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On cellular automata

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The polynomial representation of the parallel map makes us easy to discuss linear cellular automata, a subclass of cellular automata, and many results on them have been obtained .

In this report, we make a list of some problems and conjectures on cellular automata from point of linear cellular automata. For simplicity, we state the notations and the definitions needed in the following in the case of one dimensional cell space.

Let Z denote the set of all integers, called the set of cells. Let R denote a finite set (a state set at each cell). A map from Z to R is called a configuration, the set of all configurations is denoted by R^Z. A map f from R^n to R is called a local map with scope n. By acting the local map to each cell simultaneously, the parallel map f_∞ from R^Z to itself is defined as follows.

$$f_\infty(x)=y \quad \Leftrightarrow \quad f(x(i),x(i+1),\cdots,x(i+n-1))$$

In particular, when the local map f is linear, that is, R is a ring and f is written as $f=\sum a_j x_j \quad (a_j \in R)$, the parallel map f_∞ is defined as follows.

$$f_\infty(x)=y \quad \Leftrightarrow \quad y(i)=\sum_{j=1}^{n} a_j x(i+j-1)$$

Now define a shift map $\sigma : R^Z \to R^Z$ as follows. $\sigma(x)=y \quad \Leftrightarrow \quad y(i)=x(i+1)$

By using the shift map, we can write the linear parallel map as follows.

$$f_\infty=\sum_{j=1}^{n} a_j \sigma^{j-1}$$

Then replacing σ by an indeterminate X, we obtain a polynomial representation.

Therefore the following correspondence is considered.

$$f_\infty=\sum_{j=1}^{n} a_j \sigma^{j-1} \quad \leftrightarrow \quad F(X)=\sum_{j=1}^{n} a_j X^{j-1}$$

For $x \in R^Z$ and $f(x) \in R^z$, put $C(X) = \sum_{i \in Z} x(i) X^{-i}$ and $C'(X) = \sum_{i \in Z} f_\infty(x)(i) X^{-i}$.

Then we have $C'(X) = F(X)C(X)$. From this, we see that if $G(X)$ is a polynomial representation of g_∞, that is,

$$g_\infty = \sum b_j \sigma^{j-1} \quad \leftrightarrow \quad G(X) = \sum b_j X^{j-1}$$

then we have $f_\infty g_\infty \leftrightarrow F(X)G(X)$.

Thus, the polynomial representation of the parallel map for linear cellular automata makes us easy to investigate the properties of maps such as injectivity ,surjectivity and dynamical behavior of them.

Problem 1. Find a mathematical method to characterize the general cellular automata with two or more higher dimension.

Definition 1. Let f be a local map over R.

f is injective on R $\Leftrightarrow$ f_∞ is injective on R^Z.

f is surjective on R $\Leftrightarrow$ f_∞ is surjective on R^Z.

f is k to one map on R $\Leftrightarrow$ f_∞ is k to one map on R^Z.

f has a finite order $\Leftrightarrow$ $\exists k, n, f_\infty^n = \sigma^k$.

For linear cellular automata, the number of local maps in each class defined above is given as a function of $|R|$ and scope n.

Problem 2. For general cellular automata , find the each function of $|R|$ and scope n which denotes the number of the desired local maps.

It is well known that for linear cellular automata with two or more higher dimension, there is no k to 1 parallel map except injective map.

Problem 3. For general cellular automata with two or more higher dimension, does k to one parallel map always take injective one ?

For one dimensional linear cellular automata, it is well known that surjective parallel map and k to one map are equivalent, and k takes the value in the set of divisors of $|R|^{n-1}$.

Problem 4. For k to one parallel map of one dimensional general cellular automata, does k take the value in the set of divisors of $| R |^{n-1}$?

Problem 5. Is it decidable that a local map f has a finite order?
(For linear cellular automata, it is decidable.)

R^Z is also viewed as a probability space. Let $x \in R^Z$. Consider a sequence of configurations such as x , $f_\infty(x), f_\infty(x)^2, \cdots, f_\infty(x)^n, \cdots$.
A sufficient condition for a parallel map to be ergodic is already known.

Problem 6. Is it decidable whether a parallel map f_∞ is ergodic or not?
(For linear cellular automata, it is decidable.)

<u>Definition2.</u> Let $f = \sum a_j x_j$ be a local map over Z_m. We say that f has a group structure if the following statement holds.
The power set of f denoted by $< f >$ forms a group under the operation $*$ defined below and any local map belongs to $< f >$ is injective, where $< f > = \{ f^n | n \in Z \}$, $f^2 = f * f = \sum a_j^{\,2} x_j$

Therefore parallel maps are classified in the following.

$$\text{parallel map} \begin{cases} \textit{surjectivity} \begin{cases} \textit{injectivity} \begin{cases} \textit{group}-\textit{structure} \\ \textit{non}-\textit{group}-\textit{structure} \end{cases} \\ \textit{non}-\textit{injectivity} \end{cases} \\ \textit{non}-\textit{surjectivity} \end{cases}$$

Fig.1 Classification of parallel maps

The group structure appears in the process of finding its reversible linear cellular automaton and the problem of finding its inverse automaton can be reduced to that of finding its inverse element of the group. Furthermore it is known that such groups regardless of their scopes are isomorphic to each other.

Problem 7. What is the algabraic properties of cellular automata with group structure?

The parallel maps of cellular automata has the following relation as shown in Fig.2.

$$\begin{array}{ccc} f_\infty \text{ is injective} & \Leftrightarrow & f_\infty \text{ is reversible} \\ \Downarrow & & \\ f_\infty \text{ is surjective} & & \end{array}$$

Fig. 2 Richardson's relation

This relation holds only when the state set of each cell is finite. And if we put a infinite set as the state one, then above relation does not always hold even for linear cellular automata. Thus it arises a problem that what conditions for the ring are needed in order to always hold Richardson's relation.

<u>Definition 3</u> Let R be a ring. When the parallel map f_∞ defined by a local map f over R is always satisfied the following condition, we say R a cellular ring.
And if the parallel map for linear cellular automata over R is always satisfied, then We say R a linear cellular ring. Clearly the former is included in latter by definition.

$$f_\infty \text{ is injective} \Leftrightarrow f_\infty \text{ is reversible}$$

The examples of linear cellular ring as known by now are found for 0-dimensional rings only.

Problem 8. Is it valid that a commutative ring R is a linear cellular ring if and only if dim $R = 0$?

Problem 9. Decide whether a cellular ring is equivalent to a linear cellular one or not.

Problem 10. Can you see the outline of general cellular automata from the properties of linear cellular automata over Z_m ?

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