数理解析研究所講究録 1269 巻 2002 年 48-62

Local structure of Fatou mappings at an indeterminate point with homoclinic points

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Abstract

In this notes, we show that some Fatou mapping has an indeterminate point with homoclinic points. In particular, using the structure of horseshoe mappings, we show that if the homoclinic point satisfies the transversality condition, then periodic points of Fatou mapping accumulate at its indeterminate point.

1 Introduction

In this notes, we focus our study on a mapping as follows:

$$F : [t : x : y] \mapsto [aty + xy - by^2 : atx + x^2 - bxy + cy^2 : y^2],\ a \neq 0$$

which is a birational mapping of the 2-dimensional complex projective space $\mathbf{P}^2$. A rational mapping F of $\mathbf{P}^2$ is said to be a *birational mapping* if there exists another rational map G of $\mathbf{P}^2$ such that $F \circ G = id$, $G \circ F = id$, the identity mapping, on $\mathbf{P}^2$ except some algebraic sets, and G is called the inverse mapping. For our F, the inverse mapping G has the following form:

$$G : [t : x : y] \mapsto [t^2 + bty + cy^2 - xy : axy - acy^2 : aty],\ a \neq 0.$$

Here, we remark that F is conjugate to the mapping originally used by P. Fatou to exhibit a Fatou-Bierbach domain (see [1]). Therefore, we call it a *Fatou mapping* in this paper.

In order to state our Main Theorem, we introduce some notation and terminology. Let $f_i(t, x, y)$ $(i = 0, 1, 2)$ be homogeneous polynomials with degree d, $F : [t : x : y] \mapsto [f_0 : f_1 : f_2]$ a rational mapping on $\mathbf{P}^2$ and $\tilde{F} : (t, x, y) \mapsto (f_0, f_1, f_2)$ a polynomial mapping on $\mathbf{C}^3$. Then, we have $\pi \circ \tilde{F} = F \circ \pi$ on $\mathbf{C}^3$ except some analytic sets, where $\pi : \mathbf{C}^3 \setminus \{(0,0,0)\} \to \mathbf{P}^2$ is the canonical projection. A point $p \in \mathbf{P}^2$ is said to be an *indeterminate point* of F if $\tilde{F}(\tilde{p}) = (0,0,0)$ for some point $\tilde{p} \in \pi^{-1}(p)$. In general, if p is an indeterminate point, then $\bigcap_{N_p} F(N_p \setminus \{p\})$ is not a single point, where the intersection is taken over all open neighborhoods N_p of p. Hence, F is not continuous and the dynamical structure is quite complicated at such a point p. In our case, by a direct calculation, one can check that indeterminate points of the Fatou mapping F above are $I_1 = [1:0:0]$ and $I_2 = [1:-a:0]$ and G is continuous at I_1 and I_2. In particular, I_1 is a fixed point of G and eigenvalues of Jacobian matrix of G at I_1 are 0 and a. So, to see the dynamical structure near the indeterminate point I_1, it suffices to consider the behavior of G^n near the fixed point I_1. We assume that $|a| > 1$. Then I_1 is a saddle fixed point of G. By [4, Theorem 6.4.3] and a direct calculation, there exists some injective holomorphic mapping $H : \Delta_\rho = \{z \in \mathbf{C} \mid |z| < \rho\} \to \mathbf{C}^2(x, y)$ such that

$$H(0) = I_1, \quad W^u(I_1) \supset H(\Delta_\rho) \text{ and } W^s(I_1) \supset \left\{(x,y) \in \mathbf{C}^2 \,\middle|\, y = 0\right\},$$

where

$$W^s_{loc}(I_1) = \left\{q \in U \,\middle|\, G^n(q) \to I_1\right\}, \quad W^u_{loc}(I_1) = \left\{q \in U \,\middle|\, F^n(q) \to I_1\right\} \cup \{I_1\},$$

$$W^s(I_1) = \bigcup_{n \geq 0}^{\infty} G^{-n}(W^s_{loc}(I_1)) \text{ and } W^u(I_1) = \bigcup_{n \geq 0}^{\infty} G^n(W^u_{loc}(I_1))$$

are called a *local stable set, a local unstable set*, the *stable set* and the *unstable set* of I_1 for some open neighborhood U of I_1, respectively. If $W^s(I_1)$ and $W^u(I_1)$ intersect at some point other than I_1, the point is said to be a *homoclinic point*. Moreover, $q \in W^s(I_1) \cap W^u(I_1) \setminus \{I_1\}$ is said to be a *transverse homoclinic point* if $T_q(\mathbf{C}^2)$ is the direct sum of $T_qW^s(I_1)$ and $T_qW^u(I_1)$: $T_q(\mathbf{C}^2) = T_qW^s(I_1) \oplus T_qW^u(I_1)$. In general, if G is a diffeomorphism of class C^r on a differentiable manifold which has a saddle fixed point p with a transverse homoclinic point, then G^k satisfies the *horseshoe condition* for some positive integer k, and the dynamical structure near p is described by symbol dynamics (see [7]). We remark that our G is not a local diffeomorphism at I_1. However, observing the orbits of critical points of G, we have the following results:

Main Theorem. *Suppose that $|a| > 1$, $b \neq c$ and $[0:c:1] \notin W^u(I_1)$. Then, we have the following:*

(1) *I_2 is a homoclinic point of I_1.*

(2) *Moreover, suppose that I_2 is a transverse homoclinic point. Then there exist an integer $k > 0$, a set $X \subset \mathbf{P}^2$ and some homeomorphism $\Psi : X \to \{0,1\}^{\mathbf{Z}} \setminus E$ such that*

X is invariant by G and $\sigma \circ \Psi = \Psi \circ G^k$ on X, where

$$E = \left\{(\cdots, s_{-1}, s_0, s_1, \cdots) \in \{0,1\}^{\mathbf{Z}} \;\middle|\; (\cdots, s_{n_0-1}, s_{n_0}, 0, 0, \cdots)\right\}$$

and σ is the shift map on $\{0,1\}^{\mathbf{Z}} \setminus E$.
In particular, periodic points of Fatou mapping accumulate at its indeterminate point I_1.

2 Fundamental Properties of Fatou mappings

In this section, we state some properties about Fatou mappings F for later use. First, we prepare some notations and terminology. Let us fix an homogeneous coordinate system $[t : x : y]$ in $\mathbf{P}^2$. Sometimes, we identify $\mathbf{C}^2(x, y)$ with $\left\{[t : x : y] \in \mathbf{P}^2 \;\middle|\; t \neq 0\right\}$, and if (x, y) is clear from the context, we may write $\mathbf{C}^2$ instead of $\mathbf{C}^2(x, y)$. Similarly we denote the corresponding sets for $\left\{[t : x : y] \in \mathbf{P}^2 \;\middle|\; y \neq 0\right\}$ and $\left\{[t : x : y] \in \mathbf{P}^2 \;\middle|\; x \neq 0\right\}$ by $\mathbf{C}^2(t, x)$ and $\mathbf{C}^2(t, y)$, respectively. Define the norm of $\mathbf{C}^2$ by $\|(x, y)\| = \sqrt{|x|^2 + |y|^2}$. Consider holomorphic functions $f(x)$ and $F(x, y)$ on $\mathbf{C}$ and $\mathbf{C}^2$, respectively. As usual, we denote their derivatives by $f'(x) = df(x)/dx$, $F_x(x, y) = \partial F(x, y)/\partial x$ and write the Jacobian matrix of F at point p by $JF(p)$. The iteration F^n of F is defined by setting $F^1 = F$, $F^n = F \circ F^{n-1}$ for $n \geq 2$. Also, we put $F^0 = id$. Moreover, we denote the usual projection mappings $\pi_i : \mathbf{C}^2 \to \mathbf{C}$ $(i = 1, 2)$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Let us set $\Delta(p, r) = \left\{z \in \mathbf{C} \;\middle|\; |z - p| < r\right\}$, $\Delta(p, r)^* = \Delta(p, r) \setminus \{p\}$, $\Delta_r = \Delta(0, r)$, $\Delta^2(p, r) = \Delta(p, r) \times \Delta(p, r)$ and $\Delta_r^2 = \Delta^2(0, r)$. Let S be a subset of a given set X. Then we denote by $\overline{S}$ the closure of the set in X. We define the set of indeterminate points of F by I_F and set $\Im_F = \overline{\bigcup_{j=0}^{\infty} F^{-j}(I_F)}$. We denote the corresponding sets for the inverse mapping G of F by I_G and $\Im_G$. Set $I_1 = [1 : 0 : 0]$, $I_2 = [1 : -a : 0]$, $J_1 = [0 : 1 : 0]$ and $J_2 = [0 : c : 1]$. Then one can see that

$$I_F = \left\{I_1,\ I_2\right\},\quad I_F = \Im_F,\quad I_G = \left\{J_1,\ J_2\right\} \text{ and } \Im_G = \overline{\left\{J_1,\ \{F^n(J_2)\}_{n=0}^{\infty}\right\}}.$$

Moreover, setting

$$C_1 = \left\{[t : x : y] \in \mathbf{P}^2 \;\middle|\; y = 0\right\},\quad C_2 = \left\{[t : x : y] \in \mathbf{P}^2 \;\middle|\; at + x - by = 0\right\} \text{ and}$$

$$D = \left\{[t : x : y] \in \mathbf{P}^2 \;\middle|\; t = 0\right\},$$

we have the following proposition:

Proposition 2.1.

(1) $F(C_1 \setminus \{I_1, I_2\}) = J_1$, $F(C_2 \setminus \{I_2\}) = J_2$, $G(C_1 \setminus \{J_1\}) = I_1$, $G(D \setminus \{J_1, J_2\}) = I_2$.
(2) $F : \mathbf{P}^2 \setminus \{C_1 \cup C_2\} \to \mathbf{P}^2 \setminus \{C_1 \cup D\}$ *and* $G : \mathbf{P}^2 \setminus \{C_1 \cup D\} \to \mathbf{P}^2 \setminus \{C_1 \cup C_2\}$ *are biholomorphic mappings.*

3 Proof of (1) of Main Theorem

In the reminder of this paper, we always assume the conditions of Main theorem. In this section, we show that, under some conditions, I_1 has a homoclinic point I_2. The proof is preceded by several steps. For a saddle fixed point, the following result is known:

Theorem 3.1 ([4, Theorem 6.4.3]). *Let G be a holomorphic mapping with a fixed point p. Suppose that the eigenvalues α and β of $JG(p)$ satisfy the inequality $|\beta| < 1 < |\alpha|$. Then there exists a holomorphic mapping H from Δ_ρ into U such that $H(0) = p$ and $G \circ H(z) = H(\alpha z)$ for $z, \alpha z \in \Delta_\rho$, where U is an open neighborhood of p. In particular, $H(\Delta_\rho) \subset W^u(p)$.*

We apply Theorem 3.1 for our G and I_1. Then it follows from the proof of [4, Theorem 6.4.3] that $JH(0) = {}^t(0, 1)$ and H is injective on Δ_ρ. Set $G^{-n}(D) = D_{-n}$ for all positive integers $n \geq 1$. In particular, D_{-1} has the following form:

$$D_{-1} = \left\{(x, y) \in \mathbf{C}^2 \;\middle|\; 1 + by + cy^2 - xy = 0\right\}.$$

We rechoose ρ so small that $H(\Delta_\rho) \cap D_{-1} = \emptyset$. On the other hand, it is clear from the definition that $W^s(I_1) \supset C_1 \setminus \{J_1\}$. Define the mapping

$$H_n : \Delta_{a^n \rho} \to \mathbf{C}^2 \ \text{ by } \ z \mapsto H_n(z) = G^n \circ H(z/a^n).$$

Then, from the definitions of G and H_n, we have the following proposition.

Proposition 3.2. *If G^n is a holomorphic and injective mapping on $H(\Delta_\rho)$, then H_n is a well-defined holomorphic injective mapping on $\Delta_{a^n\rho}$ and has the following properties:*
(1) $G \circ H_n(z) = H_n(az)$ for $z,\ az \in \Delta_{a^n\rho}$,
(2) $H_n(z) = H_{n-1}(z)$ for $z \in \Delta_{a^{n-1}\rho}$, and $H(\Delta_\rho) \subset H_1(\Delta_{a\rho}) \subset \cdots \subset H(\Delta_{a^n\rho})$,
(3) $H_n(\Delta_{a^n\rho}) \subset W^u(I_1)$.

We have now two cases to consider.

Case1. There exists some positive integer n_0 with $D_{-n_0} \cap H(\Delta_\rho) \neq \emptyset$.

Let n_0 be the minimum satisfying this condition. By Proposition 2.1, one knows that G^n is a holomorphic and injective mapping on $\mathbf{C}^2 \setminus \bigcup_{k\geq 0}^{n_0-1} G^{-k}(D_{-1} \cup C_1)$. Here, we claim the following:

(3.1) G^n is a holomorphic and injective mapping on $H(\Delta_\rho)$ for $1 \leq n < n_0$.

To this end, it is enough to show that $\bigcup_{k\geq 0}^{n_0-2} G^{-k}(D_{-1} \cup C_1) \cap H(\Delta_\rho) = \{(0,0)\}$. Since $G^{-1}(C_1) \subset D \cup C_1 \setminus \{J_1, J_2\}$, we have $\bigcup_{k\geq 0}^{n_0-2} G^{-k}(D_{-1} \cup C_1) = \bigcup_{k\geq 0}^{n_0-2} G^{-k}(D_{-1}) \cup C_1$. From this and the facts $D_{-n_0} \cap H(\Delta_\rho) \neq \emptyset$ and $D_{-n} \cap H(\Delta_\rho) = \emptyset$ for $1 \leq n < n_0$, we have the assertion (3.1). Thus, by Proposition 3.2, one can define the mapping H_{n_0-1} on $\Delta_{a^{n_0-1}\rho}$. We take a point $p_0 \in D_{-n_0} \cap H(\Delta_\rho) \setminus \{(0,0)\}$ and set $p_n = G^n(p_0)$ for all positive integers n. Then, it follows from Proposition 3.2 that $p_{n_0-1} \in D_{-1} \cap G^{n_0-1} \circ H(\Delta_\rho) = D_{-1} \cap H_{n_0-1}(\Delta_{a^{n_0}\rho}) \subset W^u(I_1)$. Moreover, taking into account the fact $J_2 \notin W^u(I_1)$, we have $p_{n_0} = G(p_{n_0-1}) \in D \setminus \{J_1, J_2\}$ and $p_{n_0+1} = G(p_{n_0}) = I_2 \in W^s(I_1) \setminus \{I_1\}$. Therefore, we conclude that I_2 is a homoclinic point of I_1 in Case 1.

Case 2. $D_{-n} \cap H(\Delta_\rho) = \emptyset$ *for all* n.

In this case, using Proposition 3.2, we can define an injective holomorphic mapping $H : \mathbf{C} \to \mathbf{C}^2$ by $H(z) = H_n(z)$ for all n. Set $H(z) = (h_1(z), h_2(z))$. From Proposition 3.2, (3) and the fact $W^s(I_1) \subset C_1 \setminus \{I_1\}$, one knows that $H(\mathbf{C}) \subset W^u(I_1)$ and $h_2(z)$ is non-constant. Set $D_{-n} = \left\{(x, y) \in \mathbf{C}^2 \;\middle|\; P_n(x, y) = 0\right\}$. It should be remarked that $P_n(x, y)$ is a polynomial which is given by the denominator of $\pi_1 \circ G^n(x, y)$. Moreover, from the assumption of Case 2, we see that $H(\mathbf{C}) \cap D_{-i} = \emptyset$ for $i = 1, 2$. So, we have holomorphic functions $k_i(z) = P_i \circ H(z)$ on $\mathbf{C}$ which are non-zero constants or transcendental entire functions with the exceptional value 0. Suppose that both k_1 and k_2 are constants, say, $k_1 \equiv \alpha$ and $k_2 \equiv \beta$ for some α, $\beta \in \mathbf{C}^*$. Then, by using the concrete forms of P_i for $i = 1, 2$, we have the following equation:

$$a^2(c - b)\{h_2(z)\}^2 + a\{b\alpha + a(\alpha - 1)\}h_2(z) + \alpha^2 - \beta \equiv 0.$$

Clearly, this contradicts the facts that $h_2(z)$ is a non-constant holomorphic function and $b \neq c$. Therefore, at least k_1 or k_2 is a non-constant transcendental entire function and so, without loss of generality, we may assume that h_2 is a non-constant transcendental holomorphic function. In the following part, we give a proof of (1) of Main theorem which is based on an argument by Jin in [3].

Lemma 3.3. $H(\mathbf{C})$ *is not contained in any algebraic curve.*

Proof. Assume the contrary. Then, there exists some polynomial $Q(x,y)$ such that $\Sigma = \left\{(x,y) \in \mathbf{C}^2 \,\middle|\, Q(x,y) = 0\right\} \supset H(\mathbf{C})$, that means $Q(h_1(z), h_2(z)) \equiv 0$ for all $z \in \mathbf{C}$. Since h_2 is a non-constant transcendental entire function, there exist some constant γ and infinitely many distinct points $\{z_\nu\}$ such that $h_2(z_\nu) = \gamma$. Set $\delta_\nu = h_1(z_\nu)$. Then, $Q(\delta_\nu, \gamma) = 0$ for all ν and, from the injectivity of H, $\{\delta_\nu\}$ is a set of infinitely many distinct points. This contradicts the fact Q is a polynomial. □

To complete the proof of (1) of Main Theorem, we here recall the following result:

Theorem 3.4 ([5, Theorem 5.6]). *Let $H : \mathbf{C} \to \mathbf{C}^2$ be an entire mapping. Assume that the set of exceptional values of H contains algebraic curves $\Sigma_i = \left\{(x,y) \in \mathbf{C}^2 \,\middle|\, P_i(x,y) = 0\right\}$ for $i = 1,2,3$, where P_i $(i = 1,2,3)$ are non-constant, irreducible and relatively prime polynomials. Then there exists some polynomial $Q(x,y)$ such that $H(\mathbf{C}) \subset \left\{(x,y) \in \mathbf{C}^2 \,\middle|\, Q(x,y) = 0\right\}$.*

Let us return to the proof of (1) of Main Theorem. Here, we assert that the polynomials P_i defining the algebraic curves D_{-i} $(i = 1,2,3)$ are non-constant, irreducible and relatively prime, by rechoosing, if necessary, some irreducible components in place of P_i. To see this, it suffices to show that $D_{-i} \cap D_{-j} = \emptyset$ for $i = 1,2,3$. Assume the contrary that there exists some point $p \in D_{-i} \cap D_{-j}$ with $G^i(p) \in D \setminus \{J_2\}$, $G^{i+1}(p) = I_2$ and $G^{i+2}(p) = I_1$. Then, we have $G^j(p) \notin D$. Clearly, this is a contradiction. Now, apply Theorem 3.5 for H and D_{-i} $(i = 1,2,3)$. Then $H(\mathbf{C})$ is contained in some algebraic curve. This contradicts Lemma 3.4. Therefore, we conclude that the Case 2 does not occur; completing the proof of (1) of Main Theorem.

4 Proof of (2) of Main Theorem

In this section, we construct a horseshoe mapping at some neighborhood of I_1 and give the proof of (2) of Main Theorem. In [4, §7.4], one can see the construction of a horseshoe structure for Hénon mapping. Our construction is basically parallel to that in it. The proof is preceded by several lemmas.

First, for the proof of (2) of Main Theorem, we prove *λ-lemma* for G at I_1. It is known that similar results are very useful tools in smooth dynamical systems. In order to state λ-lemma, we need a few preparations. From now on, we fix an affine coordinate in $\mathbf{C}^2(x,y)$ with respect to which $I_1 = (0,0)$. Recall that $h_2'(0) \neq 0$. Then we can define the inverse mapping of h_2 on Δ_R for some constant $R > 0$, and $H(\Delta_\rho)$

is locally described as follows:

$$H(\Delta_\rho) \supset \left\{(x,y) \in \mathbf{C}^2 \,\middle|\, x = h_1 \circ\ h_2^{-1}(y) \text{ on } \Delta_R\right\}.$$

Define the maps $\phi_u : \Delta_r \to \mathbf{C}$ and $\Phi : \Delta_R^2 \to \mathbf{C}^2$ by

$$\phi_u(y) = h_1 \circ h_2^{-1}(y) \quad \text{and} \quad \Phi(x,y) = (x - \phi_u(y), y).$$

It is clear from the definitions that Φ is a biholomorphic mapping and Φ^{-1} is well-define on $\Delta_{R'}^2$ for some constant R' with $0 < R' < R$. Set $\tilde{G} = \Phi \circ G \circ \Phi^{-1}$ on $\Delta_{R'}^2$. Then, we can see that local stable and unstable sets of $(0,0)$ for $\tilde{G}$ are x-axis and y-axis, respectively. We consider an injective holomorphic mapping $\phi : \Delta_r \to \mathbf{C}^2$ with $\phi(z) = (\phi_1(z), \phi_2(z))$, $\phi(0) = 0$ and $D^u = \phi(\Delta_r)$. Then, we have the following:

Lemma 4.1 **(λ-Lemma for G at I_1).** *Assume that $D^u \cap \left\{(x,y) \in \mathbf{C}^2 \,\middle|\, y = 0\right\} = \{(0,0)\}$ and $\phi_2'(0) \neq 0$. Then, there exists a positive integer n_0 satisfying the following: For any $n \geq n_0$, there exist holomorphic functions $\phi_n : \Delta_{R'} \to \mathbf{C}$ such that $\phi_n(0) = 0$ and $\tilde{G}^n(D^u) \cap \Delta_{R'}^2 \supset \left\{(x,y) \in \Delta_{R'}^2 \,\middle|\, x = \phi_n(y) \text{ on } \Delta_{R'}\right\}$. In particular, $\{\phi_n\}$ converges locally uniformly to the constant function $x \equiv 0$ on $\Delta_{R'}$.*

The lemma is proved by similar discussion in [7, Lemma 7.1] and we omit it.

In the following part of this section, we always assume that $\Delta_{R'}^2$ is given in Lemma 4.1. From a direct calculation, we see that Φ^{-1} has the form $\Phi^{-1}(x,y) = (x + \phi_u(y), y)$. Now, we define the set $\tilde{l}_{x_0}$ and the mapping $(\Phi^{-1})_{x_0} : \Delta_{R'} \to \Delta_R^2$ for all $x_0 \in \Delta_{R'}$ by

$$\tilde{l}_{x_0} = \left\{(x,y) \in \Delta_{R'}^2 \,\middle|\, x = x_0\right\} \quad \text{and} \quad (\Phi^{-1})_{x_0} : y \mapsto (\Phi^{-1})_{x_0}(y) = \Phi^{-1}(x_0, y).$$

It is clear that $(\Phi^{-1})_{x_0}$ is an injective holomorphic mapping on $\Delta_{R'}$ and $(\Phi^{-1})_0(\Delta_{R'}) \subset W^u(I_1)$. Set $l_{x_0} = \Phi^{-1}(\tilde{l}_{x_0})$ and $U = \Phi^{-1}(\Delta_{R'}^2)$. Then, one can see that U is foliated by the leaves $\{l_{x_0}\}_{x_0 \in \Delta_{R'}}$. Here, we can take the point $\tilde{y} \in \Delta_{R'}$ such that $\Phi^{-1}(0, \tilde{y}) = p_0$, where p_0 is the point appearing in the proof of (1) of Main Theorem and it satisfies the conditions $p_0 \in H(\Delta_\rho)$ and $p_{n_0-1} = G^{n_0-1}(p_0) \in W^u(I_1) \cap D_{-1}$. By the identity theorem, we have the following lemma:

Lemma 4.2. *There exists some positive constant $r_2 > 0$ such that*

$$G^{n_0-1} \circ (\Phi^{-1})_0(\Delta(\tilde{y}, r_2)) \cap D_{-1} = \{p_{n_0-1}\}.$$

Set $K(y) = G^{n_0-1} \circ \Phi_0^{-1}(y)$ and $K(y) = (k_1(y), k_2(y))$. It follows from the previous lemma that $\tilde{y}$ is a unique zero for $P \circ K(y)$ on $\Delta(\tilde{y}, r_2)$. Therefore, there exist some holomorphic function $\tilde{K}(y)$ on $\Delta(\tilde{y}, r_2)$ and a positive integer m such that $\tilde{K}(\tilde{y}) \neq 0$ and $P \circ K(y) = \tilde{K}(y)(y - \tilde{y})$.

Lemma 4.3. *If $m = 1$, then I_2 is a transverse homoclinic point.*

Proof. From the condition $[0 : c : 1] \notin W^u(I_1)$, there is some constant $x_0 \neq c$ with $p_{n_0} = [0 : x_0 : 1]$ and $p_{n_0+1} = [1 : -a : 0]$. Here, we recall that G has the following form in the specific local charts:

$$G : \mathbf{C}^2(x, y) \to \mathbf{C}^2(t, x), \quad (x, y) \mapsto \left(\frac{1 + by + cy^2 - xy}{ay}, x - cy \right), \tag{4.1}$$

$$G : \mathbf{C}^2(t, x) \to \mathbf{C}^2(x, y), \quad (t, x) \mapsto \left(\frac{ax - ac}{t^2 + bt + c - x}, \frac{at}{t^2 + bt + c - x} \right). \tag{4.2}$$

Write G in (4.1) and (4.2) by G_1 and G_2, respectively. Then, one obtains that

$$G_1 \circ K(y) = \left(\frac{P \circ K(y)}{ak_2(y)}, k_1(y) - ck_2(y) \right) = \left(\frac{(y - \tilde{y})\tilde{K}(y)}{ak_2(y)}, k_1(y) - ck_2(y) \right),$$

$$J(G_1 \circ K)(\tilde{y}) = {}^t \left(\tilde{K}(\tilde{y})/(ak_2(\tilde{y})), k_1'(\tilde{y}) - ck_2'(\tilde{y}) \right),$$

$$J(G^2 \circ K)(\tilde{y}) = JG_2(0, x_0) J(G_1 \circ K)(\tilde{y}) = \frac{\tilde{K}(\tilde{y})}{k_2(\tilde{y})(c - x_0)} \begin{pmatrix} -b \\ 1 \end{pmatrix}.$$

This implies the assertion of Lemma. □

From now on, we only consider the case $m = 1$, that is, I_2 is a transverse homoclinic point.

For some positive constant r_1 with $0 < r_1 < R'$, we set $V_0 = \Delta_{r_1} \times \Delta_{r_2}$ and $V_1 = \Delta_{r_1} \times \Delta(\tilde{y}, r_2)$. Then V_i is foliated by the leaves $\{\tilde{l}_{x_0} \cap V_i\}_{x_0 \in \Delta_{r_1}}$ for $i = 0, 1$. Moreover, set $(l^i_{x_0})_n = G^n \circ \Phi^{-1}(\tilde{l}_{x_0} \cap V_i)$. Since the family of mappings $\{P \circ G^{n_0 - 1} \circ (\Phi^{-1})_{x_0}\}_{x_0 \in \Delta_{r_1}}$ locally uniformly converge to the mapping $P \circ G^{n_0-1} \circ (\Phi^{-1})_0$ on $\Delta(\tilde{y}, r_2)$ as $|x_0| \to 0$, Hurwitz's theorem implies the following claim:

$$\begin{cases} \text{There is a positive constant } r_1 > 0 \text{ such that for all } x_0 \in \Delta_{r_1} \\ P \circ G^{n_0-1} \circ (\Phi^{-1})_{x_0}(y) \text{ has a unique zero point } \tilde{y}_{x_0} \text{ in } \Delta(\tilde{y}, r_2). \end{cases} \tag{4.3}$$

Similar calculation in the proof of Lemma 4.3 implies that $(l^i_{x_0})_{n_0+1}$ is a 1-dimensional submanifold for each $x_0 \in \Delta_{r_1}$ and $i = 0, 1$. Set $(\tilde{l}^i_{x_0})_{n_0+1} = \Phi((l^i_{x_0})_{n_0+1})$. From Lemma 4.1, we have the following lemma:

Lemma 4.4. *There is an integer m_0 such that, for all $x_0 \in \Delta_{r_1}$ and $i = 0, 1$,*

$$\pi_1 \circ \Phi \circ G^{m_0} \circ \Phi^{-1}((\tilde{l}^i_{x_0})_{n_0+2}) \subset \Delta_{r_1/2} \text{ and } \pi_2 \circ \Phi \circ G^{m_0} \circ \Phi^{-1}((\tilde{l}^i_{x_0})_{n_0+2}) \supset \Delta_{R'}.$$

In the following, unless specified otherwise, i (and i_j) will usually denote $i = 0$ or 1. To simplify the discussion, we write $\tilde{G}$ in place of $\Phi \circ G^{n_0+m_0} \circ \Phi^{-1}$ and set $\tilde{G}^n(x,y) = (g^n_1(x,y), g^n_2(x,y))$, $\tilde{F} = \Phi \circ F^{n_0+m_0} \circ \Phi^{-1}$, $\tilde{F}^n(x,y) = (f^n_1(x,y), f^n_2(x,y))$ and $l_{x_0} = \left\{(x,y) \in \Delta^2_{R'} \,\middle|\, x = x_0\right\}$ for all $x_0 \in \Delta_{r_1}$. Let $U^i_{x_0}$ be the subset of $l_{x_0} \cap V_i$ with $\pi_2 \circ \tilde{G}(U^i_{x_0}) = \Delta_{R'}$ for all $x_0 \in \Delta_{r_1}$. Moreover, using $U^i_{x_0}$, we reset $V_i = \bigcup_{x_0 \in \Delta_{r_1}} U^i_{x_0}$, $W_i = \tilde{G}(V_i)$ and $l^i_{x_0} = \tilde{G}(V_i \cap l_{x_0})$ for every $x_0 \in \Delta_{r_1}$. From the Lemma 4.1, there exists some holomorphic function $\phi^i_{x_0}(y)$ on $\Delta_{R'}$ such that $l^i_{x_0} = \left\{(x,y) \in W_i \,\middle|\, x = \phi^i_{x_0}(y) \text{ on } \Delta_{R'}\right\}$ and

$$\tilde{G} : V_0 \cup V_1 \setminus \{\tilde{G}^{-1}(0,0)\} \to W_0 \cup W_1 \setminus \{(0,0)\}$$

is a biholomorphic mapping. So,this implies that $W_i \setminus \{(0,0)\}$ is foliated by the leaves $\{l^i_{x_0} \setminus \{(0,0)\}\}_{x_0 \in \Delta_{r_1}}$. Similarly, set $\tilde{l}_{y_0} = \left\{(x,y) \in \Delta^2_{R'} \,\middle|\, y = y_0\right\}$ for all $y_0 \in \Delta_{R'}$, $\tilde{l}^{i_1}_{y_0} = \tilde{F}(\tilde{l}_{y_0} \cap W_{i_1})$ for $y_0 \in \Delta^*_R$ and $\tilde{l}^{i_1}_0 = \tilde{G}^{-1}(\tilde{l}_0 \cap W_{i_1}) \cap V_{i_1}$. Then we have the following lemma:

Lemma 4.5. *For all $y_0 \in \Delta_{R'}$ there exist some holomorphic functions $\psi^{i_1}_{y_0}(x)$ on Δ_{r_1} such that $\tilde{l}^{i_1}_{y_0} = \left\{(x,y) \in \Delta^2_R \,\middle|\, y = \psi^{i_1}_{y_0}(x) \text{ on } \Delta_{r_1}\right\}$.*

Proof. It should be remarked here that since $\tilde{F}(\tilde{l}_{y_0} \cap W_{i_1}) = \tilde{G}^{-1}(\tilde{l}_{y_0} \cap W_{i_1})$ for $y_0 \in \Delta^*_{R'}$, $\tilde{l}^{i_1}_{y_0}$ is given by the inverse image of $\tilde{G}$ of $\bigcup_{x_0 \in \Delta_{r_1}} \tilde{l}_{y_0} \cap l^{i_1}_{x_0}$. From the previous discussion, $\tilde{l}_{y_0} \cap l^{i_1}_{x_0} = \{(\phi^{i_1}_{x_0}(y_0), y_0)\}$ for $(x_0, y_0) \in \Delta_{r_1} \times \Delta^*_{R'}$. Moreover, from the injectivity of $\tilde{G}$, $\tilde{G}^{-1}(\phi^{i_1}_{x_0}(y_0), y_0)$ is given by a single point contained in $l_{x_0} \cap V_{i_1}$. So, we set $\tilde{G}^{-1}(\phi^{i_1}_{x_0}(y_0), y_0) = (x_0, \psi^{i_1}_{y_0}(x_0))$. Repeating this process for all $x_0 \in \Delta_{r_1}$, we have

$$\tilde{l}^{i_1}_{y_0} = \tilde{G}^{-1}(l_{y_0} \cap W_{i_1}) = \left\{(x,y) \in V_{i_1} \,\middle|\, y = \psi^{i_1}_{y_0}(x) \text{ on } \Delta_{r_1}\right\}.$$

On the other hand, $\tilde{G}^{-1}(\tilde{l}_{y_0} \cap W_{i_1})$ is an analytic subset of pure dimension 1. By using this fact, it follows from [6, Theorem 4.4.1] that $y = \psi^{i_1}_{y_0}(x)$ is a holomorphic function on Δ_{r_1}. In case of $y_0 = 0$, we define ψ^i_0 by $\psi^1_0(x) = \tilde{y}_x$ which is appeared in (4.3) and $\psi^0_0 \equiv 0$. Then we see that $\tilde{l}^{i_1}_0 = \tilde{G}^{-1}(\tilde{l}_0 \cap W_{i_1}) \cap V_{i_1} = \left\{(x,y) \in V_{i_1} \,\middle|\, y = \psi^{i_1}_{y_0}(x) \text{ on } \Delta_{r_1}\right\}$, and [6, Theorem 4.4.1] implies that $\psi^{i_1}_0$ is a holomorphic function. □

Inductively, we define the set $V_{i_{n+1} i_n \ldots i_1}$ by $V_{i_{n+1}} \cap \tilde{G}^{-1}(V_{i_n \ldots i_1})$ for $n \geq 1$ and the holomorphic mapping

$$\mathcal{G}^n : V_{i_n \ldots i_1} \to \Delta_{r_1} \times \Delta_{R'} \text{ by } (x,y) \mapsto (x, g^n_2(x,y)) \text{ for } n \geq 1.$$

The map $\tilde{G}^n$ is said to satisfy the *horseshoe condition* (HC_n) if $\mathcal{G}^n$ is biholomorphic.

Lemma 4.6. *$\tilde{G}^n$ satisfies the horseshoe condition (HC_n) for all positive integers n.*

Proof. From Lemma 4.5 and the fact that W_{i_1} is foliated by the leaves $\{l_{x0}^{i_1}\}_{x0 \in \Delta_{r_1}}$, we see that $\tilde{G}$ satisfies (HC_1).

Assume that $\tilde{G}^n$ satisfies the (HC_n). Then, since $\mathcal{G}^n$ is a biholomorphic mapping, there exists some holomorphic function $\psi_{y0}^{i_n \dots i_1}(x)$ on Δ_{r_1} for each $y_0 \in \Delta_{R'}$ such that

$$\Big\{(x,y) \in V_{i_n \dots i_1} \;\Big|\; g_2^n(x,y) = y_0\Big\} = \Big\{(x,y) \in V_{i_n \dots i_1} \;\Big|\; y = \psi_{y0}^{i_n \dots i_1}(x) \text{ on } \Delta_{r_1}\Big\}.$$

Denoting this set by $\tilde{l}_{y0}^{i_n \dots i_1}$, we see that $V_{i_n \dots i_1}$ is foliated by the leaves $\{\tilde{l}_{y0}^{i_n \dots i_1}\}_{y0 \in \Delta_{R'}}$.

Define the holomorphic mapping

$$\tilde{\mathcal{G}}^n : V_{i_n \dots i_1} \cap \tilde{G}(V_{i_{n+1}}) \setminus \tilde{l}_0^{i_n \dots i_1} \to \Delta_{r_1} \times \Delta_{R'}^* \text{ by } (x,y) \mapsto (f_1^1(x,y), g_2^n(x,y)).$$

Then, we have the following assertion:

$$(4.4) \quad \left\{ \begin{array}{l} \text{For all } (x_0, y_0) \in \Delta_{r_1} \times \Delta_{R'}^*, \text{ there exists a unique point} \\ (x,y) \in V_{i_n \dots i_1} \cap \ \tilde{G}(V_{i_{n+1}}) \setminus \tilde{l}_0^{i_n \dots i_1} \text{ such that } \tilde{\mathcal{G}}^n(x,y) = (x_0, y_0). \end{array} \right.$$

To show (4.4), it is enough to see that the holomorphic function $\phi_{x0}^{i_{n+1}} \circ \psi_{y0}^{i_n \dots i_1}(x)$ on Δ_{r_1} has a unique fixed point $\tilde{x}$ with $\psi_{y0}^{i_n \dots i_1}(\tilde{x}) \neq 0$. Indeed, by Lemma 4.4, one knows that $\phi_{x0}^{i_{n+1}} \circ \psi_{y0}^{i_n \dots i_1}(\Delta_{r_1}) \subset \Delta_{r_{1/2}}$. Hence, it follows from [4, Theorem 6.3.5] that there exists a unique fixed point $\tilde{x} \in \Delta_{r_1}$ of $\phi_{x0}^{i_{n+1}} \circ \psi_{y0}^{i_n \dots i_1}$. So, $(\tilde{x}, \psi_{y0}^{i_n \dots i_1}(\tilde{x}))$ is the required point satisfying (4.4). In particular, we see that $\tilde{\mathcal{G}}^n$ is biholomorphic. Moreover, consider the following mapping:

$$\tilde{\mathcal{G}}^n \circ \tilde{G} : \tilde{G}^{-1}(V_{i_n \dots i_1} \cap \tilde{G}(V_{i_{n+1}}) \setminus \tilde{l}_0^{i_n \dots i_1}) \cap V_{i_{n+1}} \to \Delta_{r_1} \times \Delta_{R'}^*.$$

It is clear from the definition that $\tilde{\mathcal{G}}^n \circ \tilde{G} = \mathcal{G}^{n+1}$. Set $\tilde{l}_0^{i_{n+1} \dots i_1} = \tilde{G}^{-1}(\tilde{l}_0^{i_n \dots i_1} \cap V_{i_n \dots i_1} \cap \tilde{G}(V_{i_{n+1}}))$. Then, there exists some holomorphic function $\psi_0^{i_{n+1} \dots i_1}(x)$ on Δ_{r_1} such that

$$(4.5) \quad \left\{ \begin{array}{l} \tilde{l}_0^{i_{n+1} \dots i_1} = \Big\{(x,y) \in V_{i_{n+1}} \cap \tilde{F}^{-1}(V_{i_n \dots i_1}) \;\Big|\; y = \psi_0^{i_{n+1} \dots i_1}(x) \text{ on } \Delta_{r_1}\Big\} \text{ and} \\ V_{i_{n+1}} \cap \tilde{G}^{-1}(V_{i_n \dots i_1} \cap \tilde{G}(V_{i_{n+1}}) \setminus \tilde{l}_0^{i_n \dots i_1}) = V_{i_{n+1}} \cap \tilde{G}^{-1}(V_{i_n \dots i_1}) \setminus \tilde{l}_0^{i_{n+1} \dots i_1}. \end{array} \right.$$

Indeed, by the same argument as in the proof of (4.4), one can see that $l_{x0}^{i_{n+1}}$ and $\tilde{l}_0^{i_n \dots i_1}$ have a unique intersection point for any fixed $x_0 \in \Delta_{r_1}$, and we denote it by $(\phi_{x0}^{i_{n+1}}(y_0), y_0) \in V_{i_n \dots i_1} \cap \tilde{G}(V_{i_{n+1}})$. In case of $y_0 \neq 0$, $\tilde{G}^{-1}(\phi_{x0}^{i_{n+1}}(y_0), y_0)$ is given by a single point contained in $\tilde{G}^{-1}(V_{i_n \dots i_1}) \cap V_{i_{n+1}}$, so we write it by $(x_0, \psi_0^{i_{n+1} \dots i_1}(x_0))$. In case of $y_0 = 0$, from the fact $l_{x_0}^{i_{n+1}} \cap \tilde{l}_0^{i_n \dots i_1} = \{(0,0)\}$ and the argument in the proof of Lemma 4.5, we set $(x_0, \psi_0^{i_{n+1} \dots i_1}(x_0)) = (x_0, \psi_0^{i_{n+1}}(x_0))$. Since $\tilde{l}_0^{i_{n+1} \dots i_1}$ is described as

$$\tilde{l}_0^{i_{n+1} \dots i_1} = \left\{ (x,y) \in V_{i_{n+1}} \;\middle|\; y = \psi_0^{i_{n+1} \dots i_1}(x) \text{ on } \Delta_{r_1} \right\} \text{ and}$$

$$\tilde{l}_0^{i_{n+1} \dots i_1} = \tilde{G}^{-1}(\tilde{l}_0^{i_n \dots i_1} \cap V_{i_{n+1}} \cap \tilde{G}(V_{i_{n+1}})) \cap V_{i_{n+1}},$$

$\tilde{l}_0^{i_{n+1} \dots i_1}$ is an analytic set of pure dimension 1, and hence [6, Theorem 4.4.1] implies that $\psi_0^{i_{n+1} \dots i_1}$ is a holomorphic function on Δ_{r_1}. Thus, we have the assertion (4.5).

Now, one knows that

$$\mathcal{G}^{n+1} : \tilde{G}^{-1}(V_{i_n \dots i_1}) \cap V_{i_{n+1}} \setminus \tilde{l}_0^{i_{n+1} \dots i_1} \to \Delta_{r_1} \times \Delta_{R'}^*$$

is a biholomorphic mapping. Moreover, it is clear that $\mathcal{G}^{n+1}(x,y) = (x,0)$ for $(x,y) = (x,\ \psi_0^{i_{n+1} \dots i_1}(x)) \in \tilde{l}_0^{i_{n+1} \dots i_1}$, and this shows that the mapping $\mathcal{G}^{n+1} : \tilde{G}^{-1}(V_{i_n \dots i_1}) \cap V_{i_{n+1}} \to \Delta_{r_1} \times \Delta_{R'}$ is a biholomorphic mapping, proving Lemma 4.6. □

Now, we classify the points p of $\bigcap_{n=0}^{\infty} \tilde{G}^{-n}(V_0 \cup V_1)$ by using the fact that the k-th orbit of p is contained in V_0 or V_1. To this end, we introduce some notation from symbolic dynamics. A sequence $(s_0, \dots, s_{n-1})$ with terms $s_k = 0, 1$ is said to be a *symbol sequence of length n*. The set of all symbol sequences of length n is denoted by $\{0,1\}^n$. For a symbol sequence $(s_0, \dots, s_{n-1}) \in \{0,1\}^n$, we define the set $V_{s_0 \dots s_{n-1}}$ as follows:

$$V_{s_0 \dots s_{n-1}} = \left\{ (x,y) \in \Delta_{r_1} \times \Delta_{R'} \;\middle|\; \tilde{G}^j(x,y) \in V_{s_j},\ \ j = 0, \dots n-1 \right\}.$$

Then, from the previous discussion, we have the following results.

Lemma 4.7. $V_{s_0 \dots s_{n-1}} = \bigcup_{y_0 \in \Delta_R} \tilde{l}_{y_0}^{s_0 \dots s_{n-1}}$ *and* $\tilde{G}(V_{s_0 \dots s_n}) \subset V_{s_1 \dots s_n}$ *for all symbol sequences* $(s_0, \dots, s_n) \in \{0,1\}^{n+1}$.

As in [4, § 7.4], we define the space $\{0,1\}^{\mathbf{N} \cup \{0\}}$ of symbol sequneces as follows:

$$\{0,1\}^{\mathbf{N} \cup \{0\}} = \left\{ s_+ = (s_0, s_1, \dots) \;\middle|\; s_i = 0,\ 1 \right\}.$$

By setting $\Gamma(s_+) = \bigcap_{n=0}^{\infty} V_{s_0 \dots s_n}$ for all $s_+ \in \{0,1\}^{\mathbf{N} \cup \{0\}}$, we have the following lemma.

Lemma 4.8. *For all* $s_+ \in \{0,1\}^{\mathbf{N} \cup \{0\}}$ *there exist some holomorphic functions* $\psi_{s_+} : \Delta_{r_1} \to \mathbf{C}$ *such that*

$$\Gamma(s_+) = \left\{ (x,y) \in \Delta_{r_1} \times \Delta_{R'} \;\middle|\; y = \psi_{s_+}(x) \text{ on } \Delta_{r_1} \right\}.$$

Proof. First, we claim that $l_{x_0} \cap \Gamma(s_+)$ consists of a single point for all $x_0 \in \Delta_{r_1}$. Indeed, taking into account the fact $l_{x_0} \cap \Gamma(s_+) = \bigcap_{n=0}^{\infty} V_{s_0 \ldots s_n} \cap l_{x_0}$, we define the holomorphic function

$$(g_2^{n+1})_{x_0} : \pi_2(l_{x_0} \cap V_{s_0 \ldots s_n}) \to \Delta_{R'} \ \text{ by } \ y \mapsto (g_2^{n+1})_{x_0}(y) = g_2^{n+1}(x_0, y).$$

Then, it is univalnet; so that there exists its inverse function $(g_2^{n+1})_{x_0}^{-1} : \Delta_{R'} \to \pi_2(l_{x_0} \cap V_{s_0 \ldots s_{n+1}})$. From the facts that $\tilde{G}(V_{s_0 \ldots s_{n+1}}) \subset V_{s_1 \ldots s_{n+1}}$ and $\tilde{G}^{n+1}(V_{s_0 \ldots s_{n+1}}) \subset V_{s_{n+1}}$, there exists some constant R'' with $0 < R'' < R'$ such that $(g_2^{n+1})_{x_0} \circ \pi_2(l_{x_0} \cap V_{s_0 \ldots s_{n+1}}) \subset \Delta_{r_2} \cup \Delta(\tilde{y}, r_2) \subset \Delta_{R''}$. Therefore, we have the following inclusions:

$$\pi_2(l_{x_0} \cap V_{s_0 \ldots s_n}) = (g_2^{n+1})_{x_0}^{-1}(\Delta_{R'}) \supset (g_2^{n+1})_{x_0}^{-1}(\Delta_{R''}) \supset \pi_2(l_{x_0} \cap V_{s_0 \ldots s_{n+1}}).$$

It follows from [4, Lemma 6.3.7] that $\bigcap_{n=1}^{\infty} \pi_2(l_{x_0} \cap V_{s_0 \ldots s_n})$ consists of a unique point, and we denote it by $\psi_{s_+}(x_0)$. Then, $l_{x_0} \cap \Gamma(s_+) = (x_0, \psi_{s_+}(x_0))$ and

$$\Gamma(s_+) = \left\{ (x, y) \in \Delta_{r_1} \times \Delta_{R'} \;\middle|\; y = \psi_{s_+}(x) \text{ on } \Delta_{r_1} \right\}.$$

Finally, we show that $\psi_{s_+}(x)$ is a holomorphic function. To see this, set

$$\tilde{l}_0^{s_0 \ldots s_n} = \left\{ (x, y) \in \Delta_{r_1} \times \Delta_R \;\middle|\; y = \psi_0^{s_0 \ldots s_n}(x) \text{ on } \Delta_{r_1} \right\}.$$

Then, $\{\psi_0^{s_0 \ldots s_n}(x_0)\}$ converges to $\psi_{s_+}(x_0)$ as $n \to \infty$ for every fixed point $x_0 \in \Delta_{r_1}$. On the other hand, since the family of functions $\{\psi_0^{s_0 \ldots s_n}(x)\}_{n \geq 0}$ is uniformly bounded on Δ_{r_1}, it is normal. So, $\{\psi_0^{s_0 \ldots s_n}(x)\}_{n \geq 0}$ converges locally uniformly to a holomorphic function ψ_{s_+} on Δ_{r_1}. This completes the proof of the lemma. □

Put $V = \bigcup_{s_+ \in \{0,1\}^{\mathbf{N} \cup \{0\}}} \Gamma(s_+)$ and define the mappings

$$\psi^+ : V \to \{0,1\}^{\mathbf{N} \cup \{0\}} \ \text{ by } \ (x, y) \mapsto s_+, \ \text{ where } y = \psi_{s_+}(x),$$

$$\Psi^+ : V \to \Delta_{r_1} \times \{0,1\}^{\mathbf{N} \cup \{0\}} \ \text{ by } \ (x, y) \mapsto (x, \psi^+(x, y)) \text{ and}$$

$$\sigma : \{0,1\}^{\mathbf{N} \cup \{0\}} \to \{0,1\}^{\mathbf{N} \cup \{0\}} \ \text{ by } \ s_+ = (s_0, s_1, \ldots) \mapsto \sigma(s_+) = (s_1, s_2, \ldots).$$

By Lemmas 4.7 and 4.8, we have the following:

Lemma 4.9. *Ψ^+ is homeomorphism and $\sigma \circ \psi^+(x, y) = \psi^+ \circ \tilde{G}(x, y)$ for all $(x, y) \in V$.*

Now, we repeat the same process as above for $\tilde{F}$. Inductively, we define the set $\tilde{W}_{i_{n+1} \ldots i_1}$ by

$$\tilde{W}_{i_1} = W_{i_1} \setminus \{(0,0)\} \ \text{ and } \ \tilde{W}_{i_{n+1} \ldots i_1} = \tilde{W}_{i_{n+1}} \cap \tilde{F}^{-1}(\tilde{W}_{i_n \ldots i_1}) \ \text{ for } n \geq 1,$$

and define the holomorphic mapping

$$\mathcal{F}^n : \tilde{W}_{i_n \dots i_1} \to \Delta_{r_1} \times \Delta^*_{R'} \text{ by } (x,y) \mapsto (f_1^n(x,y), y) \text{ for } n \geq 1.$$

The map $\tilde{F}^n$ is said to satisfy the horseshoe condition (HC_n) if $\mathcal{F}^n$ is biholomorphic.

Lemma 4.10. *$\tilde{F}^n$ satisfies (HC_n) for any positive integer n.*

Proof. From the fact that V_{i_1} is foliated by the leaves $\{\tilde{l}^{i_1}_{y_0}\}_{y_0} \in \Delta_{R'}$, we see that $\tilde{F}$ satisfies (HC_1). Assume that $\tilde{F}^n$ satisfies (HC_n). Then, since $\mathcal{F}^n$ is a biholomorphic mapping, there exists a holomorhic function $\phi^{i_n \dots i_1}_{x_0}(y)$ on $\Delta^*_{R'}$ for all $x_0 \in \Delta_{r_1}$ such that

$$\left\{(x,y) \in \tilde{W}_{i_n \dots i_1} \,\middle|\, f_1^n(x,y) = x_0\right\} = \left\{(x,y) \in \tilde{W}_{i_n \dots i_1} \,\middle|\, x = \phi^{i_n \dots i_1}_{x_0}(y) \text{ on } \Delta^*_{R'}\right\}.$$

Denoting this set by $l^{i_n, \dots i_1}_{x_0}$, we see that $\tilde{W}_{i_n \dots i_1}$ is foliated by the leaves $\{l^{i_n \dots i_1}_{x_0}\}_{x_0 \in \Delta_{r_1}}$. Define the holomorphic mapping

$$\tilde{\mathcal{F}}^n : \tilde{W}_{i_n \dots i_1} \cap \tilde{F}(\tilde{W}_{i_{n+1}}) \to \Delta_{r_1} \times \Delta^*_{R'} \text{ by } (x,y) \to (f_1^n(x,y), g_2^1(x,y)).$$

From a similar discussion in the case of $\tilde{G}^n$, one obtains that $\tilde{\mathcal{F}}^n$ is a biholomorphic mapping. Together with the fact that

$$\tilde{F}^{-1}(\tilde{W}_{i_n \dots i_1} \cap \tilde{F}(\tilde{W}_{i_{n+1}})) = \tilde{F}^{-1}(\tilde{W}_{i_n \dots i_1}) \cap \tilde{W}_{i_{n+1}}$$

we see that $\mathcal{F}^{n+1} = \tilde{\mathcal{F}}^n \circ \tilde{F} : \tilde{W}_{i_{n+1} \dots i_1} \to \Delta_{r_1} \times \Delta^*_{R'}$ is a biholomorphic mapping. □

Define a set for a symbol sequence having a form $(s_{-1}, \dots, s_{-n})$ of length n

$$\tilde{W}_{s_{-1} \dots s_{-n}} = \left\{(x,y) \in W_0 \cup W_1 \setminus \{(0,0)\} \,\middle|\, \tilde{F}^j(x,y) \in V_{s_{-j}},\ \ j = 1, \dots, n\right\}.$$

Then, from the definitions of $\tilde{W}_{s_{-1} \dots s_{-n}}$ and $\tilde{F}$, we have easily the following lemma:

Lemma 4.11. $\tilde{W}_{s_{-1} \dots s_{-(n+1)}} \subset \tilde{W}_{s_{-1} \dots s_{-n}}$ *and* $\tilde{F}(\tilde{W}_{s_{-1} \dots s_{-n}}) \subset \tilde{W}_{s_{-2} \dots s_{-n}}$ *for all symbol sequences* $(s_{-1}, \dots, s_{-(n+1)}) \in \{0,1\}^{n+1}$.

Put $\Gamma(s_-) = \bigcap_{n=0}^{\infty} \tilde{W}_{s_{-1} \dots s_{-n}}$ for all $s_- \in \{0,1\}^{\mathbf{N}}$ with $s_- = (s_{-1}, s_{-2}, \dots)$. In exactly the same way as in the proof of Lemma 4.8, we have the following lemma.

Lemma 4.12. *For all $s_- \in \{0,1\}^{\mathbf{N}}$ there exist some holomorphic functions ϕ_{s_-} : $\Delta^*_{R'} \to \mathbf{C}$ such that*

$$\Gamma(s_-) = \left\{(x,y) \in W_0 \cup W_1 \setminus \{(0,0)\} \,\middle|\, x = \phi_{s_-}(y) \text{ on } \Delta^*_{R'}\right\}.$$

Put $W = \bigcup_{s_- \in \{0,1\}^{\mathbf{N}}} \Gamma(s_-)$ and define the mappings

$$\psi^- : W \to \{0,1\}^{\mathbf{N}} \text{ by } (x,y) \mapsto s_-, \text{ where } y = \phi_{s_-}(x),$$

$$\Psi^- : W \to \{0,1\}^{\mathbf{N}} \times \Delta^*_{R'} \text{ by } (x,y) \mapsto (\psi^-(x,y), y).$$

By a similar disscussion in the proof of Lemma 4.9, we have the following lemma.

Lemma 4.13. *Ψ^- is a homeomorphism and $\sigma \circ \psi^-(x,y) = \psi^- \circ \tilde{F}(x,y)$ for all $(x,y) \in W$.*

Put $X = V \cap W \setminus \bigcup_{n\geq 0}^{\infty} \tilde{G}^{-n}(0,0)$. Then, the following lemma holds from the definitions of V, W, $\tilde{G}$ and $\tilde{F}$.

Lemma 4.14. *$\tilde{F} : X \to X$ is a bijective mapping with $\tilde{F}(X) = X$ and $\tilde{G}(X) = X$.*

We define the space of bi-infinite symbol sequence

$$\{0,1\}^{\mathbf{Z}} = \left\{ s = (s_-, s_+) \in \{0,1\}^{\mathbf{Z}} \;\middle|\; s = (\cdots, s_{-1}, s_0, s_1, \cdots) \right\},$$

and define a subset E of $\{0,1\}^{\mathbf{Z}}$ by

$$E = \left\{ s \in \{0,1\}^{\mathbf{Z}} \;\middle|\; \text{there is an } n_0 \text{ such that } s_n = 0 \text{ for all } n \geq n_0 \right\}.$$

Moreover, define the mappings:

$$\Psi : X \to \{0,1\}^{\mathbf{Z}} \setminus E \text{ by } (x,y) \mapsto (\psi^-(x,y), \psi^+(x,y)) = (\cdots s_{-1}, s_0, s_1, \cdots) \text{ and}$$

$$\sigma : \{0,1\}^{\mathbf{Z}} \to \{0,1\}^{\mathbf{Z}} \text{ by } (\cdots s_{-1}, \hat{s_0}, s_1, \cdots) \mapsto (\cdots s_{-1}, s_0, \hat{s_1}, s_2 \cdots).$$

Then, the following lemma holds.

Lemma 4.15. *$\Psi : X \to \{0,1\}^{\mathbf{Z}} \setminus E$ is a homeomorphism such that $\sigma \circ \Psi(x,y) = \Psi \circ \tilde{G}(x,y)$ and $\sigma^{-1} \circ \Psi(x,y) = \Psi \circ \tilde{F}(x,y)$ for all $(x,y) \in X$.*

To complete the proof of (2) of Main Theorem, we only need to show its last statement. By Lemma 4.15, the periodic points of $\tilde{G}$ and those of σ are in one to one corresspondence. Since the set of periodic points of σ is dense in $\{0,1\}^{\mathbf{Z}}$, we complete the proof of (2) of Main Theorem.

Acknowledgements. The author would like to thank Mr. Teisuke Jin for his knowledge about the argument in Section 3, and Professors Shigehiro Ushiki and Mitsuhiro Shishikura for helpful suggestions and discussions.

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