = 1, \cdots, d$ *and* $\psi \in \mathcal{E}$, *the* $\mathcal{K}$*-valued function* $q_{j\mu}e^{isH_{\rm NM}}\psi$ *is strongly continuous in* $s \in \mathbb{R}$.

(A.6) *For all* $f \in \mathcal{S}_{\rm real}(\mathbb{R}^d)$, *the* $\mathcal{K}$*-valued functions* $\phi(f)e^{isH_{\rm NM}}\psi$ *and* $\pi(f)e^{isH_{\rm NM}}\psi$ *are strongly continuous in* $s \in \mathbb{R}$.

Then the Heisenberg operators $q_{j\mu}(t), p_{j\mu}(t), \phi(t,f)$ *and* $\pi(t,f)$ $(f \in \mathcal{S}_{\rm real}(\mathbb{R}^d))$ *are weakly differentiable on* $\mathcal{E}$ *and*

$$\text{w-}\frac{dq_{j\mu}(t)}{dt}\bigg|_{\mathcal{E}} = \frac{p_{j\mu}(t)}{m_j}, \tag{2.34}$$

$$\text{w-}\frac{dp_{j\mu}(t)}{dt}\bigg|_{\mathcal{E}} = -D_{j\mu}V(q(t)) - \lambda e^{itH_{\rm NM}}\phi(D_\mu\rho_j(q_j - \cdot))e^{-itH_{\rm NM}}, \tag{2.35}$$

$$\text{w-}\frac{d\phi(t,f)}{dt}\bigg|_{\mathcal{E}} = \pi(t,f), \tag{2.36}$$

$$\text{w-}\frac{d\pi(t,f)}{dt}\bigg|_{\mathcal{E}} = -\phi(t,\omega(-i\nabla)^2 f) - \lambda\sum_{j=1}^N(\rho_j * f)(q_j(t)), \tag{2.37}$$

where $q_j(t) := (q_{j1}(t), \cdots, q_{jd}(t)), q(t) := (q_1(t), \cdots, q_N(t))$.

To apply Theorem 2.7 to the SNM, we need the following condition too.

(H.3) The function ω is such that

$$\mathcal{S}_{\rm real}(\mathbb{R}^d) \subset H_\omega^{-1} \cap H_\omega^{3/2}. \tag{2.38}$$

Theorem 2.8 *Assume (H.1)–(H.3), (A.1), (A.2) and (2.29). Suppose that* $Q(V) \subset D(|q|)$ *and there exist constants* $c_1, c_2 \geq 0$ *such that for all* $u \in Q(V)$

$$\||q|u\| \leq c_1\||V|^{1/2}u\| + c_2\|u\|. \tag{2.39}$$

and that

$$D(H_{\rm p}) \subset \cap_{j,l=1}^N \cap_{\mu,\nu=1}^d [D(p_{j\mu}q_{l\nu}) \cap D(q_{l\nu}p_{j\mu})] \cap \left(\cap_{j=1}^N \cap_{\mu=1}^d Q(D_{j\mu}V)\right) \tag{2.40}$$

Then the conclusion of Theorem 2.7 holds with $H_{\rm NM} = H_{\rm SNM}, \phi(f) = \phi_{\rm F}(f), \pi(f) = \pi_{\rm F}(f)$ *and* $\mathcal{E} = D(H_{\rm SNM})$.

3 A Nelson model in a non-Fock representation

Under hypotheses (H.2) and (H.3), we can define for each $f \in \mathcal{S}_{\mathrm{real}}(\mathbb{R}^d)$

$$\tilde{\phi}(f) := \phi_{\mathrm{F}}(f) + \lambda \sum_{j=1}^{N} \int_{\mathbb{R}^d} \frac{\hat{\rho}_j(k)\hat{f}(k)}{\omega(k)^2} dk. \tag{3.1}$$

We set

$$\tilde{\pi}(f) := \pi_{\mathrm{F}}(f). \tag{3.2}$$

Proposition 3.1 *Assume (H.2) and (H.3). Then:*

(i) $\{\mathcal{F}_{\mathrm{b}}, \mathcal{F}_0, \{\tilde{\phi}(f), \tilde{\pi}(f) | f \in \mathcal{S}_{\mathrm{real}}(\mathbb{R}^d)\}\}$ *is a representation of the CCR indexed by* $\mathcal{S}_{\mathrm{real}}(\mathbb{R}^d)$.

(ii) *The representation* $\{\tilde{\phi}(f), \tilde{\pi}(f) | f \in \mathcal{S}_{\mathrm{real}}(\mathbb{R}^d)\}$ *is unitarily equivalent to the Fock representation* $\{\phi_{\mathrm{F}}(f), \pi_{\mathrm{F}}(f) | f \in \mathcal{S}_{\mathrm{real}}(\mathbb{R}^d)\}$ *if and only if* $\sum_{j=1}^{N} \rho_j \in H_{\omega}^{-3/2}$.

We now consider a Nelson model whose time-zero fields are given by $\{\tilde{\phi}(f), \tilde{\pi}(f) | f \in \mathcal{S}_{\mathrm{real}}(\mathbb{R}^d)\}$. We introduce an $L^2(\mathbb{R}^d)$-valued function G on $\mathbb{R}^{dN}$ ($G : \mathbb{R}^{dN} \to L^2(\mathbb{R}^d)$, $G(q) \in L^2(\mathbb{R}^d)$, $q \in \mathbb{R}^{dN}$) by

$$G(q)(k) := \sum_{j=1}^{N} \frac{\hat{\rho}_j(k)^*}{\sqrt{\omega(k)}} \left(e^{-iq_j k} - 1\right) \tag{3.3}$$

and a function

$$W(q) := \sum_{j,l=1}^{N} \int_{\mathbb{R}^d} \frac{\hat{\rho}_j(k)\hat{\rho}_l(k)^*}{\omega(k)^2} e^{-iq_l k} dk, \quad q \in \mathbb{R}^{dN}. \tag{3.4}$$

By reality of ρ_j and (2.3), W is real-valued.

We define a Hamiltonian by

$$H := H_0 + \lambda \Phi_{\mathrm{S}}(G(q)) - \lambda^2 W + c_0 \lambda^2, \tag{3.5}$$

where

$$c_0 := \frac{1}{2} \left\| \frac{\sum_{j=1}^{N} \hat{\rho}_j}{\omega} \right\|_{L^2(\mathbb{R}^d)}^2. \tag{3.6}$$

Lemma 3.2 *Assume (H.1) and (H.2). Then H is self-adjoint with $D(H) = D(H_0)$ and bounded from below. Moreover, H is essentially self-adjoint on each core of H_0.*

Let

$$L_{\lambda}^{\mathrm{NF}} := H_{\mathrm{b}} + \lambda \Phi_{\mathrm{S}}(G(q)) - \lambda^2 W + c_0 \lambda^2, \tag{3.7}$$

so that

$$H = H_{\mathrm{p}} + L_{\lambda}^{\mathrm{NF}}. \tag{3.8}$$

Proposition 3.3 *Assume (H.1)–(H.3). Then*

$$\mathbb{M}_{\rm NF} := \{\mathcal{H}, \mathcal{D}_0, L_\lambda^{\rm NF}, \{\widetilde{\phi}(f), \widetilde{\pi}(g) | f \in H_\omega^{-1}, g \in H_\omega^{1/2}\}\} \tag{3.9}$$

is a Nelson model.

Proposition 3.4 *Assume (H.1)–(H.3). Then* $\{H, \{\widetilde{\phi}(f), \widetilde{\pi}(f) | f \in \mathcal{S}_{\rm real}(\mathbb{R}^d)\}\}$ *is unitarily equivalent to* $\{H_{\rm SNM}, \{\phi_{\rm F}(f), \pi_{\rm F}(f) | f \in \mathcal{S}_{\rm real}(\mathbb{R}^d)\}\}$ *if and only if*

$$\sum_{j=1}^N \rho_j \in H_\omega^{-3/2}. \tag{3.10}$$

Proposition 3.4 shows that, under (H.1)–(H.3) and the condition that $\sum_{j=1}^N \rho_j \in H_\omega^{-3/2}$, the Nelson model $\mathbb{M}_{\rm NF}$ is equivalent to the SNM. In this case, under suitable additional conditions, $H_{\rm SNM}$ has a ground state [7] and so does H. Thus we are interested in the case

$$\sum_{j=1}^N \rho_j \notin H_\omega^{-3/2}. \tag{3.11}$$

This condition is called an *infrared singularity condition.* In this paper, we say that *the Nelson model* $\mathbb{M}_{\rm Nelson}$ *has no infrared cutoff* if (3.11) holds. Under condition (3.11), the Nelson model $\mathbb{M}_{\rm NF}$ is not unitarily equivalent to the SNM. Note that (3.11) and the natural condition $\sum_{j=1}^N \rho_j/\sqrt{\omega} \in L^2(\mathbb{R}^d)$ imply

$$\text{ess.inf}_{k \in \mathbb{R}^d}\, \omega(k) = 0 \tag{3.12}$$

("ess.inf" means essential infimum), i.e., the quantum scalar field under consideration is "massless".

4 Existence of a ground state of the Nelson model $\mathbb{M}_{\rm NF}$ without infrared cutoff

Let T be a self-adjoint operator on a Hilbert space and bounded from below. We say that T has a ground state if there exists a non-zero vector $\psi \in D(T)$ such that $T\psi = E_0(T)\psi$, where $E_0(T) := \inf \sigma(T)$ is the infimum of the spectrum $\sigma(T)$ of T. In that case ψ is called a ground state of T.

To ensure the existence of a ground state of the model $\mathbb{M}_{\rm NF}$ without infrared cutoff, we need some additional conditions.

(H.4) The function ω is continuous on $\mathbb{R}^d$ satisfying (3.12) and the following conditions:

$$D_\mu \omega \in L^\infty(\mathbb{R}^d), \ \mu = 1, \cdots, d,$$
$$\lim_{|k| \to \infty} \omega(k) = \infty,$$

(H.5) For $j = 1, \cdots, N$,

$$\int_{\mathbb{R}^d} \frac{|k|^2 |\hat{\rho}_j(k)|^2}{\omega(k)^3} dk < \infty. \tag{4.1}$$

(H.6) There exist constants $c_1, c_2 \geq 0$ such that

$$|q|^2 \leq c_1 V(q) + c_2, \quad \text{a.e.} q \in \mathbb{R}^{dN}. \tag{4.2}$$

An infrared-cutoff Hamiltonian of the SNM is defined by

$$H_{\mathrm{SNM},\sigma} := H_0 + \lambda \sum_{j=1}^{N} \Phi_{\mathrm{S}} \left(e^{-ikq_j} \frac{\chi_{\omega \geq \sigma} \hat{\rho}_j{}^*}{\sqrt{\omega}} \right), \tag{4.3}$$

where $\sigma > 0$ is an infrared cutoff parameter and χ_S is a characteristic function of the set S.

Under (H.2), we can define for all $\sigma > 0$ a unitary operator

$$U_\sigma := \exp \left(-i\lambda \Phi_{\mathrm{S}} \left(i \frac{\sum_{j=1}^{N} \chi_{\omega \geq \sigma} \hat{\rho}_j}{\omega^{3/2}} \right) \right). \tag{4.4}$$

Theorem 4.1 *Assume (H.1)–(H.6). Then H has a ground state ψ_0 which has the following property: there exists a sequence $\{\phi_{\sigma_n}\}_{n=1}^{\infty}$ of unit vectors in $D(H_0)$ such that $\sigma_n > 0$, $n \in \mathbb{N}$, $\lim_{n\to\infty} \sigma_n = 0$, each ϕ_{σ_n} is a ground state of $H_{\mathrm{SNM},\sigma_n}$ and*

$$\text{w-}\lim_{n\to\infty} U_{\sigma_n}^{-1} \phi_{\sigma_n} = \psi_0, \tag{4.5}$$

where "w-lim" means weak limit.

Remark 4.1 Consider the physical case $\omega = \omega_0$ ((2.4) with $m = 0$). Suppose that

$$\sum_{j=1}^{N} \frac{\hat{\rho}_j(\cdot)}{|k|^{3/2}} \notin L^2(\mathbb{R}^d). \tag{4.6}$$

Then (3.11), (H.2), (H.4) and (H.5) hold with $\omega = \omega_0$. Hence Theorem 4.1 holds in the physical case without infrared cutoff.

Remark 4.2 Assume (H.3) and (3.11). Then we have

$$\text{w-}\lim_{\sigma\to 0} U_\sigma = 0. \tag{4.7}$$

This is an expression of infrared divergence. Hence the relation w-$\lim_{n\to\infty} U_{\sigma_n}^{-1} \phi_{\sigma_n} = \psi_0$ in Theorem 4.1 suggests that w-$\lim_{n\to\infty} \phi_{\sigma_n} = 0$.

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