数理解析研究所講究録 1267 巻 2002 年 48-61

Teichmüller groupoids

佐賀大学理工学部数理科学科　市川尚志 (Takashi Ichikawa)
Department of Mathematics, Faculty of Science and Engineering
Saga University

Introduction

The aim of this paper is to review results in [I2] without giving their proof.

Provided that $2g+n-2>0$, let $\mathcal{M}_{g,n}$ denote the moduli stack over $\mathbf{Z}$ classifying proper and smooth n-pointed curves of genus g. Then by a result of Oda [O2], the algebraic (orbispace) fundamental group of $\mathcal{M}_{g,n}\otimes_{\mathbf{Z}}\overline{\mathbf{Q}}$ ($\overline{\mathbf{Q}}$: the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$) is isomorphic to the profinite completion $\widehat{\Pi}_{g,n}$ of the Teichmüller modular group $\Pi_{g,n}$, the fundamental group of $\mathcal{M}_{g,n}(\mathbf{C})$. Hence one can consider the Galois action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $\widehat{\Pi}_{g,n}$ taking an appropriate base point in $\mathcal{M}_{g,n}$. As an extension of the Teichmüller modular group, the topological Teichmüller groupoid for $\mathcal{M}_{g,n}(\mathbf{C})$ is defined to be its fundamental groupoid whose base points are the points at "infinity" corresponding to maximally degenerate pointed complex curves. In [G], Grothendieck considered this groupoid in the category of arithmetic geometry, which we call the Teichmüller groupoid for $\mathcal{M}_{g,n}$, and he proposed a conjecture on a "game of Lego-Teichmüller" which states, roughly speaking, that the Teichmüller groupoid for $\mathcal{M}_{g,n}$ will have generators attached to $\mathcal{M}_{0,4}$, $\mathcal{M}_{1,1}$ with relations induced from $\mathcal{M}_{0,5}$, $\mathcal{M}_{1,2}$. This conjecture implies the prediction that each profinite Teichmüller groupoid with the Galois action can be described by these "basic" groupoids.

We note that important work has been done in this subject. In connection with conformal field theory, Moore and Seiberg [MS] stated the "completeness theorem" (see [HLS], [BK] for its accurate formulation and proof) which can be regarded as a realization of the game of Lego-Teichmüller in topological category. Drinfeld [Dr] studied the case that $\mathcal{M}_{g,n}=\mathcal{M}_{0,5}$, and obtained a distinguished relation on the basic Galois action corresponding to the pentagon relation in conformal field theory, which was rewritten as the "5-cycle relation" by Ihara [Ih1], [Ih2]. Using this relation and other supplementary relations, Drinfeld introduced the "profinite" Grothendieck-Teichmüller group $\widehat{GT}$ concerned with the free profinite group of rank 2, which contains $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by a result of Belyi [Be].

The case that $\mathcal{M}_{g,n} = \mathcal{M}_{0,n}$ for any n was also studied by Drinfeld [Dr], and by Ihara and Matsumoto [IhM]. To give an affirmative answer to Oda's prediction stated in [O1] on the fields of definition of the Teichmüller modular towers with various genus, Ihara and Nakamura [IhN] constructed Schottky-Mumford uniformized universal deformations of maximally degenerate pointed curves consisting of smooth pointed projective lines, and they studied the Galois action on these algebraic fundamental groups. Using this result, Nakamura [N2], [N3] described the Galois action on $\widehat{\Pi}_{g,n}$ by the basic Galois action, and obtained some relations between the Galois action on $\widehat{\Pi}_{0,4}$, $\widehat{\Pi}_{0,5}$, $\widehat{\Pi}_{1,1}$ and $\widehat{\Pi}_{1,2}$. Furthermore, in the (partially joint) work of Lochak, Nakamura and Schneps (cf. [LNS], [NS]), using a relation presented in [N2] and a new relation for the basic Galois action, they introduced a remarkable subgroup $\mathbb{I}\Gamma$ of $\widehat{GT}$ which contains $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and they constructed systematic representations from $\mathbb{I}\Gamma$ into the automorphism group of $\widehat{\Pi}_{g,n}$ which is an extension of the Galois representation.

In this paper, we will apply a theory given in [I1] on Schottky-Mumford uniformized universal deformations of degenerate curves to studying Grothendieck's conjecture. First, using these deformations we construct a $(3g+n-3)$-dimensional real orbifold $\mathcal{L}$ contained in $\mathcal{M}_{g,n}(\mathbf{R})$ as a union of fusing moves and simple moves, which can be considered as an appropriate base set of the Teichmüller groupoid for $\mathcal{M}_{g,n}$. Second, we describe the Galois action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on the profinite Teichmüller groupoid for $\mathcal{M}_{g,n}$ in terms of the Galois action on the profinite Teichmüller groupoid for $\mathcal{M}_{0,4}$ and for the moduli stack $\mathcal{M}'_{1,1}$ which classifies proper and smooth 1-pointed curves of genus 1 with first-order infinitesimal structure. From the topological viewpoint, fusing moves, which are also called associativity moves or A-moves, represent different sewing procedures from two 3-holed spheres to one 4-holed sphere, and simple moves, which are also called S-moves, represent different sewing procedures from one 3-holed sphere to one 1-holed real surface of genus 1. By the results in [MS], [HLS] and [BK], the Teichmüller groupoid for the base set $\mathcal{L}$ has the fundamental generators associated with $\mathcal{M}_{0,4}$ and $\mathcal{M}'_{1,1}$; fusing moves, simple moves and Dehn (half-)twists which satisfy the relations (including the pentagon relation) induced from $\mathcal{M}_{0,5}$ and $\mathcal{M}_{1,2}$. Our construction of $\mathcal{L}$ enables us to approximate fusing moves and simple moves in $\mathcal{L}$ by those in $\mathcal{M}_{0,4}(\mathbf{R})$ and $\mathcal{M}'_{1,1}(\mathbf{R})$ respectively in the category of arithmetic geometry unifying complex and formal geometry. The intersection of $\mathcal{L}$ with an etale neighborhood U of each point at infinity becomes that of $\mathcal{M}_{g,n}(\mathbf{R})$ with U, and it gives 2^{2g+n-3} tangential base points (this notion is due to Deligne [D]). Using the approximation, we describe the Galois action on fusing and simple moves in $\mathcal{L}$ with respect to these tangential base points (the Galois action on Dehn (half-)twists are easily seen to be described in terms of the cyclotomic character).

We state our result in each section. In §1, generalizing the result in [IhN] for maximally degenerate pointed curves of restricted types, we construct Schottky-Mumford uniformized universal deformations of all degenerate pointed curves. In §2, we consider a uniformized deformation as a stable n-pointed curve C of genus g over

$$O_{\mathcal{M}_{0,4}}[[y_1,...,y_{3g+n-4}]] = \mathbf{Z}\left[x, \frac{1}{x}, \frac{1}{1-x}\right][[y_1,...,y_{3g+n-4}]],$$

and compare these moduli and deformation parameters x and y_i with the deformation parameters of the maximally degenerate pointed curves obtained as the degeneration of the closed fiber of C. The results in §1 and §2 are essential to construct the base set $\mathcal{L}$ globally and to describe the Galois actions on fusing moves in $\mathcal{L}$. In §3, using these deformations we construct $\mathcal{L}$ and tangential base points around any points at infinity which are connected by $\mathcal{L}$. In the genus 0 case, $\mathcal{L}$ becomes the base set in $\mathcal{M}_{0,n}(\mathbf{R})$ given in [IhM], and for any genus g, there is another approach by Schneps' school to the construction of such a base set in any $\mathcal{M}_{g,n}$ using geodesic lines. Our result in §4 describes the Galois action (not only on profinite Teichmüller modular groups but also) on profinite Teichmüller groupoids of all types.

Acknowledgments. I wish to thank Professor H. Nakamura since he kindly informed me about results of [LNS], [NS].

1 Uniformized deformations

1.1. We recall the well known correspondence between certain graphs and degenerate pointed curves, where a curve (resp. pointed curve) is called *degenerate* if it is a stable curve (resp. stable pointed curve) and the normalization of its irreducible components are all projective lines (resp. pointed projective lines). A *graph* $\Delta = (V, E, T)$ means a collection of 3 finite sets V of vertices, E of edges, T of tails such that each tail has one boundary (terminal) vertex and that each edge has one or two boundary vertices according to whether the edge is a loop or not. We consider only connected graphs, and a graph Δ is called *stable* if its each vertex has at least 3 branches. Denote by $\sharp X$ the number of elements of a finite set X. Under fixing a bijection $\nu : T \xrightarrow{\sim} \{1, ..., \sharp T\}$, which we call a numbering of T, a stable graph $\Delta = (V, E, T)$ becomes the dual graph of a degenerate $\sharp T$-pointed curve C of genus $\mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z})$ by the correspondences $V \leftrightarrow$ {irreducible components of C}, $E \leftrightarrow$ {singular points on C},

$T \leftrightarrow$ {marked points on C} such that an edge (resp. a tail) has a vertex as its boundary if the corresponding singular (resp. marked) point belongs to the corresponding component, and that each tail $h \in T$ corresponds to the $\nu(h)$-th marked point. If Δ is trivalent, i.e. all its vertices have just 3 branches, then the associated pointed curve is maximally degenerate.

An *orientation* of $\Delta = (V, E, T)$ means giving an orientation of each $e \in E$. Under an orientation of Δ, denote by $\pm E = \{e, -e \mid e \in E\}$ the set of oriented edges, by v_h the terminal vertex of $h \in \pm E \cup T$, and by $|h| \in E$ the edge h without orientation for each $h \in \pm E$. Let $\Delta = (V, E)$ be a stable graph without tail. Fix an orientation of Δ, and take a subset $\mathcal{E}$ of $\pm E$ whose complement $\mathcal{E}_\infty$ satisfies the condition that $\mathcal{E}_\infty \cap \{-h \mid h \in \mathcal{E}_\infty\} = \emptyset$, and that $v_h \neq v_{h'}$ for any distinct $h, h' \in \mathcal{E}_\infty$. We attach variables x_h for $h \in \mathcal{E}$ and $y_e = y_{-e}$ for $e \in E$. Let A_0 be the $\mathbf{Z}$-algebra generated by x_h $(h \in \mathcal{E})$, $1/(x_e - x_{-e})$ $(e, -e \in \mathcal{E})$ and $1/(x_h - x_{h'})$ $(h, h' \in \mathcal{E}$ with $h \neq h'$ and $v_h = v_{h'})$, and let $A = A_0[[y_e\ (e \in E)]]$. According to [I1], §2, we construct the universal Schottky group Γ associated with oriented Δ and $\mathcal{E}$ as follows. Put

$$B = A\left[\prod_{e \in E} \frac{1}{y_e}\right],$$

and for $h \in \pm E$, let ϕ_h be the element of $PGL_2(B)$ given by

$$\phi_h = \frac{1}{x_h - x_{-h}} \begin{pmatrix} x_h - x_{-h}y_h & -x_h x_{-h}(1 - y_h) \\ 1 - y_h & -x_{-h} + x_h y_h \end{pmatrix} \bmod(B^\times),$$

where x_h (resp. x_{-h}) means ∞ if h (resp. $-h$) belongs to $\mathcal{E}_\infty$. Denote by Ω the quotient field of A, and let $PGL_2(B)$ act on $\mathbf{P}^1(\Omega)$ by linear fractional transformation. Then ϕ_h satisfies that $\phi_{-h} = \phi_h^{-1}$ and that

$$\frac{\phi_h(z) - x_h}{z - x_h} = y_h \frac{\phi_h(z) - x_{-h}}{z - x_{-h}} \quad (z \in \mathbf{P}^1(\Omega)).$$

For any reduced path $\rho = h(1) \cdot h(2) \cdots h(l)$ which is the product of oriented edges $h(1), \ldots, h(l)$, we associate an element ρ^* of $PGL_2(B)$ having reduced expression $\phi_{h(l)}\phi_{h(l-1)} \cdots \phi_{h(1)}$. Fix a base point v_b on V, and consider the fundamental group $\pi_1(\Delta, v_b)$ which is a free group of rank $g = \mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z})$. Then the correspondence $\rho \mapsto \rho^*$ gives an injective anti-homomorphism $\pi_1(\Delta, v_b) \to PGL_2(B)$ whose image is denoted by Γ.

It is shown in [I1], §3 (and had been shown in [IhN], §2 when Δ is trivalent and has no loop) that for any stable graph $\Delta = (V, E)$ without tail, there exists a stable curve C_Δ of genus g over A which satisfies the following:

- The closed fiber $C_\Delta \otimes_A A_0$ of C_Δ obtained by substituting $y_e = 0$ $(e \in E)$ becomes the degenerate curve over A_0 with dual graph Δ which is obtained from the collection of $P_v := \mathbf{P}^1_{A_0}$ $(v \in V)$ by identifying the points $x_e \in P_{v_e}$ and $x_{-e} \in P_{v_{-e}}$ $(e \in E)$, where x_h denotes ∞ if $h \in \mathcal{E}_\infty$.

- C_Δ gives a universal deformation of degenerate curves with dual graph Δ, i.e. if R is a noetherian and normal complete local ring with residue field k, and C is a stable curve over R with nonsingular generic fiber such that the closed fiber $C \otimes_R k$ is a degenerate curve with dual graph Δ, in which all double points are k-rational, then there exists a ring homomorphism $A \to R$ giving rise to $C_\Delta \otimes_A R \cong C$.

- $C_\Delta \otimes_A B$ is smooth over B and is Mumford uniformized (cf. [Mu]) by Γ.

- Let α_h $(h \in \mathcal{E})$ be complex numbers such that $\alpha_e \neq \alpha_{-e}$ and that $\alpha_h \neq \alpha_{h'}$ if $h \neq h'$ and $v_h = v_{h'}$. Then for nonzero complex numbers β_e $(e \in E)$ with sufficiently small absolute value, by substituting $x_h = \alpha_h$ and $y_e = \beta_e$, C_Δ becomes a Riemann surface, i.e. a proper smooth curve over $\mathbf{C}$, which is Schottky uniformized (cf. [S]) by the Schottky group $\Gamma|_{x_h=\alpha_h, y_e=\beta_e}$ over $\mathbf{C}$.

1.2. We apply the above result to construct a uniformized deformation of a degenerate pointed curve which had been done by Ihara and Nakamura (cf. [IhN], §2, Theorems 1 and 1') when the degenerate pointed curve is maximally degenerate and consists of smooth pointed projective lines. To obtain explicit local coordinates on the moduli stack of stable pointed curves using the universal deformation, we will rigidify a coordinate on each projective line appearing as an irreducible component of the base degenerate curve. In the maximally degenerate case, this process is considered in [IhN] using the notion of "tangential structure".

Let $\Delta = (V, E, T)$ be a stable graph with numbering ν of T. We define the *extension* $\tilde{\Delta} = (\tilde{V}, \tilde{E})$ of Δ as a stable graph without tail by adding a vertex with a loop to the end different from v_h for each tail $h \in T$. A *rigidification* of an oriented stable graph $\Delta = (V, E, T)$ with numbering ν of T means a collection $\tau = (\tau_v)_{v \in V}$ of injective maps $\tau_v : \{0, 1, \infty\} \to \{h \in \pm E \cup T \mid v_h = v\}$ such that $\tau_v(a) \neq -\tau_{v'}(a)$ for any $a \in \{0, 1, \infty\}$ and distinct elements $v, v' \in V$ with $\tau_v(a), \tau_{v'}(a) \in \pm E$. One can see that any stable graph has a rigidification by the induction on the number of edges and tails, and that a rigidification τ of Δ can be extended to a rigidification $\tilde{\tau}$ of $\tilde{\Delta}$ for which one of $h, -h$ $(h \in T)$ belongs to $\tilde{\mathcal{E}}_\infty = \{\tilde{\tau}_v(\infty) \mid v \in \tilde{V}\}$. Assume that a rigidification τ of Δ and its extension $\tilde{\tau}$ to $\tilde{\Delta}$ as above are given. Then the subset $\tilde{\mathcal{E}}_\infty$ of $\pm\tilde{E}$ satisfies the condition in 1.1, and hence by the substitution $x_h = a$ for $h = \tau_v(a)$ $(a \in \{0, 1\})$, $C_{\tilde{\Delta}}$

gives rise to a stable curve $C_{(\widetilde{\Delta},\widetilde{\tau})}$. It is clear that the substitution $y_e = 0$ for all $e \in \widetilde{E} - E$ makes $C_{(\widetilde{\Delta},\widetilde{\tau})}$ a deformation by the parameters y_e $(e \in E)$ of the degenerate curve with dual graph $\widetilde{\Delta}$. Therefore, by replacing the self-intersecting projective lines which correspond to v_{-h} $(h \in T)$ with the $\nu(h)$-th marked points on the closed fiber of $C_{(\widetilde{\Delta},\widetilde{\tau})}$ obtained by $y_e = 0$ $(e \in \widetilde{E} - E)$, we have a stable $\sharp T$-pointed curve, which we denote by $C_{(\Delta,\tau)}$, as a deformation by y_e $(e \in E)$ of the degenerate $\sharp T$-pointed curve with dual graph Δ. It is seen that $C_{(\Delta,\tau)}$ can be characterized as a deformation of its closed fiber as is done in [IhN], 2.3.9, and that the deformation can be regarded as the quotient by $\pi_1(\Delta)$ of the glued scheme of pointed projective lines which is associated with the universal cover of Δ. By the property of $\widetilde{\tau}$, the base ring $A_{(\Delta,\tau)}$ of $C_{(\Delta,\tau)}$ becomes the formal power series ring of y_e $(e \in E)$ over the $\mathbf{Z}$-algebra which is generated by x_h $(h \in \mathcal{E})$, $1/(x_e - x_{-e})$ $(e, -e \in \mathcal{E} - T)$ and $1/(x_h - x_{h'})$ $(h, h' \in \mathcal{E}$ with $h \neq h'$ and $v_h = v_{h'})$, where $\mathcal{E} = \pm E \cup T - \{\tau_v(\infty) \mid v \in V\}$, $x_h = a$ for $h = \tau_v(a)$ $(a \in \{0, 1\})$ and x_h are variables for the other $h \in \mathcal{E}$.

Let τ be a rigidification of an oriented stable graph $\Delta = (V, E, T)$ with numbering of T, and put

$$\mathcal{E}_\tau \ = \ \pm E \cup T - \bigcup_{v \in V} \mathrm{Im}(\tau_v).$$

Then x_h $(h \in \mathcal{E}_\tau)$ and y_e $(e \in E)$ give effective parameters of the moduli and the deformation of degenerate $\sharp T$-pointed curves with dual graph Δ respectively. Therefore, $(x_h\ (h \in \mathcal{E}_\tau),\ y_e\ (e \in E))$ gives a system of formal coordinates on an etale neighborhood of Z_Δ, where Z_Δ denotes the moduli stack of $\overline{\mathcal{M}}_{g,n}$ classifying degenerate $\sharp T$-pointed curves with dual graph Δ. Furthermore, by the result mentioned in 1.1, this system gives local coordinates on an etale neighborhood of the complex orbifold $Z_\Delta(\mathbf{C})$. In particular, if Δ is trivalent, then for any rigidification τ of Δ, $\pm E \cup T = \bigcup_{v \in V} \mathrm{Im}(\tau_v)$, and hence $A_{(\Delta,\tau)}$ is the formal power series ring over $\mathbf{Z}$ of variables y_e $(e \in E)$.

2 Comparison of deformations

Let $\Delta = (V, E, T)$ be a stable graph with numbering of T such that only one vertex, which we denote by v_0, has 4 branches and that the other vertices have 3 branches. Fix an orientation of Δ, and denote by h_1, h_2, h_3 and h_4 the mutually different elements of $\pm E \cup T$ with terminal vertex v_0. Then one can take a rigidification $\tau = (\tau_v)_{v \in V}$ of Δ such that $\tau_{v_0}(0) = h_2$, $\tau_{v_0}(1) = h_3$, $\tau_{v_0}(\infty) = h_4$, and hence $x = x_{h_1}$ gives the coordinate on $\mathbf{P}^1_{\mathbf{Z}} - \{0, 1, \infty\}$. Denote by $C_{(\Delta,\tau)}$ the uniformized deformation given in 1.2 which is a stable $\sharp T$-pointed curve over

$$A_{(\Delta,\tau)} = \mathbf{Z}\left[x, \frac{1}{x}, \frac{1}{1-x}\right][[y_e\ (e \in E)]].$$

Let $\Delta' = (V', E', T')$ (resp. $\Delta'' = (V'', E'', T'')$) be the trivalent graph obtained by replacing v_0 with an edge e_0' (resp. e_0'') having two boundary vertices one of which is a boundary of h_1, h_2 (resp. h_1, h_3) and another is a boundary of h_3, h_4 (resp. h_2, h_4). Then one can identify T', T'' with T naturally, and it is easy to see that according as $x \to 0$ (resp. $x \to 1$), the degenerate $\sharp T$-pointed curve corresponding to x becomes the maximally degenerate $\sharp T$-pointed curve with dual graph Δ' (resp. Δ''). Let Δ' (resp. Δ'') without e_0' (resp. e_0'') have the orientation naturally induced from that of Δ, and let h_0' (resp. h_0'') be the edge e_0' (resp. e_0'') with orientation. For $1 \leq i \leq 4$, we denote by h_i' (resp. h_i'') the oriented edge in Δ' (resp. Δ'') corresponding to h_i, and identify the invariant part $E^{\mathrm{inv}} = E - \{|h_i|\ ;\ 1 \leq i \leq 4\}$ of E as that of E' and of E''. Then as seen in 1.2, for a rigidification τ' (resp. τ'') of Δ' (resp. Δ''), we have the uniformized deformation $C_{(\Delta',\tau')}$ (resp. $C_{(\Delta'',\tau'')}$) which is a stable $\sharp T$-pointed curve over

$$A_{(\Delta',\tau')} = \mathbf{Z}[[s_{e'}\ (e' \in E')]] \quad (\text{resp. } A_{(\Delta'',\tau'')} = \mathbf{Z}[[t_{e''}\ (e'' \in E'')]]).$$

Then we will consider two isomorphisms of $C_{(\Delta,\tau)}$ to $C_{(\Delta',\tau')}$ and to $C_{(\Delta'',\tau'')}$. By comparing deformation parameters depending only on the conjugate classes of the associated Schottky groups, we have the following:

Theorem 1. *Put $I = \{1 \leq i \leq 4 \mid h_i \in \pm E\}$, denote by y_i the deformation parameters associated with h_i for $i \in I$, and denote by s_j (resp. t_j) the deformation parameters associated with h_j' (resp. h_j'') for $j \in \{0\} \cup I$. Then*

(1) *Over $\mathbf{Z}((x))[[y_e\ (e \in E)]]$, $C_{(\Delta,\tau)}$ is isomorphic to $C_{(\Delta',\tau')}$, where under the isomorphism, the variables of the base rings $A_{(\Delta,\tau)}$ and $A_{(\Delta',\tau')}$ are related as*

$$\frac{x}{s_0},\ \frac{y_i}{s_0 s_i}\ (i \in \{1,2\} \cap I),\ \frac{y_i}{s_i}\ (i \in \{3,4\} \cap I),\ \frac{y_e}{s_e}\ (e \in E^{\mathrm{inv}})$$

belong to $(A_{(\Delta',\tau')})^\times$ if $|h_i|$ $(1 \leq i \leq 4)$ are mutually different, and

$$\frac{x}{s_0},\ \frac{y_i}{s_i}\ (i \in I),\ \frac{y_e}{s_e}\ (e \in E^{\mathrm{inv}})$$

belong to $(A_{(\Delta',\tau')})^\times$ if $|h_1| = |h_2|$.

(2) *Over $\mathbf{Z}((1-x))[[y_e\ (e \in E)]]$, $C_{(\Delta,\tau)}$ is isomorphic to $C_{(\Delta'',\tau'')}$, where under the isomorphism, the variables of the base rings $A_{(\Delta,\tau)}$ and $A_{(\Delta'',\tau'')}$ are related as*

$$\frac{1-x}{t_0},\ \frac{y_i}{t_0 t_i}\ (i \in \{1,3\} \cap I),\ \frac{y_i}{t_i}\ (i \in \{2,4\} \cap I),\ \frac{y_e}{t_e}\ (e \in E^{\mathrm{inv}})$$

belong to $(A_{(\Delta'',\tau'')})^\times$ *if* $|h_i|$ $(1 \leq i \leq 4)$ *are mutually different,*

$$\frac{1-x}{t_0},\ \frac{y_i}{t_0 t_i}\ (i \in \{1,2,3\} \cap I),\ \frac{y_i}{t_i}\ (i \in \{4\} \cap I),\ \frac{y_e}{t_e}\ (e \in E^{\mathrm{inv}})$$

belong to $(A_{(\Delta'',\tau'')})^\times$ *if* $|h_1| = |h_2|$, $|h_3| \neq |h_4|$, *and*

$$\frac{1-x}{t_0},\ \frac{y_i}{t_0 t_i}\ (i \in I),\ \frac{y_e}{t_e}\ (e \in E^{\mathrm{inv}})$$

belong to $(A_{(\Delta'',\tau'')})^\times$ *if* $|h_1| = |h_2|$, $|h_3| = |h_4|$.

Remark 1. The constant terms of the above ratios in $(A_{(\Delta',\tau')})^\times$, $(A_{(\Delta'',\tau'')})^\times$ are clearly ± 1, and these signs can be easily determined from the data of rigidifications. If $|h_1| = |h_2|$ in (2), then $y_1/(t_0 t_1)$ belongs to $\left\{(A_{(\Delta'',\tau'')})^\times\right\}^2$ and hence this constant term is 1 because by Proposition 1.3 in [I1], the reduced element $\phi_{h(l)}\phi_{h(l-1)} \cdots \phi_{h(1)}$ has the multiplier in $\prod_{i=1}^{l} y_{h(i)} \cdot (A^\times)^2$ if $h(1) \neq -h(l)$.

Remark 2. From the result in 1.1, one can see that the assertion in (1) (resp. (2)) holds in the category of complex geometry when x, y_e and $s_{e'}$ (resp. $1-x$, y_e and $t_{e''}$) are sufficiently small.

3 Teichmüller groupoids

Provided that $2g + n - 2 > 0$, denote by $\overline{\mathcal{M}}_{g,n}$ the moduli stack over $\mathbf{Z}$ classifying stable n-pointed curves of genus g, and denote by $\mathcal{M}_{g,n}$ its open substack classifying smooth n-pointed curves (cf. [DM], [K]). In this section, using the result in §2 we construct an appropriate base set of the Teichmüller groupoid for $\mathcal{M}_{g,n}$ as a union of fusing moves and simple moves.

3.1. First, we construct fusing moves as tubular neighborhoods in $\mathcal{M}_{g,n}(\mathbf{R})$ of real 1-dimensional paths on the 1-dimensional locus in $\overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ which corresponds to degenerate pointed curves with dual graphs considered in §2. Let $\Delta = (V, E, T)$ be a stable graph considered in §2 such that $\mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z}) = g$, $\sharp T = n$, and take a numbering of T and a rigidification τ of the graph Δ with orientation. In particular, V has only one vertex v_0 with 4 branches and the other vertices have 3 branches. Then as is described in 1.2, one can construct a uniformized deformation $C_{(\Delta,\tau)}$ which is a stable n-pointed curve of genus g over $A_{(\Delta,\tau)} = \mathbf{Z}\left[x, 1/x, 1/(1-x)\right]\left[\left[y_e\ (e \in E)\right]\right]$, where x is the coordinate corresponding to only one element of $(\pm E \cup T) - \mathrm{Im}(\tau_{v_0})$ with terminal vertex v_0.

Here we treat the case that all branches starting from v_0 are non-loop edges, and let h_1, $h_2 = \tau_{v_0}(0)$, $h_3 = \tau_{v_0}(1)$ and $h_4 = \tau_{v_0}(\infty)$ denote the oriented edges with terminal vertex v_0. We put $u_0 = x$ and attach variables u_i $(1 \leq i \leq G := 3g+n-4)$ to edges of Δ such that

$$u_i = \begin{cases} y_{h_1}/x(1-x) & \text{or} \quad -y_{h_1}/x(1-x) \quad (i=1) \\ y_{h_2}/x & \text{or} \quad -y_{h_2}/x \quad (i=2) \\ y_{h_3}/(1-x) & \text{or} \quad -y_{h_3}/(1-x) \quad (i=3) \\ y_{h_4} & \text{or} \quad -y_{h_4} \quad (i=4) \end{cases}$$

and that u_i $(i \geq 5)$ are obtained by specifying one of y_e and $-y_e$ for each $e \in E^{\mathrm{inv}}$. Moreover, we assume that for any closed path $\rho = h(1) \cdot h(2) \cdots h(l)$ in Δ, the product of the signs of these variables attached to $h(j)$ $(1 \leq j \leq l)$ is $+1$ under regarding the sign of $y_{h_1}/x(1-x)$, y_{h_2}/x, $y_{h_3}/(1-x)$, y_{h_4}, $y_e(e \in E^{\mathrm{inv}})$ as $+1$ (there are 2^{2g+n-4} ways of choosing such variables). Then by Theorem 1, under $u_0 \to 0$ (resp. 1), the variables u_0 (resp. $1-u_0$) and $(u_i)_{1 \leq i \leq G}$ are deformation parameters over $\mathbf{Z}$ of the maximally degenerate n-pointed curve C_0' (resp. C_0'') with dual graph Δ' (resp. Δ'') given in §2, and hence these variables give a basis of the tangent space over $\mathbf{Z}$ at the point P' (resp. P'') of $\overline{\mathcal{M}}_{g,n}$ corresponding to C_0' (resp. C_0''). In the case that there are tails or loops with boundary vertex v_0, one can take appropriate variables satisfying this property by Theorem 1, and it is easy to see that the following argument can be applicable similarly.

Denote by (a, b) the open interval between two real numbers a, b with $a < b$. Then for each choice of these variables u_i, by the result in 1.1, there exists a (sufficiently small) positive real number ε such that $C_{(\Delta,\tau)}$ becomes a proper and smooth n-pointed curve over $\mathbf{R}$ for any $u_0 \in (0,1)$ and $u_i \in (0, \varepsilon)$ $(1 \leq i \leq G)$. Hence one can define a fusing move $f(\varepsilon)$ as the sublocus of $\mathcal{M}_{g,n}(\mathbf{R})$ induced from $C_{(\Delta,\tau)}$ with $u_0 \in (0,1)$ and $u_i \in (0,\varepsilon)$ $(i \geq 1)$.

3.2. Second, we construct simple moves as tubular neighborhoods in $\mathcal{M}_{g,n}(\mathbf{R})$ of real 1-dimensional paths on the 1-dimensional locus in $\overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$ which corresponds to stable pointed curves obtained by attaching elliptic curves to one of the marked points on maximally degenerate pointed curves. We consider a trivalent graph $\Delta = (V, E, T)$ such that $\mathrm{rank}_{\mathbf{Z}} H_1(\Delta, \mathbf{Z}) = g$, $\sharp T = n$, and take a numbering of T and a rigidification τ of the graph Δ with orientation. Assume that there is a loop in E, and denote this by e_0. Then as is described in 1.2, one can construct a uniformized deformation $C_{(\Delta,\tau)}$ of the maximally degenerate n-pointed curve C_0 with dual graph Δ, which is a stable n-pointed curve of genus g over $A_{(\Delta,\tau)} = \mathbf{Z}\left[\left[y_e \ (e \in E)\right]\right]$. We put $u_0 = y_{e_0}$ and attach variables u_i $(1 \leq i \leq G)$ to elements of $E - \{e_0\}$ which are obtained by specifying one of y_e and $-y_e$ such that for any closed path $\rho = h(1) \cdot h(2) \cdots h(l)$ in Δ, the product of the signs of these variables attached to $h(j)$ $(1 \leq j \leq l)$ is $+1$ under regarding the sign of each $y_{h(j)}$ as $+1$ (there are 2^{2g+n-3} ways of choosing such variables).

Fix a positive real number r such that $\exp(4\pi^2/\log(r)) < r < 1$. For each choice of these variables u_i, by the result in 1.1, there exists $\varepsilon > 0$ such that $C_{(\Delta,\tau)}$ becomes a proper and smooth n-pointed curves over $\mathbf{R}$ for any $u_0 \in (0, r)$ and $u_i \in (0, \varepsilon)$ $(1 \leq i \leq G)$. Hence we have an etale morphism from

$$\{(u_0, u_1, ..., u_G) \mid 0 < u_0 < r,\ 0 < u_i < \varepsilon\ (i \geq 1)\}$$

into $\mathcal{M}_{g,n}(\mathbf{R})$, and denote this image by $s_1(\varepsilon)$. Similarly, we define $s_2(\varepsilon)$ as the image of

$$\left\{(u_0, u_1, ..., u_G) \;\middle|\; \exp(4\pi^2/\log(r)) < u_0 < 1,\ 0 < u_i < \varepsilon\ (i \geq 1)\right\}$$

by the composite of the morphism obtained by replacing u_0 with u_0', and the transformation $u_0 \mapsto u_0' = \exp\left(4\pi^2/\log(u_0)\right)$ which corresponds to the transformation $\tau \mapsto -1/\tau$ ($\tau \in$ the Poincaré upper half-plane) of periods of elliptic curves over $\mathbf{C}$ because u_0 can be regarded as multiplicative periods given by $\exp(2\pi\sqrt{-1}\tau)$. Then we define a simple move $s(\varepsilon)$ as the union $s_1(\varepsilon) \cup s_2(\varepsilon)$.

3.3. From 3.1 and 3.2, we have the subloci $f(\varepsilon)$ and $s(\varepsilon)$ of $\mathcal{M}_{g,n}(\mathbf{R})$ associated with each fusing move and simple move respectively. Then by Remarks 1, 2 of Theorem 1, moving graphs Δ, rigidifications τ and variables u_i, and taking the union of all $f(\varepsilon)$ and $s(\varepsilon)$ for sufficiently small $\varepsilon > 0$, we obtain a sublocus $\mathcal{L}$ of $\mathcal{M}_{g,n}(\mathbf{R})$ which consists of fusing moves and simple moves. Since the above variables u_i $(0 \leq i \leq G)$ give systems of formal coordinates over $\mathbf{Z}$ and local complex coordinates on etale neighborhoods of points at infinity, $\mathcal{L}$ becomes $\mathcal{M}_{g,n}(\mathbf{R})$ on the neighborhoods, and defines tangential base points. Furthermore, a 4-pointed projective line degenerates maximally in three ways and these degenerations are connected by fusing moves. From this and the consideration in 3.1 and 3.2, we have the following:

Theorem 2. *$\mathcal{L}$ becomes a real orbifold of dimension $3g + n - 3$, and for an etale neighborhood $\pi : U \to \overline{\mathcal{M}}_{g,n}(\mathbf{C})$ of each point at infinity corresponding to a maximally degenerate pointed curve, $\pi^{-1}(\mathcal{L}) \cap U = \pi^{-1}(\mathcal{M}_{g,n}(\mathbf{R})) \cap U$ decomposes into 2^{2g+n-3} simply connected pieces, which are regarded as tangential base points of the point at infinity.*

As the completeness theorem in [MS] (see [HLS], [BK] for its accurate formulation and proof), it is shown that the topological Teichmüller groupoid with base set $\mathcal{L}$ has the following generators of four types associated with $\mathcal{M}_{0,4}$ and $\mathcal{M}_{1,1}'$:

fusing moves, simple moves and Dehn twists attached to loops, Dehn half-twists attached to non-loop edges which are in the dual graphs of maximally degenerate n-pointed curves of genus g. Furthermore, it is also shown that these generators satisfy the relations induced from $\mathcal{M}_{0,5}$ and $\mathcal{M}_{1,2}$.

4 Galois action

4.1. We consider the Galois action on generators of the Teichmüller groupoid for $\mathcal{M}_{g,n}$. In this subsection, we treat fusing moves, and hence let the notation be as in §2 and 3.1. Furthermore, assume that all branches starting from v_0 are non-loop edges. Even in the case that there are tails or loops with boundary vertex v_0, using Theorem 1 one can take appropriate variables u_i to construct a fusing move, and it is easy to see that the following argument can be applicable similarly.

Let q be a path in $f(\varepsilon)$ obtained by moving the variable $u_0 \in (0,1)$. Then for each element σ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, $\sigma(q) \cdot q^{-1}$ can be regarded as an element of the algebraic fundamental group of $\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ with respect to the tangential base point from $u_0 = 0$ to $u_0 = 1$ in q. Hence by a result of Oda [O2], we have $\sigma(q) \cdot q^{-1} \in \widehat{\Pi}_{g,n}$, where $\widehat{\Pi}_{g,n}$ denotes the profinite completion of the fundamental group $\Pi_{g,n}$ of $\mathcal{M}_{g,n}(\mathbf{C})$ with respect to this tangential base point. We prepare some notations (cf. [Ih1], [Ih2]) to describe $\sigma(q) \cdot q^{-1}$. Let Π be the fundamental group of $\mathcal{M}_{0,4}(\mathbf{C}) = \mathbf{P}^1(\mathbf{C}) - \{0, 1, \infty\}$ with tangential base point $\overrightarrow{01}$ which becomes a free group with generators α, β defined by positive simple loops going counterclockwise around $0, 1$ respectively. Let p be an element of the fundamental groupoid of $\mathcal{M}_{0,4}(\mathbf{C})$ with tangential base point set $\{\overrightarrow{01}, \overrightarrow{10}\}$ which is defined by the open interval $(0,1)$ with orientation from 0 to 1. Then $f_\sigma = \sigma(p) \cdot p^{-1}$ belongs to the algebraic fundamental group of $\mathcal{M}_{0,4} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ with tangential base point $\overrightarrow{01}$, and hence f_σ becomes an element $f_\sigma(\alpha, \beta)$ of the free profinite group generated by α, β. Then using the above description of fusing moves and the method of Puiseux series expansions by Anderson and Ihara (cf. [AIh], [Ih1], [Ih2]), we have:

Theorem 3. *Let the notation be as above. Then we have*

$$\sigma(q) \cdot q^{-1} \;=\; f_\sigma(\delta_1, \delta_2),$$

where δ_1 and δ_2 are the Dehn twists in $\Pi_{g,n}$ represented by α and β on the u_0-projective line respectively, and $f_\sigma(\delta_1, \delta_2)$ denotes the element of $\widehat{\Pi}_{g,n}$ obtained from $f_\sigma(\alpha, \beta)$ by the substitution $\alpha = \delta_1$, $\beta = \delta_2$.

4.2. In this subsection, we treat simple moves, and hence let the notation be as in 3.2. In particular, the trivalent graph Δ has a loop which we denote by e_0. Let s be a path in $s(\varepsilon)$ obtained by moving the variable $u_0 \in (0,1)$. Then for each $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, $\sigma(s)$ belongs to the algebraic fundamental group of $\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}$ with respect to the tangential base point ξ in s around $u_0 = u_1 = \cdots = u_G = 0$, and hence $\sigma(s) \in \pi_1\left(\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}, \xi\right) = \widehat{\Pi}_{g,n}$. Let $\mathcal{M}'_{1,1}$ denote the moduli stack classifying proper and smooth 1-pointed curves of genus 1 with first-order infinitesimal structure, and let $\Pi'_{1,1}$ denote the topological fundamental group of $\mathcal{M}'_{1,1}(\mathbf{C})$. Then we have a natural exact sequence

$$1 \longrightarrow \mathbf{Z} \longrightarrow \Pi'_{1,1} \longrightarrow \Pi_{1,1} \cong SL_2(\mathbf{Z}) \longrightarrow 1,$$

where the generator 1 of $\mathbf{Z}$ is sent to the Dehn twist around the marked point with first-order infinitesimal structure. Furthermore, regarding the loop e_0 with 1-tail as the dual graph of the degenerate 1-pointed curve of genus 1, we have a group homomorphism $\Pi'_{1,1} \to \Pi_{g,n}$ which gives a homomorphism $\rho : \widehat{\Pi}'_{1,1} \to \widehat{\Pi}_{g,n}$ of profinite groups. Let r be a path in $\mathcal{M}'_{1,1}(\mathbf{R})$ defined by the same way as for s, where u_0 denotes the Tate parameter and u_1 denotes the first-order infinitesimal structure on the marked point. Then $\rho(r) = s$ and $\sigma(r) \in \pi_1\left(\mathcal{M}'_{1,1} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}, \zeta\right) = \widehat{\Pi}'_{1,1}$, where ζ is the tangential base point on $\mathcal{M}'_{1,1}$ around $u_0 = u_1 = 0$. Then using the theory of tame fundamental groups (cf. [GM]), we have the following result which seems to be substantially obtained in [N1-3], however cannot be found in this form:

Theorem 4. *Let the notation be as above. Then we have* $\sigma(s) = \rho\left(\sigma(r)\right)$.

4.3. Finally, we mention the Galois action on the other generators of the Teichmüller groupoid for $\mathcal{M}_{g,n}$. Let δ be a Dehn twist or a Dehn half-twist. Then from the argument in 4.1, it is easy to show that for any $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, we have $\sigma(\delta) = \delta^{\chi(\sigma)}$, where $\chi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \widehat{\mathbf{Z}}^\times$ denotes the cyclotomic character.

References

[AIh] *G. Anderson* and *Y. Ihara*, Pro-l branched coverings of $\mathbf{P}^1$ and higher circular l-units, Ann. of Math. **128** (1988), 271-293; Part 2, Internat. J. Math. **1** (1990), 119-148.

[BK] *B. Bakalov* and *A. Kirillov,* On the Lego-Teichmüller game, mathGT/9809057, 33 pages.

[Be] *G. V. Belyi,* On Galois extensions of a maximal cyclotomic field, Math. USSR Izv. **14** (1980), 247-256.

[D] *P. Deligne,* Le groupe fondamental de la droite projective moins trois points, in: Y. Ihara, K. Ribet and J. P. Serre, (eds.), Galois groups over **Q**, Publ. MSRI 16, Springer-Verlag, 1989, pp. 79-298.

[DM] *P. Deligne* and *D. Mumford,* The irreducibility of the space of curves of given genus, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75-109.

[Dr] *V. G. Drinfeld,* On quasi-triangular quasi-Hopf algebras and a group closely connected with Gal($\overline{\mathbf{Q}}/\mathbf{Q}$), Algebra i Analiz **2** (1990), 114-148; English transl. Leningrad Math. J. **2** (1991), 829-860.

[G] *A. Grothendieck,* Esquisse d'un programme, Mimeographed Note (1984), in: P. Lochak and L. Schneps, (eds.), Geometric Galois actions I, London Math. Soc. Lect. Note Ser. 242, Cambridge Univ. Press, 1997, pp. 5-48.

[GM] *A. Grothendieck* and *J. P. Murre,* The tame fundamental group of a formal neighborhood of a divisor with normal crossings on a scheme, Lect. Notes Math. 208, Springer-Verlag, 1971.

[HLS] *A. Hatcher, P. Lochak* and *L. Schneps,* On the Teichmüller tower of mapping class groups, J. Reine Angew. Math. **521** (2000), 1-24.

[I1] *T. Ichikawa,* Generalized Tate curve and integral Teichmüller modular forms, Amer. J. Math. **122** (2000), 1139-1174.

[I2] *T. Ichikawa,* Teichmüller groupoids and Galois action, Preprint.

[Ih1] *Y. Ihara,* Braids, Galois groups, and some arithmetic functions, in: Proc. ICM, Kyoto 1990, vol. 1, Springer-Verlag, 1991, pp. 99-120.

[Ih2] *Y. Ihara,* On the embedding of Gal($\overline{\mathbf{Q}}/\mathbf{Q}$) into $\widehat{GT}$, in: L. Schneps, (ed.), The Grothendieck theory of dessin's d'enfants, London Math. Soc. Lect. Note Ser. 200, Cambridge Univ. Press, 1994, pp. 289-305.

[IhM] *Y. Ihara* and *M. Matsumoto,* On Galois actions on profinite completions of braid groups, in: M. Fried et al., (eds.), Recent developments in the inverse Galois problem, Contemp. Math. 186, Amer. Math. Soc., 1995, pp. 173-200.

[IhN] *Y. Ihara* and *H. Nakamura,* On deformation of maximally degenerate stable marked curves and Oda's problem, J. Reine Angew. Math. **487** (1997), 125-151.

[K] *F. Knudsen,* The projectivity of the moduli space of stable curves II, Math. Scand. **52** (1983), 161-199.

[LNS] *P. Lochak, H. Nakamura* and *L. Schneps,* On a new version of the Grothendieck-Teichmüller group, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), 11-16.

[MS] *G. Moore* and *N. Seiberg,* Classical and quantum conformal field theory, Commun. Math. Phys. **123** (1989), 177-254.

[Mu] *D. Mumford,* An analytic construction of degenerating curves over complete local rings, Compos. Math. **24** (1972), 129-174.

[N1] *H. Nakamura,* Coupling of universal monodromy representations of Galois-Teichmüller modular groups, Math. Ann. **304** (1996), 99-119.

[N2] *H. Nakamura,* Galois representations in the profinite Teichmüller modular groups, in: P. Lochak and L. Schneps, (eds.), Geometric Galois actions I, London Math. Soc. Lect. Note Ser. 242, Cambridge Univ. Press, 1997, pp. 159-173.

[N3] *H. Nakamura,* Limits of Galois representations in fundamental groups along maximal degeneration of marked curves, I, Amer. J. Math. **121** (1999), 315-358; II, to appear.

[NS] *H. Nakamura* and *L. Schneps,* On a subgroup of Grothendieck-Teichmüller group acting on the tower of profinite Teichmüller modular groups, Invent. Math. **141** (2000), 503-560.

[O1] *Takayuki Oda,* The universal monodromy representations on the pro-nilpotent fundamental groups of algebraic curves, Math. Arbeitstagung (Neue Serie) 9-15 Juni 1993, Max Planck Institut preprint MPI/93-57.

[O2] *Takayuki Oda,* Etale homotopy type of the moduli spaces of algebraic curves, in: P. Lochak and L. Schneps, (eds.), Geometric Galois actions I, London Math. Soc. Lect. Note Ser. 242, Cambridge Univ. Press, 1997, pp. 85-95.

[S] *F. Schottky,* Über eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Arguments univerändert bleibt, J. Reine Angew. Math. **101** (1887), 227-272.