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Duality and Symmetry Breaking

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1 Introduction: Problems and Results

The *superselection theory* of Doplicher-Haag-Roberts (DHR) [1] and of Doplicher-Roberts (DR) [2] in algebraic quantum field theory (QFT) [3] gives a general scheme for understanding the relations between a symmetry and its observable consequences in relativistic QFT. It tells us that, *if* the internal symmetry of the theory under consideration is, i) described by a gauge group G of the *1st kind* (i.e., *global* gauge symmetry), and is, ii) *unbroken*, the basic structure of the standard QFT can be recovered totally from the data encoded in *observables* $\mathfrak{A}$ which are defined as G-invariant combinations of field operators (i.e., $\mathfrak{A} = \mathfrak{F}^G$: fixed-point subalgebra of the field algebra $\mathfrak{F}$ under G) and constitute a *net* $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$ *of local subalgebras of observables* satisfying the *local commutativity* (i.e., Einstein causality). This implies, in particular, that the Bose/Fermi statistics of the basic fields is automatically derived without necessity of introducing from the outset *unobservable* field operators such as *fermionic fields* subject to local *anti*commutativity *violating Einstein causality*, which shows that they are simple mathematical devices for bookkeeping of half-integer spin states. While all the non-trivial spacetime behaviours are described here by the observable net $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$, the internal symmetry aspects are encoded in the *superselection structure*, which also originates from the observable net as explained later. We focus here on *internal symmetries,* in use, as tools, of the basic ingredients related to spacetime aspects (such as local commutativity, relativistic covariance, etc.) in algebraic QFT.

Before going into the details, it may be appropriate here to answer the question "Why *Symmetry Breaking* and *Duality*?" First, the duality means here the group duality between a group (and later, a homogeneous space) and its representations, which gives a key for understanding the superselection structure. Next, the reason for considering the *symmetry breaking* is that the symmetry arising from DHR-DR theory is always *unbroken* excluding the situation of spontaneous symmetry breakdown (SSB), which poses a question about the "**stability**" of this method as remarked by the late Moshé Flato (1996). Indeed we know that many (actually, almost all) of the "sacred symmetries" in nature can be broken (explicitly or spontaneously) in various situations: e.g.,

- SSB's of chiral symmetry in the electro-weak theory based upon $SU(2)\times U(1)$, electromagnetic $U(1)$ in the superconductivity, and the rotation symmetry $SO(3)$ in ferromagnetism, etc.,
- Lorentz invariance is broken spontaneously in thermal equilibria with $T \neq 0°K$ (I.O. '86 [4]),
- supersymmetry is shown to be unbroken only in the vacuum states (Buchholz-I.O. '97 [5]).

So, the problem as to whether this theory can incorporate systematically the cases of SSB or not is worth examining seriously, and if the answer is yes, what kind of superselection structure is realized in that case is a non-trivial interesting question. This sort of investigation is expected also to give us some important clues for getting rid of another restriction of *global* gauge symmetries so as to incorporate local gauge symmetries.

The essential contents and necessary ingredients are briefly outlined as follows:

1. From field algebra $\mathfrak{F}$ to observable algebra $\mathfrak{A}$:

- General description of a symmetry under a gauge group G is given by an automorphic action τ, $G \overset{\tau}{\curvearrowright} \mathfrak{F}$, of G on the field algebra $\mathfrak{F}$ (which is technically taken as a C*-algebra satisfying certain set of axioms to characterize a relativistic QFT).
- The algebra $\mathfrak{A}$ of observables is defined as the *fixed-point subalgebra* $\mathfrak{A} \equiv \mathfrak{F}^G$ consisting of G-invariants.
- Notion of *superselection sectors*:
  Historically the term "*superselection rule*" has been understood as a *restriction* on the validity of quantum mechanical *superposition principle*: e.g., $c_1\psi_P + c_2\psi_N$ is "meaningless" with wave functions ψ_P and ψ_N for proton and neutron. In its presence, a "*superselection sector*" (also called a "*coherent subspace*") is defined in the state vector space by a subspace where any superpositions are "meaningful". According to [1] we adopt here a more precise characterization of a

superselection rule in reference to the above triplet $(\mathfrak{F}, G, \mathfrak{A})$, as *mutually disjoint* irreducible (or more generally, factor) representations π_γ in Hilbert spaces $\mathfrak{H}_\gamma$ [$(\pi_\gamma, \mathfrak{H}_\gamma)$, for short] *of* $\mathfrak{A}$, in 1-1 correspondence with mutually disjoint irreducible unitary representations U_γ of G in Hilbert spaces V_γ.

2. From $\mathfrak{A}$ to $\mathfrak{F}$?:
The most non-trivial step in DR theory is the opposite direction to the above:

$$\mathfrak{A} \Longrightarrow (\mathfrak{F}, G),$$

to derive $\mathfrak{F}$ and G (both are *unobservable*!!) starting from the information on the observable algebra $\mathfrak{A}$ (achieved partially in [1] ('74), completed by [2] ('90) in the vacuum situation).

Mathematically, this is a kind of Galois extension of a C*-algebra $\mathfrak{A}$ by the gauge group G (of 1st kind) as a Galois group, $G = Aut_{\mathfrak{A}}(\mathfrak{F}) =: Gal(\mathfrak{F}/\mathfrak{A})$ [i.e., a subgroup of the automorphism group $Aut(\mathfrak{F})$ fixing $\mathfrak{A}$ pointwise], in which, however, we do not know either of $\mathfrak{F}$ or G beforehand!! Actually, the information on G is supplied by the *DHR selection criterion* [1] for physically relevant states of $\mathfrak{A}$; it defines a C*-tensor category $\mathcal{T}$ consisting of local endomorphisms ρ of $\mathfrak{A}$, which turns out to be isomorphic to a category *Rep*(G) of unitary representations of a *compact* group G, on the basis of a non-commutative and abstract reformulation of *Tannaka-Krein duality* found and formulated in [2]. Namely,

- *selection criterion* $\Longrightarrow$ C*-tensor category $\mathcal{T}$ $(\subset End(\mathfrak{A}))$,
- $\mathcal{T} \overset{\mathrm{DR}}{\simeq} Rep(G) \overset{\text{Tannaka-Krein}}{\leftrightarrow} G = End_{\otimes}(V)$: compact Lie group $\subset SU(d)$ where d is determined by $\mathcal{T}$,
- $\mathfrak{F} = \mathfrak{A} \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d \curvearrowleft G = Gal(\mathfrak{F}/\mathfrak{A})$, $\mathfrak{F}^G = \mathfrak{A}$, where $\mathcal{O}_d$ is a Cuntz algebra [6] of d-tuple of isometries.

3. Broken vs. unbroken symmetry?
Through the construction of a field algebra $\mathfrak{F}$ due to [2] (DR construction, for short), the symmetry described by G is always found to be *unbroken* (=unitarily implemented in a factor representation), without suffering from any spontaneous breaking. What is known so far about the structural feature signalling SSB in the D(H)R scheme is found to be the *violation of the Haag duality*: $\mathfrak{A} \neq \mathfrak{A}^d$ [7]. Then, the C*-tensor category $\mathcal{T}_0$ arising from the

selection criterion is to be constructed in $End(\mathfrak{A}^d)$ instead of $End(\mathfrak{A})$, which yields a representation category $Rep(H) \overset{\text{DR}}{\simeq} \mathcal{T}_0$ with H ($\subset SU(d_0)$) being the Tannaka-Krein dual of a compact Lie group of *unbroken* symmtry. Then repeating the DR construction to the pair of $\mathfrak{A}^d$ and H, we obtain a field algebra $\mathfrak{F} := \mathfrak{A}^d \underset{\mathcal{O}_{d_0}^H}{\otimes} \mathcal{O}_{d_0}$, $\mathfrak{A}^d = \mathfrak{F}^H$. For the discussion of SSB along this line, the definition has been proposed of the "largest" gauge group Γ of internal symmetry as the Galois group $\Gamma := Gal(\mathfrak{F}/\mathfrak{A})$ in [8]. As is easily seen, however, the meaning of "largest" remains empty without examining the *uniqueness* problem of the so-constructed $\mathfrak{F}$ (i.e., not only whether unique or not, but also in which sense, etc.), and such important questions have remained untouched so far as

i) Is the choice $\mathfrak{F} = \mathfrak{A}^d \underset{\mathcal{O}_{d_0}^H}{\otimes} \mathcal{O}_{d_0}$ consistent, universal and stable (also, against what)?

ii) If i) is settled then we can ask how to define a bigger group $G(\supset H)$, broken spontaneously.

iii) Identifying physically relevant quantities such as ***order parameters*** *to distinguish "degenerate vacua"* and *Goldstone bosons* in the above mathematical scheme, we should clarify which kind of sector structure is to be found in the theory with SSB.

In view of prevailing occurrence of SSB, what we are concerned with is these problems in order to incorporate SSB systematically in the superselection theory. For this purpose, we need to specify the precise meaning of SSB in the present context in the following form:

Definition 1 *A symmetry described by a (strongly continous) automorphic action τ of G on the field algebra $\mathfrak{F}$ is said to be* ***unbroken*** *in a given representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$ if each* ***factor*** *subrepresentation $(\sigma, \mathfrak{H}_\sigma)$, $\sigma(\mathfrak{F})' \cap \sigma(\mathfrak{F})'' = \mathbb{C}1_{\mathfrak{H}_\sigma}$, appearing in the central decomposition of $(\pi, \mathfrak{H})$ admits a* ***covariant representation*** *of the dynamical system $G \overset{\tau}{\curvearrowright} \mathfrak{F}$ in the sense that there exists a (strongly continuous) unitary representation $(U_\sigma, \mathfrak{H}_\sigma)$ of G verifying the relation $\sigma(\tau_g(F)) = U_\sigma(g)\sigma(F)U_\sigma(g)^*$ for $\forall g \in G, \forall F \in \mathfrak{F}$. If the symmetry is not unbroken, it is said to be* ***broken spontaneously****.*

Note that the above characterization of unbroken symmetry can be reformulated equivalently as the pointwise invariance of the spectrum of centre of $\pi(\mathfrak{F})''$ under G. Therefore, SSB means in short the ***conflict between the unitary implementability and a factor representation with trivial centre.*** The situation with SSB is seen to exhibit the features of the so-called ***"infrared instability"*** under the action of G, because it does not stabilize the spectrum of centre which can be viewed physically as *macroscopic order parameters* emerging in the infrared regions.

Since the above definition of SSB still allows the mixture of unbroken and broken *sub*representations of a given π, we need to decompose the spectrum of centre of $\pi(\mathfrak{F})''$ into domains each of which is *ergodic* under G (central ergodicity). Then, π is decomposed into the direct sum (or, direct integral) of unbroken factor representations and broken non-factor representations, each component of which is stable under G. Thus we obtain a *phase diagram.*

Aside from physical significance, this leads us also to some interesting mathematical notions and tools, such as

- a homotopically fibered category of endomorphisms, unifying *finite-dimensional inductions* of group representations from unbroken remaining H to broken G,
- extension of Tannaka-Krein duality for homogeneous spaces G/H,
- a category $\mathfrak{CB}$ consisting of *crossed products of Cuntz subalgebras* with bundle structures,
- a functor from $\mathfrak{CB}$ to category of field algebras, and so on.

Finally, we state the (partial) answers to the above questions i)-iii) on the basis of a joint work (S. Maumary, T. Nozawa and I.O., in preparation):

to ii) Similarly to the above H, we can construct a bigger compact Lie group $G := End_{\otimes}(V \circ i) \supset H$ with $i : \mathcal{T} := \mathcal{T}_0 \cap End(\mathfrak{A}^d, \mathfrak{A}) \hookrightarrow \mathcal{T}_0$, where $End(\mathfrak{A}^d, \mathfrak{A})$ is a subcategory of $End(\mathfrak{A}^d)$ consisting of endomorphisms of $\mathfrak{A}^d$ stabilizing $\mathfrak{A}$ globally and of morphisms belonging to $\mathfrak{A}$. If $\Gamma = Gal(\mathfrak{F}/\mathfrak{A}) := Aut_{\mathfrak{A}}(\mathfrak{F})$ is compact, we have $\Gamma = G$.

to i) Definition of field algebra, $\mathfrak{F} := \mathfrak{A}^d \underset{\mathcal{O}_{d_0}^H}{\otimes} \mathcal{O}_{d_0}$, is consistent with the action of G in the sense that the result of constructing $\mathfrak{F}$ starting from $(g(\mathfrak{A}^d), gHg^{-1})$ with $\forall g \in G$ is isomorphic with that from $(\mathfrak{A}^d, H)$. If $\Gamma = G \subset SU(d)$ holds, equalities

$$\mathfrak{F} = \mathfrak{A}^d \underset{\mathcal{O}_{d_0}^H}{\otimes} \mathcal{O}_{d_0} = \mathfrak{A}^d \underset{\mathcal{O}_{d}^H}{\otimes} \mathcal{O}_{d} = g(\mathfrak{A}^d) \underset{\mathcal{O}_{d}^{gHg^{-1}}}{\otimes} \mathcal{O}_{d}, \quad \mathfrak{F}^G = \mathfrak{A}$$

hold, which shows the stability and universality of Doplicher-Roberts construction method extended to the situation with SSB. If $\Gamma = Gal(\mathfrak{F}/$ is non-compact, $\mathfrak{A} \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d$ is to be replaced by $\mathfrak{A} \rtimes_\delta G$ where δ is the coaction of G on $\mathfrak{A}$.

to iii) For a compact pair $H \subset G$, the pair $(\mathcal{T}, \mathcal{T}_0)$ describes a natural extension of Tannaka-Krein duality to the homogeneous space G/H, which yields also the finite-dimensional version of Mackey induction and of Frobenius reciprocity (in a homotopical sense).
$\Longrightarrow$ Order parameters and Goldstone multiplets.

2 Basic Formulation

Now we try to explain some details of the discussions in the introduction. A net, $\mathfrak{A} : \mathcal{K} \ni \mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$, of local observables in the Minkowski space is defined [3] for $\mathcal{K} := \{\text{double cones } (a+V_+) \cap (b-V_+); a, b \in \mathbb{R}^4\}$ with $\mathfrak{A}(\mathcal{O})$ for $\mathcal{O} \in \mathcal{K}$ being W*-algebras satisfying the following 1, 2, 3:

1. *Isotony*: $\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$, owing to which quasi-local observable algebra $\mathfrak{A}$ (denoted by same letter $\mathfrak{A}$) is defined by $\mathfrak{A} := \overline{\bigcup_{\mathcal{O} \in \mathcal{K}} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|}$ as a C*-inductive limit.

2. Poincaré *covariance*: Poincaré group $\mathcal{P}_+^\uparrow := \mathbb{R}^4 \rtimes L_+^\uparrow$ (consisting of spacetime translations and Lorentz transformations) acts (strongly continuously) on $\mathfrak{A}$ by *-automorphisms $\alpha : \mathcal{P}_+^\uparrow \to Aut(\mathfrak{A})$ s.t. for $(a, \Lambda) \in \mathcal{P}_+^\uparrow$

$$\alpha_{(a,\Lambda)}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\Lambda\mathcal{O} + a).$$

3. *Local commutativity*: $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$ for spacelike separated $\mathcal{O}_1, \mathcal{O}_2$. With $\mathcal{O}' := \{y \in \mathbb{R}^4; (x-y)^2 < 0 \ \forall x \in \mathcal{O}\}$: causal complement of $\mathcal{O} \in \mathcal{K}$, this is written (symbolically) as $\mathfrak{A}(\mathcal{O}') := \overline{\bigcup_{\mathcal{O}_1 \subset \mathcal{O}'} \mathfrak{A}(\mathcal{O}_1)}^{\|\cdot\|} \subset \mathfrak{A}(\mathcal{O})'$ (RHS meaningful in a representation Hilbert space of $\mathfrak{A}$).

4. Algebraic description of symmetries acting upon a field algebra $\mathfrak{F}$:

i) Consider an action of G on field algebra $\mathfrak{F}$ by algebraic automorphisms. For instance, if some basic multiplet $\{F^i\}$ of fields is available, the action would be given in such a form as $\tau_g(F^i) =$

$\sum_j \gamma(g^{-1})^i_j F^j$, verifying the conditions $\tau_{g_1} \circ \tau_{g_2} = \tau_{g_1 g_2}, \tau_e = id_{\mathfrak{F}}$, i.e., $\tau : G \to Aut(\mathfrak{F})$ is a group homomorphism.

ii) Existence of unitary operators, $G \ni g \longmapsto U(g)$ s.t. $U(g)F^i U(g)^* = \sum_j \gamma(g^{-1})^i_j F^j$, depends on representations: SSB means the *absence* of such $U(g)$'s, but algebraic transformations τ_g still meaningful.

iii) *Internal* symmetry (gauge symmetry of 1st kind) is characterized by the commutativity $\tau_g \alpha_{(a,\Lambda)} = \alpha_{(a,\Lambda)} \tau_g$ for $\forall g \in G$, $\forall (a, \Lambda) \in \mathcal{P}^{\uparrow}_{+}$.

iv) The *fixed-point subalgebra* $\mathfrak{F}^G \equiv \mathfrak{A}$ is regarded as net of local observables describing experimentally measurable consequences of a given theory.

Remark 2 *If all the characteristic features of a given theory of QFT can be encoded in $\mathfrak{A}$, notions involving $\mathfrak{F}$ and G may become mathematical tools for convenience. Actually this has not been fully attained, especially in theories with local gauge invariance.*

5. Mathematical definition of *states* ω on $\mathfrak{A}$ is given by *normalized positive linear functionals* on $\mathfrak{A}$, i.e.,

$$\begin{aligned} \omega(c_1 A_1 + c_2 A_2) &= c_1 \omega(A_1) + c_2 \omega(A_2) \quad \text{for } \forall A_1, A_2 \in \mathfrak{A}, \forall c_1, c_2 \in \mathbb{C}, \\ \omega(A^* A) &\geq 0, \ \ \omega(\mathbf{1}) = 1, \end{aligned}$$

among which *physically meaningful states* should be selected out by certain criteria. For instance, a vacuum state ω_0 is defined by translation *invariance* $\omega_0 \circ \alpha_{(a,\mathbf{1})} = \omega_0$ and *spectral condition*: $Spec(U_{\omega_0}(\mathbb{R}^4)) \subset \overline{V_+}$, where U_{ω_0}: unitary representation of translations $\mathbb{R}^4$ in GNS representation $(H_{\omega_0}, \pi_{\omega_0}, \Omega_{\omega_0})$ [s.t. $\omega_0(A) = \langle \Omega_{\omega_0} | \pi_{\omega_0}(A) \Omega_{\omega_0} \rangle$, $\overline{\pi_{\omega_0}(\mathfrak{A}) \Omega_{\omega_0}} = H_{\omega_0}$] defined by

$$U_{\omega_0}(a) \pi_{\omega_0}(A) \Omega_{\omega_0} = \pi_{\omega_0}(\alpha_{(a,\mathbf{1})}(A)) \Omega_{\omega_0}$$

which satisfies

$$\pi_{\omega_0}(\alpha_{(a,\mathbf{1})}(A)) = U_{\omega_0}(a) \pi_{\omega_0}(A) U_{\omega_0}(a)^*$$

and is written by Stone theorem as $U_{\omega_0}(a) = e^{iP_\mu a^\mu}$ and hence $Spec(U_{\omega_0}(\mathbb{R}^4)) = Spec\{P_\mu\} \subset \overline{V_+}$.

If G leaves a state ω invariant, GNS representation $(\pi_\omega, \mathfrak{H}_\omega, \Omega_\omega)$ of $\mathfrak{F}$ provides a unitary representation V_ω of G similarly to the above U_{ω_0}.

Superselection sectors of $(\mathfrak{F}, G, \mathfrak{A})$ can be understood [1] as mutually disjoint irreducible representations (π_γ, V_γ) of $\mathfrak{A}$ in 1-1 correspondence with mutually disjoint irreducible representations (U_γ, V_γ) of G, appearing in an irreducible vacuum representation $(\pi, \mathfrak{H})$ of $\mathfrak{F}$ on $\mathfrak{H} = \oplus_{\gamma \in \hat{G}} (\mathfrak{H}_\gamma \otimes V_\gamma)$ as

$$\pi(\mathfrak{A}) = \bigoplus_{\gamma \in \hat{G}} (\pi_\gamma(\mathfrak{A}) \otimes \mathbf{1}_{V_\gamma}), \qquad U(G) = \bigoplus_{\gamma \in \hat{G}} (\mathbf{1}_{\mathfrak{H}_\gamma} \otimes U_\gamma(G)).$$

where $\hat{G}$ is the *group dual* of G consisting of all mutually disjoint irreducible unitary representations of G.

Remark 3 *GNS construction shows that a Hilbert space and of unitary representations of a group are secondary notions not of fundamental importance for quantum theory, but simply a mathematical device for convenience, in contrast to an algebra and its automorphism group.*

3 From Observables to Fields: DR Construction

3.1 Selection criterion for physically relevant states

To go over from the observable net $\mathfrak{A}$ to the field algebra $\mathfrak{F}$ acted on by an internal symmetry group G, the selection of *physically relevant states* [1] becomes crucial. In the case of "charged" sectors generated from the vacuum ω_0 by certain *localizable charges*, candidate states ω are known to be characterized by the condition that there exists $\mathcal{O} \in \mathcal{K}$ s.t.

$$\omega(A) = \omega_0(A) \quad \text{for } \forall A \in \mathfrak{A}(\mathcal{O}'),$$

which means that the state ω is different from the vacuum ω_0 only within a local region $\mathcal{O}$ (and its causal shaddow). This condition is also known to be equivalent to the existence of *local endomorphism* $\rho \in End(\mathfrak{A})$, local in the sense of

$$\rho(A) = A \quad \text{for } \forall A \in \mathfrak{A}(\mathcal{O}'),$$

such that $\omega(A) = \omega_0(\rho(A))$. In this situation, the double cone $\mathcal{O} \in \mathcal{K}$ is called a *support* of ω or ρ. In terms of the GNS representations π_0 and π_ω corresponding to ω_0 and ω, respectively, we have $\pi_\omega = \pi_0 \circ \rho$.

3.2 Sectors as local endomorphisms

Such an endomorphism ρ is known to correspond to a choice of a unitary representation of G. To see it, *suppose* that we have attained the field algebra $\mathfrak{F}$ and a group G so that $\mathfrak{F}^G = \mathfrak{A}$ and $\mathfrak{A}' \cap \mathfrak{F} = \mathbb{C}\mathbf{1}$ hold. Then, a Hilbert space V_ρ in $\mathfrak{F}$ is defined by

$$V_\rho := \{\psi \in \mathfrak{F}; \psi A = \rho(A)\psi\},$$

because of $\psi_1^*\psi_2 =: \langle\psi_1|\psi_2\rangle\mathbf{1} \in \mathfrak{A}' \cap \mathfrak{F} = \mathbb{C}\mathbf{1}$. The stability of V_ρ under G is verified by $\tau_g(\psi)A = \rho(A)\tau_g(\psi)$ following from $\psi A = \rho(A)\psi$, and the unitarity of the representation $\tau|_{V_\rho} := \gamma_\rho$ of G is also easily checked: $\langle\psi_1|\psi_2\rangle\mathbf{1} = \tau_g(\psi_1^*\psi_2) = \tau_g(\psi_1)^*\tau_g(\psi_2) = \langle\tau_g(\psi_1)|\tau_g(\psi_2)\rangle\mathbf{1}$.

Consider a category $\mathcal{T}$ with such ρ's as objects and morphisms $T \in \mathfrak{A}$ from one such ρ_1 to another ρ_2 defined by the relation

$$T\rho_1(A) = \rho_2(A)T.$$

Then the above correspondence extends further to the level of *intertwiners* between representations of G: For $\forall\psi \in V_{\rho_1}$, we have $T\psi \in V_{\rho_2}$, because $(T\psi)A = T\rho_1(A)\psi = \rho_2(A)T\psi$, and hence, $\gamma_{\rho_2}(g)T\psi = \tau_g(T\psi) = \tau_g(T)\tau_g(\psi) = T\gamma_{\rho_1}(g)\psi$, namely,

$$T\gamma_{\rho_1}(g) = \gamma_{\rho_2}(g)T.$$

Thus, the map $\rho \longmapsto (V_\rho, \gamma_\rho)$ establishes the equivalence of categories $\mathcal{T}$ (defined as a full subcategory of $End(\mathfrak{A})$ divided by the unitary equivalence w.r.t. spacetime translations) and $Rep(G)$ of unitary representations of G. This equivalence holds actually as *C^*-tensor categories* involving the *tensor structures* of both. Sectors of the theory are then given by $\mathfrak{H}_\rho := V_\rho\Omega_{\omega_0}$ in the vacuum representation of $\mathfrak{F}$.

Thus, once $\mathfrak{F}$ and G are constructed s.t. $\mathfrak{F}^G = \mathfrak{A}$ and $\mathfrak{A}' \cap \mathfrak{F} = \mathbb{C}\mathbf{1}$, then the sector structure is fully understood as above. But how can $\mathfrak{F}$ and G be constructed?

3.3 Crossed product with Cuntz algebras

We need intermediate steps intervening between $(\mathfrak{A}, \mathcal{T})$ and $(\mathfrak{F}, G)$ which involves certain technicalities. While we do not fully elaborate on these subtleties here, let us just mention a few basic ingredients [3]:

1. *Haag duality*: $\mathfrak{A}^d(\mathcal{O}) := \mathfrak{A}(\mathcal{O}')' = \mathfrak{A}(\mathcal{O})$, which enables various important quantities to be identified inside of the local net $\mathfrak{A}$.

2. *Property B* (following from local commutativity and spectral condition): A net $\mathfrak{A}$ satisfies *Property B* if given such $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ that. $\mathcal{O}, \mathcal{O}_2$ are spacelike separated and that $\mathcal{O}$, $\mathcal{O}_2 \subset \mathcal{O}_1$ and such a projection $0 \neq E \in \mathfrak{A}(\mathcal{O})$, there is an isometry $W \in \mathfrak{A}(\mathcal{O}_1)$ with $WW^* = E$. This endows $\mathcal{T}$ with stability under *direct sums and subobjects* (allowing the Cuntz algebra $\mathcal{O}_{d=2}$ to be embedded in local observable algebras).

3. Assumptions on $\mathcal{T}$ of its being *specially directed*, and of the existence of *conjugates* (details of which are omitted here). This last condition is equivalent to the finiteness of *statistical dimension*.

Under these assumptions, one has at hand a strict symmetric C*-tensor category $\mathcal{T}$ equipped with associative tensor operation $\mathcal{T} \times \mathcal{T} \ni (\rho, \sigma) \longmapsto \rho\sigma \in \mathcal{T}$ defined for objects and $\mathcal{T}(\rho_1, \rho_2) \times \mathcal{T}(\sigma_1, \sigma_2) \ni (S, T) \longmapsto S \times T := S\rho_1(T) \in \mathcal{T}(\rho_1\sigma_1, \rho_2\sigma_2)$ for morphisms, and the symmetry $\varepsilon(\rho, \sigma) \in \mathcal{T}(\rho\sigma, \sigma\rho)$ interchanging the ordering. Then, according to the techniques developed in [2] one can find the "prototype" of $\mathfrak{F}$ in the Cuntz algebra $\mathcal{O}_d$ which reflects only the internal symmetry aspects of $\mathfrak{F}$ and G but which forgets all the spacetime aspects coming from $\mathfrak{A}$. The above assumption 3 implies the existence of $R \in (\iota, \rho^d) \subset \mathfrak{A}$ playing the role of determinant so that the gauge group G to be determined next is a closed subgroup of $SU(d)$. Here the Cuntz algebra $\mathcal{O}_d$ [6] is defined as a unique C*-algebra generated by d-tuple of isometries $\{s_i\}_{1 \leq i \leq d}$, $s_i^* s_j = \delta_{ij}\mathbf{1}$, $\sum_i s_i s_i^* = \mathbf{1}$, with natural actions of $U(d)$ and its subgroups, $G \subset SU(d) \subset U(d) \curvearrowright \mathcal{O}_d$. $\mathcal{O}_d^G$ denotes the fixed-point subalgebra of $\mathcal{O}_d$ under G.

Then, $\mathfrak{F}$ can be recovered from $\mathfrak{A}$ and $\mathcal{O}_d$ by taking their crossed product:

$$\mathfrak{F} := \mathfrak{A} \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d.$$

As a linear space this is defined as the solution of universality problem

$$\begin{array}{ccc} \mathfrak{A} & \dashrightarrow & \mathfrak{F} \\ \uparrow & & \vdots \\ \mathcal{O}_d^G & \hookrightarrow & \mathcal{O}_d \end{array}.$$

Its multiplication structure is determined essentially by the relation $(\mathbf{1} \underset{\mathcal{O}_d^G}{\otimes} \psi)(A \underset{\mathcal{O}_d^G}{\otimes} \mathbf{1}) = (\rho(A) \underset{\mathcal{O}_d^G}{\otimes} \psi)$ for $A \in \mathfrak{A}$, $\psi \in \mathcal{O}_d$, in addition to the corresponding formula with ψ replaced by ψ^* which heavily depends upon the above determinant element $R \in \mathcal{T}(\iota, \rho^d)$.

The *statistical dimension* $d = d(\rho)[= Ind(\mathfrak{A} : \rho(\mathfrak{A}))^{1/2}$: Jones index] is determined by $d(\rho)\mathbf{1} = R_\rho^* \circ R_\rho = \bar{R}_\rho^* \circ \bar{R}_\rho$ in terms of the intertwiners $R_\rho \in \mathcal{T}(\iota, \bar{\rho}\rho)$, $\bar{R}_\rho \in \mathcal{T}(\iota, \rho\bar{\rho})$ which obey

$$\begin{aligned} \bar{R}^* \otimes 1_\rho \circ 1_\rho \otimes R &= 1_\rho \\ R^* \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \bar{R} &= 1_{\bar{\rho}} \end{aligned}$$

with $\bar{R} = \varepsilon(\bar{\rho}, \rho) \circ R$. These are the existence conditions for *conjugate* $\bar{\rho}$ to ρ with ι contained only once in $\bar{\rho}\rho$.

Next how can G be determined?

3.4 Determination of G: Tannaka-Krein duality

The identification of G is done by

$$G := End_\otimes(V),$$

where $End_\otimes(V)$ denotes the group consisting of natural unitary transformations from a functor V to itself, i.e., $u \in End_\otimes(V)$ consists of a family of unitaries $u_\rho \in \mathcal{U}(V_\rho)$ in Hilbert space V_ρ parametrized by $\rho \in \mathcal{T}$, s.t., for $\forall$morphism $T \in \mathcal{T}(\rho, \sigma)(\subset \mathfrak{A})$ defined by $T\rho(A) = \sigma(A)T$, commutativity $Tu_\rho = u_\sigma T$ holds:

$$\begin{array}{ccccc} \rho & & V_\rho & \overset{u_\rho}{\longrightarrow} & V_\rho \\ T\downarrow & & T\downarrow & \circlearrowleft & \downarrow T \\ \sigma & & V_\sigma & \underset{u_\sigma}{\longrightarrow} & V_\sigma \end{array}.$$

Then the relations arising in this formulation justify the identification $u_\rho = \gamma_\rho(u)$ with $u \in G$ and γ_ρ as a unitary representation of G corresponding to local endomorphism $\rho \in \mathcal{T}$. The essence of Tannaka-Krein duality is contained just in this formula: Here $V : \mathcal{T} \hookrightarrow \boldsymbol{Hilb}$ is the embedding functor of $\mathcal{T}$ into category $\boldsymbol{Hilb}$ of all finite-dimsional Hilbert spaces constructed in use of Cuntz algebras (details omitted). Then $V(\mathcal{T}) \simeq Rep(G)$ reproduces the usual Tannaka-Krein duality. Considering a bundle of Hilbert $\mathfrak{A}$-modules over $\mathcal{T}$ (viewed as a *graph*), G can be identified also as a holonomy group.

4 Spontaneous Symmetry Breaking

As stated in Sec.1, we need to give up the Haag duality, $\mathfrak{A} = \mathfrak{A}^d$, to treat the situation with SSB. Instead, we can assume, without loss of generality, that the *essential duality* $\mathfrak{A}^{dd} = \mathfrak{A}^d$ is valid.

1. *Finite-dimensional induction* for compact pairs $H \hookrightarrow G$:
   Any representation (η, W) of H can be extended to a representation (γ, V) of G by taking a direct sum $\gamma|_H \cong \eta \oplus \eta'$ with a suitable representation (η', W') of H (for proof, see pp.14-15 of [9]).

2. Stability and consistency of field algebra construction in SSB:
   In use of the above result, one can verify the *stability* of the crossed product construction of the field algebra under the change of Cuntz algebras as the isomorphism between $\mathfrak{F}$ due to the original DR construction from the dual net $\mathfrak{A}^d$ and the crossed product of $\mathfrak{A}^d$ with a Cuntz algebra $\mathcal{O}_d$ for any $d > d_0$:

$$\mathfrak{F} := \mathfrak{A}^d \underset{\mathcal{O}_{d_0}^H}{\otimes} \mathcal{O}_{d_0} \cong \mathfrak{A}^d \underset{\mathcal{O}_d^H}{\otimes} \mathcal{O}_d \overset{?}{=} \mathfrak{A} \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d$$

   (up to some minor point in the last equality $\overset{?}{=}$). While the relation $g(\mathfrak{A}^d) = \mathfrak{A}^d = \mathfrak{F}^H$ for $g \in G$ requires $g \in N_H$, the normalizer of unbroken H in $\Gamma = Gal(\mathfrak{F}/\mathfrak{A})$, the equality $g(\mathfrak{A}^d) \underset{\mathcal{O}_d^{gHg^{-1}}}{\otimes} \mathcal{O}_d = g(\mathfrak{A}^d \underset{\mathcal{O}_d^H}{\otimes} \mathcal{O}_d) = \mathfrak{F}$ can be verified even for such $g \in G$ that $g \notin N_H$, which shows the *consistency* of the construction method with the action of G bigger than H (how to determine the spontaneously broken G is explained next).

   While the relation $Gal(\mathfrak{A}^d/\mathfrak{A}) = N_H/H$ was verified in [8], the analysis of SSB there was restricted only to N_H in order to avoid $g(\mathfrak{A}^d) \neq \mathfrak{A}^d$. In the physically interesting situations involving Lie groups, however, the reductivity of a compact Lie group $H \hookrightarrow \Gamma$ implies that N_H/H is abelian and/or discrete with a vanishing Lie brackets, which does not seem to be relevant to the physically meaningful contexts.

3. *Duality for homogeneous spaces* and its endomorphism version:

   For a compact group pair $H \hookrightarrow G$, the definition of $\boldsymbol{Rep}_{G/H}$ and the mutual relations among $\boldsymbol{Rep}_G$, $\boldsymbol{Rep}_H$ and $\boldsymbol{Rep}_{G/H}$ can be described in terms of a homotopy-fibre category $\boldsymbol{Rep}_G$ over $\boldsymbol{Rep}_H$ with $\boldsymbol{Rep}_{G/H}$ as homotopy fibre: Over $\eta \in \boldsymbol{Rep}_H$ a homotopy fibre (h-fibre for short) is given by a category $\eta/\boldsymbol{Rep}_G$ (which is called a comma category under η [10] whose objects are pairs (γ, T) of $\gamma \in \boldsymbol{Rep}_G$ and $T \in \boldsymbol{Rep}_H(\eta, \gamma|_H)$ and whose morphisms $\phi : (\gamma, T) \to (\gamma', T')$ are given by

$\phi \in Rep_G(\gamma, \gamma')$ s.t. $T' = \phi \circ T$:

$$\begin{array}{ccccc} & & \eta & & \\ & \swarrow T & & T' \searrow & \\ \gamma|_H & -- & \underset{\phi|_H}{-} & \rightarrow & \gamma'|_H \\ i_H \uparrow & & & & \uparrow i_H \\ \gamma & -- & \underset{\phi}{-} & \rightarrow & \gamma' \end{array} \tag{1}$$

(To be more precise, the comma category η/Rep_G is to be understood as η/i_H where the functor $i_H : Rep_G \rightarrow Rep_H$ is the restriction of G-representations to the subgroup H of G.)

The h-fibre over the trivial representation $\eta = \iota \in Rep_H$ of H is nothing but the category of *linear representations of G/H* due to Iwahori-Sugiura [11], to which any other h-fibres can be shown to be homotopically equivalent.

The version in terms of endomorphisms dual to the above h-fibre category is given as follows:

$$\begin{array}{lll} End(\mathfrak{A}^d) & \supset \mathcal{T}_0 & \longleftrightarrow Rep_H \\ End(\mathfrak{A}^d, \mathfrak{A}) & \supset \mathcal{T} = \{\rho \in \mathcal{T}; \rho(\mathfrak{A}) \subset \mathfrak{A}\} & \longleftrightarrow Rep_G \\ & \mathcal{T}_0/\mathcal{T} & \longleftrightarrow Rep_{G/H} \end{array}$$

The h-fibre category over $\rho \in \mathcal{T}_0$ is given by $\rho/\mathcal{T}$ [or, more precisely, $\rho/\mathcal{D}$ with $\mathcal{D}$ being the functor $\mathcal{T} \ni \sigma \longmapsto \tilde{\sigma} \in \mathcal{T}_0$ extending endomorphisms from $\mathfrak{A}$ to $\mathfrak{A}^d$] with the object set $\{(\sigma, T); \sigma \in \mathcal{T}, T \in (\rho, \tilde{\sigma}) \subset \mathfrak{A}^d\}$ and with the set of morphisms $\{\phi : (\sigma, T) \rightarrow (\sigma', T'); \phi \in \mathcal{T}(\sigma, \sigma') \subset \mathfrak{A}, T' = \phi \circ T \in (\rho, \tilde{\sigma}') \subset \mathfrak{A}^d\}$(: semidirect product of $\mathcal{T}$ and $\mathfrak{A}^d$), where $\tilde{\sigma} = \sigma \circ j'_{\mathfrak{A}(\mathcal{O})} \circ j_{\mathfrak{A}^d(\mathcal{O})}$ gives the extension of endomorphism $\sigma \in \mathcal{T}$ of $\mathfrak{A}(\mathcal{O})$ to $\mathfrak{A}^d(\mathcal{O})$ for $\mathcal{O}$ =suppσ and $j'_{\mathfrak{A}(\mathcal{O})}$, $j_{\mathfrak{A}^d(\mathcal{O})}$ are the modular conjugations of von Neumann algebras $\mathfrak{A}(\mathcal{O})'$ and $\mathfrak{A}^d(\mathcal{O})$, respectively: $\mathfrak{A}^d(\mathcal{O}) \overset{j_{\mathfrak{A}^d(\mathcal{O})}}{\rightarrow} \mathfrak{A}^d(\mathcal{O})' \subset \mathfrak{A}(\mathcal{O})' \overset{j'_{\mathfrak{A}(\mathcal{O})}}{\rightarrow} \mathfrak{A}(\mathcal{O})'' = \mathfrak{A}(\mathcal{O})$.

Corresponding to $\mathcal{T} \overset{i}{\hookrightarrow} \mathcal{T}_0 \overset{V}{\hookrightarrow} Hilb$, we have an embedding map $H = End_{\otimes}(V) \overset{j}{\hookrightarrow} End_{\otimes}(V \circ i) \equiv G$, as a result of which the bigger group G suffering from SSB is determined.

$\because$) For any $u \in H = End_{\otimes}(V)$, $\forall \rho \in \mathcal{T}_0$, $\exists u_\rho : V_\rho \rightarrow V_\rho$ s.t. for $\forall T \in (\rho_1, \rho_2)$ $V_T \circ u_{\rho_1} = u_{\rho_2} \circ V_T$. Then, for any $\sigma \in \mathcal{T}$, $i(\sigma) = \tilde{\sigma} \in \mathcal{T}_0$ and $\forall S \in (\sigma_1, \sigma_2)$ $i(S) \in (i(\sigma_1), i(\sigma_2)) \subset \mathfrak{A} \subset \mathfrak{A}^d$, $V_{i(S)} \circ u_{i(\sigma_1)} = u_{i(\sigma_2)} \circ$

$V_{i(S)}$, which means $j(u) = u_{i(\cdot)} : \mathcal{T} \to \mathcal{U}(V_{i(\cdot)})$ is a natural unitary transformation from the functor $V \circ i = i^*(V)$ to itself, belonging to $End_{\otimes}(V \circ i) = G$.

Then, for each $\sigma \in \mathcal{T}$, we obtain $\gamma_\sigma|_H = \gamma_\sigma \circ j = \eta_{i(\sigma)}$, which states that for each H-representation of the form $\eta_{i(\sigma)}$ $(\sigma \in \mathcal{T})$, there is a G-representation γ_σ whose restriction to H is $\eta_{i(\sigma)} = \gamma_\sigma|_H$. This is just the categorical dual formulation of the finite-dimensional induction in 1. (The structures of Goldstone multiplets and order parameters can be clarified in use of this result combined with the next items 4 and 5, which will be reported elsewhere).

4. Cuntz bundle category $\mathfrak{CB}$:

   Bundle structures: $\mathcal{O}_d^H \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d = \Gamma(G \underset{H}{\times} \mathcal{O}_d)$, $Spec(\mathfrak{Z}(\mathcal{O}_d^H \underset{\mathcal{O}_d^G}{\otimes} \mathcal{O}_d)) = G/H$.

   Associativity (as algebras): $(\mathcal{O}^{(1)} \underset{\mathcal{O}^{(2)}}{\otimes} \mathcal{O}^{(3)}) \underset{\mathcal{O}^{(4)}}{\otimes} \mathcal{O}^{(5)} \simeq \mathcal{O}^{(1)} \underset{\mathcal{O}^{(2)}}{\otimes} (\mathcal{O}^{(3)} \underset{\mathcal{O}^{(4)}}{\otimes} \mathcal{O}^{(5)})$ valid for such subalgebras $\mathcal{O}^{(i)}$ of Cuntz algebra $\mathcal{O}_2$ that $\mathcal{O}^{(1)} \supset \mathcal{O}^{(2)} \subset \mathcal{O}^{(3)}$, $\mathcal{O}^{(3)} \supset \mathcal{O}^{(4)} \subset \mathcal{O}^{(5)}$ and $\exists G^{(2)} \subset SU(d_2), G^{(4)} \subset SU(d_4)$ s.t. $\mathcal{O}_{d_2}^{G^{(2)}} \subset \mathcal{O}^{(2)}, \mathcal{O}_{d_4}^{G^{(4)}} \subset \mathcal{O}^{(4)}$.

   Functoriality of crossed product: $\mathfrak{A} \underset{\mathcal{O}_d^G}{\otimes} (-)$ is a *functor* from $\mathfrak{CB}$ to local nets if $\mathfrak{A}$ is a local net.

5. Generalization of "sectors": So far, only discrete sectors are recognized as genuine ones. In SSB case, order parameters describes continuous family of disjoint states (of $\mathfrak{F}$) parametrized by G/H and in thermal situations, (inverse) temperatures $\beta[=(\beta^\mu)]$ discriminate also disjoint KMS states (of $\mathfrak{A}$). We can also formulate variety of non-equilibrium local states (Buchholz-I.O.-Roos '01 [12]). They need to be unified in a similar way to the unified treatment of discrete and continuous spectra of self-adjoint operators.

6. Open problems: Is $\Gamma = Gal(\mathfrak{F}/\mathfrak{A})$ compact or non-compact?

   The difficulty consists in the fact that all the relevant important information is contained between $\mathfrak{A} = \mathfrak{F}^\Gamma (\subset \mathfrak{F}^G) \subset \mathfrak{A}^d = \mathfrak{F}^H$.

   If $\Gamma = Gal(\mathfrak{F}/\mathfrak{A})$ is non-compact, but still a semi-simple Lie group, then we can take advantage of the Iwasawa decomposition, $\Gamma = KAN^+$, with K: maximal compact subgroup, A: maximal abelian subgroup, N^+: nilpotent subgroup (e.g., if $\Gamma = KA$, it is an affine group of the form $ax + b$). If the appearance of non-compact Galois groups is unavoidable, the essence of DR theory needs to be extended to such contexts.

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