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Harmonic equations in the Grothendieck-Teichmüller group (Grothendieck-Teichmüller 群内での調和方程式)

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§0. Introduction in Japanese (日本語序).

本稿の目的は、論文 [NT] の内容の紹介である。V.G.Drinfel'd [Dr] によって導入された Grothendieck-Teichmüller 群 $\widehat{GT}$ は、2 元 x,y を生成元とする階数 2 の副有限自由群 $\hat{F}_2$ の自己同型群 Aut$\hat{F}_2$ の部分群であり、2-,3-,5-cycle relation と呼ばれる関係式で特徴付けられる群である。$\hat{F}_2$ を $\mathbf{P}^1_{\mathbb{Q}}\setminus\{0,1,\infty\}$ の副有限基本群と同一視する時、有理数体 $\mathbb{Q}$ の絶対 Galois 群 $G_{\mathbb{Q}}$ は $\hat{F}_2$ に忠実に作用し、その作用射 $G_{\mathbb{Q}}\to \mathrm{Aut}\hat{F}_2$ の像は $\widehat{GT}$ に含まれることが知られている。この意味で $\widehat{GT}$ は組合せ的な表示を持ちつつ $G_{\mathbb{Q}}$ を含む興味深い群である。基本的な未解決問題「$\widehat{GT}=G_{\mathbb{Q}}$ であるか?」に関して、本稿では次の幾何的対象への Galois 作用に着目し、$G_{\mathbb{Q}}$ の像の各元が $\widehat{GT}$ 内で満たすべき新たな形の関係式を得た。$\mathbf{P}^1$ から調和点集合 $\{0,\pm1,\infty\}$ (resp. 等非調和点集合 $\{0,1,\rho,\rho^{-1},\infty\}$ $(\rho=e^{2\pi\sqrt{-1}/6})$) を除いた曲線は、2 種の $\mathbf{P}^1\setminus\{0,1,\infty\}$ への射— 包含射と 2 次 (resp. 3 次) の被覆と — を有する。$G_{\mathbb{Q}}$ はこれらの射に関して同変的に作用するので、そのことから従う条件を $\widehat{GT}$ の標準的な座標 (λ,f) に関する関係式として記述した。この関係式が真に「新しい」か (即ち $\widehat{GT}$ と $G_{\mathbb{Q}}$ とが一致しないことを示すか) は依然として今後の課題である。

§1. Introduction.

The purpose of this short note is to summarize the results of our paper [NT]. For details, see [NT]. The Grothendieck-Teichmüller group $\widehat{GT}$ introduced by V.G.Drinfeld [Dr] is defined as a subgroup of the automorphism group Aut$\hat{F}_2$ of the free profinite group $\hat{F}_2$ of rank 2 with free generators x,y satisfying the so called the 2-,3-,5-cycle relations.

It is well known that the absolute Galois group $G_{\mathbb{Q}}$ faithfully acts on the $\hat{F}_2$ identified with the profinite fundamental group $\mathbf{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}$ and that the image of $G_{\mathbb{Q}} \to \mathrm{Aut}\hat{F}_2$ is contained in $\widehat{GT}$. In this sense, $\widehat{GT}$ is an interesting subject which has combinatorial presentation together with arithmetic content $G_{\mathbb{Q}}$.

It is still unknown whether "$\widehat{GT} = G_{\mathbb{Q}}$" or not. In this note, we obtain several newtype equations satisfied by $G_{\mathbb{Q}}$ in $\widehat{GT}$ by focusing on Galois actions on the following geometric objects. Namely, the open line obtained by removing the harmonic points $\{0, \pm 1, \infty\}$ (resp. equianharmonic points $\{0, 1, \rho, \rho^{-1}, \infty\}$ ($\rho = e^{2\pi\sqrt{-1}/6}$)) from $\mathbf{P}^1$ has two sorts of morphisms to $\mathbf{P}^1 \setminus \{0,1,\infty\}$ — open inclusion and double (resp. triple) covering. We will describe the condition that $G_{\mathbb{Q}}$-actions must respect homomorphisms of π_1 induced from these morphisms, and get several equations satisfied by the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ in terms of the standard coordinates (λ, f) of $\widehat{GT}$. It is still a difficult open problem to determine whether these equations give proper subgroup of $\widehat{GT}$.

To be more precise, we shall review the definition of the (profinite) Grothendieck-Teichmüller group $\widehat{GT}$. Let $(\lambda, f) : \widehat{GT} \hookrightarrow \hat{\mathbb{Z}} \times \hat{F}_2$ be the standard parametrization of $\widehat{GT}$ into the (set-)product of free profinite groups of rank 1 and 2. Here λ is a homomorphism into $\hat{\mathbb{Z}}^\times$ extending the cyclotomic character on $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) (\subset \widehat{GT})$, and f is a certain 1-cocycle into (the commutator subgroup of) the free profinite group $\hat{F}_2$ of rank 2 often identified with the profinite fundamental group of $\mathbf{P}^1 - \{0,1,\infty\}$. The latter group has certain standard free generators x, y, on which $\widehat{GT}$ acts via $x \mapsto x^\lambda$, $y \mapsto f(x,y)^{-1} y^\lambda f(x,y)$ (see V.G.Drinfeld [Dr], Y.Ihara [I1]). Recall then that $\widehat{GT}$ was introduced in [Dr] by the three equations:

$$\text{(I)} \qquad f(x,y)f(y,x) = 1,$$

$$\text{(II)} \qquad f(x,y)x^{\frac{\lambda-1}{2}} f(z,x) z^{\frac{\lambda-1}{2}} f(y,z) y^{\frac{\lambda-1}{2}} = 1,$$

$$\text{(III)} \qquad f(x_{12}, x_{23}x_{24}) f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34}) f(x_{12}x_{13}, x_{24}x_{34}) f(x_{12}, x_{23}),$$

where $z = (xy)^{-1}$, and the x_{ij} in (III) are certain standard elements of the (profinite) braid group $\hat{B}_4$ with 4 strings.

Theorem 1. *Let $\hat{B}_3$ be the profinite braid group generated by the symbols τ_1, τ_2 with the defining relation $\tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$. For an integer $a > 1$, let $\rho_a : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}$ be the Kummer 1-cocycle defined by $(\sqrt[n]{a})^{\sigma-1} = \zeta_n^{\rho_a(\sigma)}$ ($\sigma \in G_{\mathbb{Q}}$, $n \geq 1$, $\zeta_n = \exp(2\pi i/n)$). Then, the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ satisfies the following equations:*

(I′) (Harmonic equation) $\quad f(\tau_1^2, \tau_2^2) = \tau_2^{-4\rho_2} f(\tau_2^2, \eta)^{-1} f(\tau_1^2, \eta)\tau_1^{4\rho_2}$.

(II′) (Equianharmonic equation)

$$f(\tau_1^2, \tau_2^2) = \tau_2^{-3\rho_3 - \frac{\lambda-1}{2}} f(\tau_2^2, \tau_1\tau_2)^{-1}(\tau_1\tau_2)^{\frac{\lambda-1}{2}} f(\tau_1^2, \tau_1\tau_2)\tau_1^{3\rho_3 - \frac{\lambda-1}{2}}.$$

$$(\mathrm{IV}'_{\mathrm{bis}}) \qquad f(\tau_1, \tau_1\tau_2) = (\tau_1\tau_2)^{-\rho_2} f(\tau_1^2, \tau_1\tau_2)\tau_1^{2\rho_2},$$

$$(\mathrm{V}) \qquad f(\tau_1, \eta) = \eta^{\rho_3 - 2\rho_2} f(\tau_1^2, \eta)\tau_1^{6\rho_2 - 3\rho_3}.$$

In [LS2], P.Lochak and L.Schneps introduced remarkable new 1-cocycles $g, h : \widehat{GT} \to \hat{F}_2$ which decompose the principal parameter f of $\widehat{GT}$ with respect to certain automorphisms of $\hat{F}_2$. They considered automorphisms θ, ω of $\hat{F}_2$ of finite order such that $\theta(x) = y$, $\theta(y) = x$; $\omega(x) = y$, $\omega(y) = z$, $\omega(z) = x$ (after setting $z = (xy)^{-1}$), and determined the nonabelian cohomology sets $H^1(\langle\theta\rangle, \hat{F}_2), H^1(\langle\omega\rangle, \hat{F}_2)$. In the process of getting this result, they showed that each $(\lambda, f) \in \widehat{GT}$ has unique prowords $g, h \in \hat{F}_2$ satisfying

$$(1.1) \qquad f = \theta(g)^{-1}g = \begin{cases} y^{-\frac{\lambda-1}{2}}\omega(h)^{-1}h, & \lambda \equiv 1 \mod 6, \\ y^{-\frac{\lambda-1}{2}}\omega(h)^{-1}y^{-1}h, & \lambda \equiv -1 \mod 6. \end{cases}$$

(There seems some inconsistency in the presentation of [LS2]. For example, they use the same symbol ω to denote different automorphisms on §1 (p.571) and §2 (p.578). See also C.Scheiderer [Sc], J.-P.Serre [Se]). Moreover, the restrictions of these new 1-cocycles g and h on the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ were interpreted as "Galois transformation factors" of certain explicit chains on $\mathbf{P}^1 - \{0, 1, \infty\}$, as similar to the original case of f (cf. [I1] and §3 below). In what follows, we keep the notations of Theorem 1.

Theorem 2. *The image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ satisfies the following equations:*

$$(\mathrm{GF}_0) \qquad g(\tau_1^2, \tau_2^2) = \eta^{2\rho_2 - \rho_3} f(\tau_1, \eta)\tau_1^{-2\rho_2 + 3\rho_3},$$

$$(\mathrm{GF}_1) \qquad g(\tau_1^2, \tau_2^2) = f(\tau_1^2, \eta)\tau_1^{4\rho_2},$$

$$(\mathrm{HF}_0) \qquad h(\tau_1^2, \tau_2^2) = (\xi_\pm)^{\rho_2 + \frac{\lambda\mp1-6\rho_3}{4}} f(\tau_1, \xi_\pm)\tau_1^{3\rho_3 - 2\rho_2 - \frac{\lambda\mp1}{2}},$$

$$(\mathrm{HF}_1) \qquad h(\tau_1^2, \tau_2^2) = (\xi_\pm)^{\frac{\lambda\mp1-6\rho_3}{4}} f(\tau_1^2, \xi_\pm)\tau_1^{3\rho_3 - \frac{\lambda\mp1}{2}},$$

where, in the first two equations η denotes $\tau_1\tau_2\tau_1$, and in the last two equations, ξ_+, ξ_- denote $\tau_1\tau_2$, $\tau_2\tau_1$ respectively, and the sign $\mp$ is taken according as $\lambda \equiv \pm 1$ mod 6 respectively.

Note that, since $\{\tau_1^2, \tau_2^2\}$ generates a free profinite subgroup of rank 2 in $\hat{B}_3$, the above equations in Theorem A determine g, h completely as prowords. Equating the left hand sides of (HF_0), (HF_1) and of (GF_0), (GF_1) respectively, we obtain the equations ($\mathrm{IV}'_{\mathrm{bis}}$) and (V) respectively. We can also prove (I$'$)$\Leftrightarrow$ (GF_1) and (II$'$) $\Leftrightarrow$ (HF_1) (see [NT] Prop. (4.3),(5.3) respectively). The following corollary also follows from the above result (see [NT] Prop. (6.1)).

Corollary. *On the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$,*

$$(\mathrm{GF}_{n-1}) \qquad g(\tau_1^n, \tau_2^n) = f(\tau_1^n, \eta)\tau_1^{2n\rho_2} \qquad (n \geq 2),$$

$$(\mathrm{GG}) \qquad g(\tau_1^2, \tau_2^2) = \eta^{2\rho_2 - \rho_3} g(\tau_1, \tau_2)\tau_1^{-4\rho_2+3\rho_3},$$

$$(\mathrm{FF}) \qquad f(\tau_1^2, \tau_2^2) = \tau_2^{4\rho_2 - 3\rho_3} f(\tau_1, \tau_2)\tau_1^{-4\rho_2+3\rho_3}.$$

The group $\widehat{GT}$ acts universally on the tower of Artin braid groups $\{B_n\}$ and the tower of mapping class groups of genus zero. In the process of examining the case of higher genus mapping class groups, we encountered refinements of the defining equations of $\widehat{GT}$ ([N],[LNS],[NS]):

$$(\mathrm{IV}) \qquad f(\tau_1, \tau_2^4) = \tau_2^{8\rho_2} f(\tau_1^2, \tau_2^2)\tau_1^{4\rho_2}(\tau_1\tau_2)^{-6\rho_2};$$

$$(\mathrm{IV}') \qquad f(\tau_1, \tau_2^2) = \tau_2^{4\rho_2} f(\tau_1^2, \tau_2^2)\tau_1^{2\rho_2}(\tau_1\tau_2^2)^{-2\rho_2}$$
$$= \tau_2^{-4\rho_2} f(\tau_1, \tau_2^4)\tau_1^{-2\rho_2}(\tau_1\tau_2^2)^{2\rho_2};$$

$$(\mathrm{III}') \qquad f(\tau_1\tau_3, \tau_2^2) = g(x_{45}, x_{51}) f(x_{12}, x_{23}) f(x_{34}, x_{45}).$$

The above ($\mathrm{IV}'_{\mathrm{bis}}$) follows from (IV$'$). Putting the equations (GF_1), (HF_1) of Theorem 2 back to the original coboundary-like definitions (1.1) of g, h, we get Theorem 1. The equations (I), (II) are easily implied by the above (I$'$), (II$'$) respectively, while (III) is implied by the equation (III$'$). Thus, as the consequence of the present paper, we turn out to have five equations (I$'$),(II$'$),(III$'$),(IV$'$) and (V) as a set of (seemingly independent) equations restricting the image of $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$. But recently, our last named author [T] found that, besides (III$'$), a number of more equations can hold in $\hat{B}_4$ from his close study of the geometry of the moduli space of the 5-point marked projective lines. Moreover, Y.Ihara [I2], with his independent method, investigated series of infinitely many "arithmetic relations" satisfied by the image of $G_{\mathbb{Q}}$. (Especially, he extended the Kummer 1-cocycles ρ_a $(a > 0)$ to the whole $\widehat{GT}$ in a uniform way. See §5.) On the other hand, F.Pop [P] recently indicated a remarkable evidence asserting that certain restricted families of "geometric" homomorphisms between fundamental groups of algebraic varieties over $\mathbb{Q}$ are enough to characterize $G_{\mathbb{Q}}$. These new results around the injection $G_{\mathbb{Q}} \hookrightarrow \widehat{GT}$ have increased the variety of scales for measuring the possible gap between $G_{\mathbb{Q}}$ and $\widehat{GT}$ in our hands, although currently they still leave us with the basic

Open problem: Do these newtype relations on $G_{\mathbb{Q}}$ never hold on the whole $\widehat{GT}$?

§2. Legendre-Jacobi covering and its subcoverings.

We shall consider the quotient line of $\mathbf{P}^1_t - \{0,1,\infty\}$ by the S_3-symmetry. One can can introduce the coordinate s for such a quotient line by

$$s = \phi(t) = \frac{27}{4}\frac{t^2(t-1)^2}{(t^2-t+1)^3}, \tag{3.1}$$

where the ramification points are normalized so that $\phi^{-1}(0) = \{0,1,\infty\}$, $\phi^{-1}(1) = \{\frac{1}{2}, -1, 2\}$ and $\phi^{-1}(\infty) = \{\rho, \rho^{-1}\}$ hold. Let $X_s = \mathbf{P}^1_s - \{0,1,\infty\}$. We call $\phi : \mathbf{P}^1_t \to \mathbf{P}^1_s$ the *Legendre-Jacobi covering.*

We also introduce the following two subcoverings. One is the *harmonic line* $\mathbf{P}^1_u$ between $\mathbf{P}^1_t$ and $\mathbf{P}^1_s$ given by

$$u = 4t(1-t) \text{ and } s = \frac{27u^2}{(4-u)^3}.$$

The covering map $\psi : \mathbf{P}^1_t \to \mathbf{P}^1_u$ is ramified only at $t = 0, \frac{1}{2}$ (over $u = 0, 1$ respectively). Letting $X_u = \mathbf{P}^1_u - \{0,1,\infty\}$, we may consider $\pi_1(\mathbf{P}^1_t - \{0,1,\infty\}, \mathfrak{B})$ as a subgroupoid of $\pi_1(X_u, e_1|2)$ which classifies the etale covers of X_u with ramification indices over $u = 1$ dividing 2.

Another intermediate line to be considered is the *equianharmonic line* $\mathbf{P}^1_v$ between $\mathbf{P}^1_t$ and $\mathbf{P}^1_s$. Let us introduce its coordinate v by

$$v=\varphi(t)=\left(\frac{t-\rho}{t-\rho^{-1}}\right)^3, \quad s=\frac{-4v}{(v-1)^2}.$$

Notice here that the covering morphism $\varphi:\mathbf{P}^1_t\to\mathbf{P}^1_v$ is defined only over $\mathbb{Q}(\rho)$ $(\rho=\exp(2\pi i/6))$. In fact, if we change the variable v by $v'=\frac{3(v-\rho^2)}{\rho(v-1)}$, then φ can be defined over $\mathbb{Q}$ as $t\mapsto v'=t+\frac{1}{1-t}+\frac{t-1}{t}$. Still in this paper we make use of v instead of v'.

§3. Geometric interpretation of the cocycles g,h.

For each $\sigma\in G_{\mathbb{Q}}$, we denote by λ_σ, f_σ, g_σ, h_σ the images of σ by λ,f,g,h respectively. Letting $\mathbf{P}^1_t$ denote the projective line with a fixed coordinate t, we shall consider the etale fundamental groupoid of $X_t=\mathbf{P}^1_t-\{0,1,\infty\}$ with specific set of base points

$$\mathfrak{B}=\{\overrightarrow{01},\overrightarrow{10},\overrightarrow{1\infty},\overrightarrow{\infty 1},\overrightarrow{\infty 0},\overrightarrow{0\infty}\}\cup\{-1,\frac{1}{2},2\}\cup\{\rho,\rho^{-1}\}$$
$$(\rho=\exp(2\pi i/6)).$$

Here $\overrightarrow{ab}$ $(a,b\in\{0,1,\infty\})$ denote the tangential base points introduced by Deligne [De], Anderson-Ihara [AI]. Introduce some basic paths q,r,ε in $\pi_1(X_t,\mathfrak{B})$ as in Figure 1:

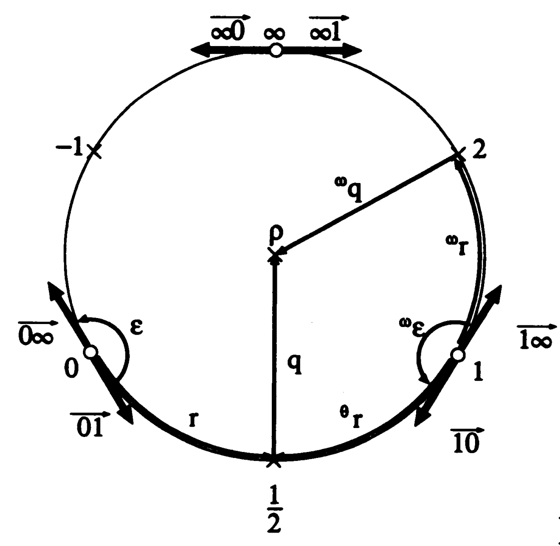

Figure 1

The symmetric group S_3 on $\{0,1,\infty\}$ acts naturally on the paths in $\pi_1(X_t, \mathfrak{B})$. We write ${}^\theta\alpha$, ${}^\omega\alpha$, ${}^{\bar\omega}\alpha$ to denote the images of a path α by the permutations (01), (01∞), (0∞1) of S_3 respectively. Under these notations, for example, the standard path p connecting $\overrightarrow{01}$ and $\overrightarrow{10}$ can be written as $p = r({}^\theta r)^{-1}$, and the standard generators x, y of $\pi_1(X_t, \overrightarrow{01})$ are introduced as:

$$x = \varepsilon({}^{\theta\omega}\varepsilon), \quad y = p({}^\theta\varepsilon)({}^\omega\varepsilon)p^{-1}.$$

The geometric interpretation of f_σ, g_σ and h_σ for $\sigma \in G_{\mathbb{Q}}$ are then given by:

$$\begin{aligned}
&\sigma(p) = f_\sigma(x,y)^{-1}p, \\
(2.1) \qquad &\sigma(r) = g_\sigma(x,y)^{-1}r, \\
&\sigma(rq) = \begin{cases} h_\sigma(x,y)^{-1}rq & (\lambda_\sigma \equiv 1 \mod 6), \\ h_\sigma(x,y)^{-1}r({}^\theta q) & (\lambda_\sigma \equiv -1 \mod 6). \end{cases}
\end{aligned}$$

Note that λ_σ is the cyclotomic character, hence $\lambda_\sigma \equiv 1 \mod 6$ if and only if σ fixes $\rho = \exp(2\pi i/6)$. These $f_\sigma, g_\sigma, h_\sigma$ have values in the geometric fundamental group $\bar\pi_1(X_t, \overrightarrow{01})$ $(= \pi_1(X_t \otimes \overline{\mathbb{Q}}, \overrightarrow{01}))$ regarded as the free profinite group $\hat F_2$ with two free generators (corresponding to) x, y.

We summarize basic knowledge on the abelianization of these 1-cocycles here: Let $[\hat F_2, \hat F_2]$ denote the commutator subgroup of $\hat F_2 = \bar\pi_1(X_t, \overrightarrow{01})$. Then, for each $\sigma \in G_{\mathbb{Q}}$, the following congruences hold modulo $[\hat F_2, \hat F_2]$.

$$(2.2) \qquad f_\sigma(x,y) \equiv 1,$$

$$(2.3) \qquad g_\sigma(x,y) \equiv (xy)^{\rho_2(\sigma)},$$

$$(2.4) \qquad h_\sigma(x,y) \equiv \begin{cases} x^{-\frac{\lambda_\sigma - 1}{6}} y^{\frac{\lambda_\sigma - 1}{6}}, & (\lambda_\sigma \equiv 1 \mod 6), \\ x^{-\frac{\lambda_\sigma + 1}{6}} y^{\frac{\lambda_\sigma + 1}{6}} & (\lambda_\sigma \equiv -1 \mod 6). \end{cases}$$

§4. Sketch of proof of (GF_0).

In this note, we only illustrate the proof of (GF_0). We observe what happens when we transform the geometric interpretation formula in §3 by the covering map ϕ of §2 which is defined over $\mathbb{Q}$. For other equations of Theorem 1 and of Theorem 2, see [NT] where other subcoverings of §2 and paths of §3 are examined to prove them.

Now, the Taylor expansion of ϕ in t and $t - \frac{1}{2}$ show their principal terms as

$$s \sim \frac{27}{4}t^2, \quad (1-s) \sim 12(t - \frac{1}{2})^2.$$

This means that, in view of the effects of Galois actions, we should regard

$$\phi(\overrightarrow{01}_t) = \frac{4}{27}\overrightarrow{01}_s, \quad \phi((\tfrac{1}{2})_t) = \frac{1}{12}\overrightarrow{10}_s.$$

Write δ_1, δ_2 for the canonical paths from $\frac{4}{27}\overrightarrow{01}_s$ to $\overrightarrow{01}_s$ and from $\frac{1}{12}\overrightarrow{10}_s$ to $\overrightarrow{10}_s$ along the real axis respectively. Then, we have

$$\begin{cases} \sigma(r_t) = g(x_t, y_t)^{-1} r_t, \\ \sigma(p_s) = f(x_s, y_s)^{-1} p_s, \end{cases} \quad \text{and} \quad \begin{cases} \sigma(\delta_1) = \delta_1 x_s^{2\rho_2(\sigma) - 3\rho_3(\sigma)}, \\ \sigma(\delta_2) = \delta_2({}^{\theta}x_s)^{-2\rho_2(\sigma) - \rho_3(\sigma)}. \end{cases}$$

for $\sigma \in G_{\mathbb{Q}}$. Putting these together into the commutative diagram

$$\begin{array}{ccc} \frac{4}{27}\overrightarrow{01}_s & \xrightarrow{\delta_1} & \overrightarrow{01}_s \\ {\scriptstyle \phi(r_t)}\downarrow & & \downarrow{\scriptstyle p_s} \\ \frac{1}{12}\overrightarrow{10}_s & \xrightarrow[\delta_2]{} & \overrightarrow{10}_s, \end{array}$$

we obtain the equation

$$\delta_1^{-1} g_\sigma(\phi(x_t), \phi(y_t))\delta_1 = y_s^{-2\rho_2(\sigma) - \rho_3(\sigma)} f_\sigma(x_s, y_s) x_s^{-2\rho_2(\sigma) + 3\rho_3(\sigma)}$$

in the fundamental group $\pi_1(X_s, e_1|2, e_\infty|3, \overrightarrow{01}_s)$. Note here that this fundamental group is generated by x_s, y_s, z_s with the defining relations $x_s y_s z_s = y_s^2 = z_s^3 = 1$, and that the map ϕ can be described by $\phi(x_t) = x_s^2$, $\phi(y_t) = y_s^{-1} x_s^2 y_s$. Now there is an exact sequence of profinite groups

$$1 \longrightarrow \langle \eta^2 \rangle \longrightarrow \hat{B}_3 \longrightarrow \bar{\pi}_1(X_s, e_1|2, e_\infty|3, \overrightarrow{01}_s) \longrightarrow 1,$$

where the latter surjection is defined by $\tau_1 \mapsto x_s, \tau_2 \mapsto y_s x_s y_s^{-1}, \eta = \tau_1\tau_2\tau_1 \mapsto y_s$. From this, we see that there exists some constant $c \in \hat{\mathbb{Z}}$ such that

$$g_\sigma(\tau_1^2, \tau_2^2) = \eta^{2c}\eta^{-2\rho_2(\sigma)-\rho_3(\sigma)} f_\sigma(\tau_1, \eta)\tau_1^{-2\rho_2(\sigma)+3\rho_3(\sigma)}.$$

To determine c, one may apply the surjection of $\hat{B}_3$ onto $\hat{\mathbb{Z}}$ sending τ_1, τ_2 to 1. Noticing that $f_\sigma \equiv 0$, $g_\sigma(x, y) \equiv (xy)^{\rho_2(\sigma)}$ modulo $[\hat{F}_2, \hat{F}_2]$ (cf. Prop.(2.2)), we obtain $c = 2\rho_2(\sigma)$. This proves (GF_0).

§5. Kummer 1-cocyles, H.Furusho's work.

Ihara [I2-3] invented a beautiful theory of the (hyper-)adelic beta and gamma functions defined on the whole Grothendieck-Teichmüller group $\widehat{GT}$. He considered n-cyclic Kummer coverings of $\mathbf{P}^1 - \{0, 1, \infty\}$ ($n \in \mathbb{N}$), and defined a system of 1-cocycles including the $-\Psi_n^{(0)} : \widehat{GT} \to \hat{\mathbb{Z}}(1)$ ($n \in \mathbb{N}$) which extend the Kummer 1-cocycles ρ_n on $G_\mathbb{Q}$ respectively (ρ_n is defined by $\sigma(\sqrt[k]{n}) = \sqrt[k]{n}\,\zeta_k^{\rho_n(\sigma)}$ for $k \geq 1$, $\sigma \in G_\mathbb{Q}$). Using these functions, Ihara introduced certain subgroups GTA, GTK of $\widehat{GT}$ containing $G_\mathbb{Q}$ and discussed their relationships. More recently, H.Furusho examined relations between Ihara's work [I2-3] and our work [N, NS, NT] and corelated each other by showing "$\mathbb{I}\Gamma \cap GTK \subset GTA_{2^\infty}$". See [F1,2] for details. Furusho's result may be interpreted as indicating future possibilities that the "arithmetic relations" of GTA may be captured by somewhat complicated combinations of various types of "geometric relations" including what appeared in GTK or in our works [N, NS, NT].

Let us review how Ihara extended the Kummer 1-cocycle ρ_n on $G_\mathbb{Q}$ to $\widehat{GT}$: For a positive integer n, let H_n be the kernel of the homomorphism $\hat{F}_2 \to \mathbb{Z}/n\mathbb{Z}$ defined by $x \mapsto 1, y \mapsto 0$, which is a free profinite group of rank $n+1$ generated by the $x^i y x^{-i}$ ($i = 0, \dots, n-1$) and x^n. Since, for any $\sigma = (\lambda, f) \in \widehat{GT}$, $f = f_\sigma$ belongs to $[\hat{F}_2, \hat{F}_2] \subset H_n$, one can consider the image (denoted $\Psi_n^{(0)}(\sigma)$) of f_σ by the homomorphism $H_n \to \hat{\mathbb{Z}}$ defined by $x^n, x^i y x^{-i} \mapsto 0$ ($i = 1, \dots, n-1$) and $y \mapsto 1$. Then, Ihara proved that $-\Psi_n^{(0)} : \widehat{GT} \to \hat{\mathbb{Z}}(1)$ is a 1-cocycle extending the Kummer 1-cocycle ρ_n ([I3] Theorem 1).

On the other hand, for $n = 2$, we have another 1-cocycle on $\widehat{GT}$ extending the Kummer 1-cocycle ρ_2 on $G_\mathbb{Q}$. As we mentioned in §1, the

1-cocycle $g : \widehat{GT} \to \hat{F}_2$ introduced by P.Lochak and L.Schneps [LS2] is defined not only on $G_{\mathbb{Q}}$ but also on whole $\widehat{GT}$. If we define $\tilde{\rho}_2 : \widehat{GT} \to \hat{\mathbb{Z}}(1)$ by $g(x,y) \equiv (xy)^{\tilde{\rho}_2(\sigma)} \bmod [\hat{F}_2, \hat{F}_2]$, $\tilde{\rho}_2$ extends the Kummer 1-cocycle ρ_2 on $G_{\mathbb{Q}}$.

Proposition 5.1. *If $\sigma = (\lambda, f) \in \widehat{GT}$ satisfies the equation*

$$(\mathrm{GF}_1) \qquad g(\tau_1^2, \tau_2^2) = f(\tau_1^2, \eta)\tau_1^{4\tilde{\rho}_2(\sigma)},$$

or equivalently

$$(\mathrm{I}') \qquad f(\tau_1^2, \tau_2^2) = \tau_2^{-4\tilde{\rho}_2(\sigma)} f(\tau_2^2, \eta)^{-1} f(\tau_1^2, \eta)\tau_1^{4\tilde{\rho}_2(\sigma)},$$

in $\hat{B}_3$, then it holds that $-\Psi_2^{(0)}(\sigma) = \tilde{\rho}_2(\sigma)$. In other words, under (GF_1) *(or* (I')*), two cocycles $-\Psi_2^{(0)}$ and $\tilde{\rho}_2$ coincide with each other.*

Proof. Define a homomorphism from a subgroup $\langle \tau_1^2, \tau_2^2, \eta \rangle$ of $\hat{B}_3$ onto $\hat{F}_2 / \langle\langle y^2 \rangle\rangle$ ($\hat{F}_2 = \langle x, y \rangle$), where $\langle\langle y^2 \rangle\rangle$ denotes the normal closure in $\hat{F}_2$, by $\tau_1^2 \mapsto x, \tau_2^2 \mapsto yxy^{-1}, \eta \mapsto y$. Then, by applying this homomorphism to both sides of the equation (GF_1), we have

$$g(x, yxy^{-1}) = f(x,y)x^{2\tilde{\rho}_2(\sigma)}$$

in $\hat{F}_2 / \langle\langle y^2 \rangle\rangle$. On the other hand, the proword $f(x,y) \in \hat{F}_2$ lies in the commutator subgroup $[\hat{F}_2, \hat{F}_2]$, hence in particular, in the normal (free profinite) subgroup $\hat{F}_3 = \langle x', y', z' \rangle$ of $\hat{F}_2$ with $x' = x, y' = yxy^{-1}, z' = y^2$. This means that there exists a unique proword $f^{(2)}(x', y', z') \in \hat{F}_3$ such that

$$f(x,y) = f^{(2)}(x, yxy^{-1}, y^2)$$

holds in $\hat{F}_2$ (cf. [M] 3.3). The above equation can be written in the present notations as

$$g(x', y') \equiv f^{(2)}(x', y', z')(x')^{2\tilde{\rho}_2(\sigma)} \mod \langle\langle z' \rangle\rangle.$$

By using $g(x,y) \equiv (xy)^{\tilde{\rho}_2(\sigma)} \bmod [\hat{F}_2, \hat{F}_2]$, and the fact that $f = f^{(2)}$ belongs to $[\hat{F}_2, \hat{F}_2]$, we can determine the abelianization of $f^{(2)}$ as follows:

$$f_\sigma^{(2)}(x', y', z') \equiv (x')^{-\tilde{\rho}_2(\sigma)}(y')^{\tilde{\rho}_2(\sigma)} \mod [\hat{F}_3, \hat{F}_3].$$

The assertion follows by comparing this and the definition of $\Psi_2^{(0)}$. $\square$

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