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NOTE ON THE PAPER "A NEW TREE ASSOCIATED WITH HERMAN RINGS"

MITSUHIRO SHISHIKURA

The folowing paper "A new tree associated with Herman rings" was originally written and submitted for the proceedings of a conference on complex dyanmical systems, held in Cornell University (Ithaca, NY, USA) in August 1987, organized by J. H. Hubbard. However the proceedings have never been published, and the paper did not have a chance to appear. On th occasion of conference "Complex dynamics and related fields" held at Research Institute for Mathematical Sciences, Kyoto University, October 9-12, 2002, the author decided to include the paper in the Proceedings "Kokyuroku", although the paper itself was not presented during the meeting. The only modification made was to include the volume number for the reference [2].

This paper was an early report on the results which were later published in

[2] M. Shishikura, Trees associated with the configuration of Herman rings, Ergod. Th. & Dynam. Sys. 9 (1989), p.543-560.

This paper also contains original results such as a lower bound on the degree of rational maps realizing given trees (§5 Theorem 3) and many examples (§3). In this respect, this paper complements the paper [2].

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY, KYOTO 606-8502, JAPAN
E-mail address: mitsu@kusm.kyoto-u.ac.jp

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A NEW TREE ASSOCIATED WITH HERMAN RINGS

Mitsuhiro Shishikura

Department of Mathematics
Kyoto University
Kyoto, 606, Japan

ABSTRACT

A certain kind of tree and piecewise linear map on it are defined, in connection with the configuration of Herman rings of rational functions. Their properties are investigated and several examples of the tree are shown. A sufficient condition for trees to be realized by rational functions is given. The construction of the tree can be generalized to rational functions with attractive periodic points. These trees are completely different from Douady-Hubbard's tree.

INTRODUCTION

Let $f:\bar{\mathbf{C}} \to \bar{\mathbf{C}}$ be a rational function, where $\bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ is the Riemann sphere. Suppose that f has a cycle of Herman rings $A_0, A_1, \ldots, A_{p-1}$ $(p \geq 1)$; i.e., A_j are distinct connected components of $\bar{\mathbf{C}} - J_f$ (J_f = the Julia set of f), $f(A_j) = A_{j+1}$ $(j=0,\ldots,p-1;\ A_p = A_0)$, and there exists a conformal mapping $\phi: A_0 \to \{z \in \mathbf{C}:\ r_0 < |z| < 1\}$ such that $\phi \circ f^p \circ \phi^{-1}(z) = e^{2\pi i\theta} \cdot z$ $(0 < r_0 < 1,\ \theta \in \mathbf{R} - \mathbf{Q})$.

In this paper, we are concerned with the configuration of Herman rings. A *configuration* means a cyclically ordered collection of oriented

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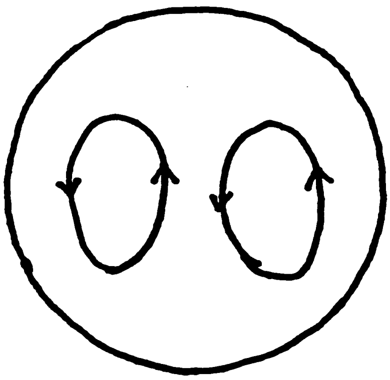
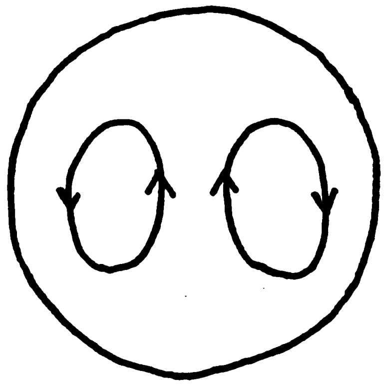
Jordan curves $\gamma_0, \gamma_1, \ldots, \gamma_{p-1}$ (and $\gamma_p = \gamma_0$) in $\overline{\mathbb{C}}$, considered up to orientation preserving homeomorphisms of $\overline{\mathbb{C}}$, and up tp simultaneous inversion of the orientation of γ_j. For example, if $p = 2$, there are two possibilities, which are indicated below.



Configuration A



Configuration B

Figure 1

If $p = 3$, there are 6 possibilities (see §3), and there are two even apart from the cyclic order and the orientations of γ_j.

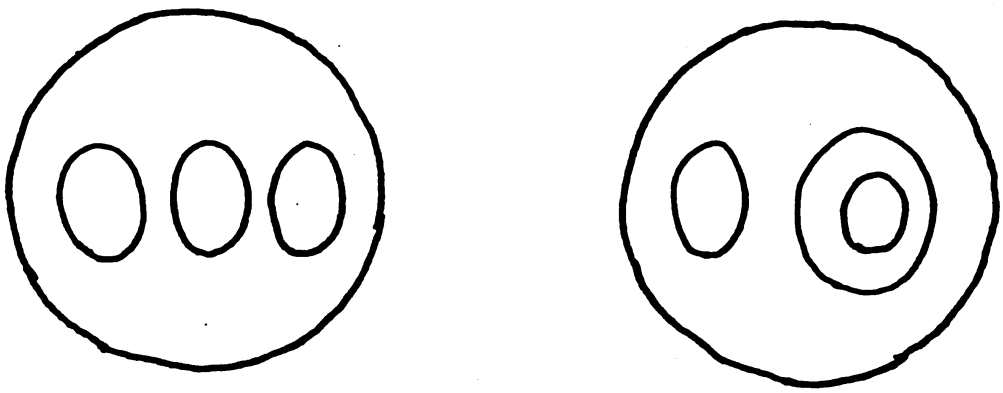


Figure 2

The cycle of Herman rings defines a configuration as follows: Fix r so that $r_0 < r < 1$, and define $\gamma_0 = \phi^{-1}\{z\in\mathbb{C}: |z| = r\}$ and $\gamma_j = f^j(\gamma_0)$. If one fix an orientation of γ_0, then f induces equivariant orientations of γ_j. Then $\{\gamma_j\}$ defines a configuration, which is independent of the choice of r and the orientation of γ_0.

The surgery done in [1,§6] suggests that not only the Herman rings themselves but also their preimages play an important role in the dynamics of f. To investigate the configuration of Herman rings and their preimages, we introduce in §1 a tree T_f and a piecewise linear map f_* on it, whose "derivative" is integer.

The primitive idea of the construction of the tree is the following. Suppose that there are disjoint annuli (doubly connected regions) on $\bar{\mathbb{C}}$. Then one can define a graph:

— vertices are connected components of the complement of the annuli;

— edges are the annuli;

— two vertices are connected by an edge if and only if the corresponding components have a common annulus whose boundary is contained in them.

This graph does not contain any non-trivial loop. (Since $\bar{\mathbb{C}}$ is simply connected!) So it can be considered as a tree.

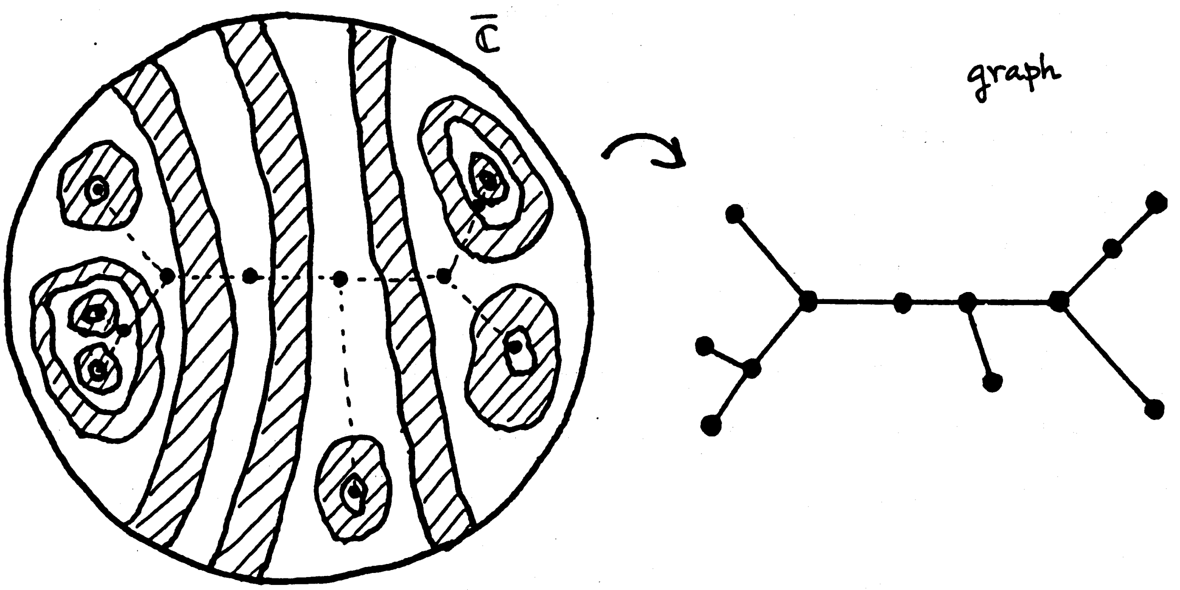


Figure 3

In the actual construction in §1, we take into consideration all "essential" preimages of Herman rings (so they might be infinite in general) and also some quantitative information such as the moduli of annuli. Conversely, it is possible to construct a rational function from such a tree under certain conditions (§4). As another application, we give an estimate from below on the degree of rational function realizing a tree (§5). For example, the configuration B above cannot be realized by any rational function of degree 3. Finally in §6, the tree construction is generalized to a rational function with super-attractive fixed points.

The proofs the results are not given in this paper. For the proofs, see [2] and [3].

1. DEFINITION OF THE TREE

Let f and A_j etc. be as in Introduction. Define

$C = \{f^n(z) \mid z$ is critical points of $f,\ n \geq 0\}$,

$\mathcal{A}_0 = \{$connected components of $(A_0 - \text{closure}(A_0 \cap C))\}$,

$\mathcal{A}' = \{$connected components of $f^{-n}(A) \mid A \in \mathcal{A}_0,\ n \geq 0\}$,

$B = \partial A_0 \cup \partial A_1 \cup \ldots \cup \partial A_{p-1}$, and

$\mathcal{A} = \{A \in \mathcal{A}' \mid f^n(A)$ separates B for all $n \geq 0\}$.

Note that $\mathcal{A}_0$, $\mathcal{A}'$ consist of annuli, hence the definition of $\mathcal{A}$ makes sense. (An annulus A *separates* a set B, if both components of $\bar{\mathbb{C}}-A$ intersect B.) For each annulus $A \in \mathcal{A}$, there exists a conformal map

$$\phi_A : A \to \{z \in \mathbb{C} \mid e^{-2\pi m_A} < |z| < 1\},$$

where $m_A > 0$. For $x, y \in \overline{\mathbb{C}}$, define

$$d(x,y) = \sum_{A \in \mathcal{A}} |\{s \in (0,m_A) |\ \phi_A^{-1}(e^{-2\pi s}) \text{ separartes } x \text{ and } y\}|,$$

where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$.

LEMMA 1 $d(x,y) < \infty$ *and* $d(\cdot,\cdot)$ *is continuous on* $\overline{\mathbb{C}}\times\overline{\mathbb{C}}$. *Moreover,* d *is a pseudo-metric on* $\overline{\mathbb{C}}$, *i.e., satisfies* $d(x,x) = 0$, $d(x,y) = d(y,x)$ *and* $d(x,z) \leq d(x,y) + d(y,z)$.

DEFINITION $$T_f = \overline{\mathbb{C}}/\sim,$$

where $x \sim y$ if and only if $d(x,y) = 0$. T_f is called the *tree associated with Herman rings*. Let $\pi:\overline{\mathbb{C}} \to T_f$ be the natural projection, and define $\tilde{d}(x,y) = d(\tilde{x},\tilde{y})$, where $\tilde{x} \in \pi^{-1}(x)$ and $\tilde{y} \in \pi^{-1}(y)$.

LEMMA 2 T_f *is a topologically finite tree (i.e., connected, one-dimensional finite simplicial complex containing no loop), and* $d(\cdot,\cdot)$ *is a linear metric on it (i.e., every arc in* T_f *is isometric to an interval).*

The fact that T_f becomes a tree is understood as in the construction of the graph in Introduction (see Figure 3). Although there might be an infinite number of annuli, they do not cause infinite branching. This is because each annulus separates the B, which consists of a finite number of connected components.

DEFINITION Define $f_*:T_f \to T_f$ by

$$f_*(x) = \pi \circ f(\partial\pi^{-1}(x)),$$

where $\partial\pi^{-1}(x)$ is the boundary of $\pi^{-1}(x)$ in $\overline{\mathbb{C}}$.

LEMMA 3 *f_* is well-defined and continuous.*

Note that for general $x \in T_f$, $\pi\circ f(\pi^{-1}(x))$ is not necessarily a point.

2. BASIC PROPERTIES

THEOREM 1 *Write $T = T_f$, $d = \bar{d}$, $F = f_*$. Then (T,d,F) has the following properties.*

(a) *(T,d) is a topologically finite tree with a linear metric.*

(b) *$F:T \to T$ is continuous.*

(c) *There exist a finite subset $\mathrm{Sing}(T,F)$ of T and a positive integer-valued function DF on $T-\mathrm{Sing}(T,F)$ which is constant on each component, such that:*

if x and y are in the same component of $T-\mathrm{Sing}(T,F)$, then

$$d(F(x),F(y)) = DF(x)\cdot d(x,y).$$

(d) *There exist arcs I_j $(j=0,\dots,p-1;\ I_p = I_0)$ in T such that:*

int I_j *(interior of I_j) are mutually disjoint and contain no branch points;*

$$F(I_j) = I_{j+1}, \quad \textit{and} \quad F^p|_{I_j} = \mathrm{id}.$$

(e) $T = \overline{\bigcup_{j,n\geq 0} F^{-n}(\mathrm{int}\ I_j)}$.

(f) *Every end point of T is an end point of an I_j.*

Here

$\mathrm{Sing}(T,F) = \{$end points, branch points of $T_f\}\cup\pi\{$critical points of $f\}$,

$I_j = \pi(A_j)$, called *periodic arcs*.

From Lemmas in the previous section, (a) and (b) follow. The assertions (d), (e) and (f) follow immediately from the construction and the properties of Herman rings. Let us see how (c) holds. Let $J \subset T_f$ be an arc which contains neither branch point nor the image by π of a critical point of f, and suppose that $\partial J \subset \bigcup_{A \in \mathcal{A}} \pi(A)$. Then $R = \pi^{-1}(J)$ is an annulus and $f: R \to f(R)$ is a covering map onto an annulus. If the degree of the covering is k, then the modulus of annulus in R is multiplied by k. Hence f_* satisfies $d(f_*(x), f_*(y)) = k \cdot d(x,y)$ for $x, y \in J$. So $DF = k$ on J.

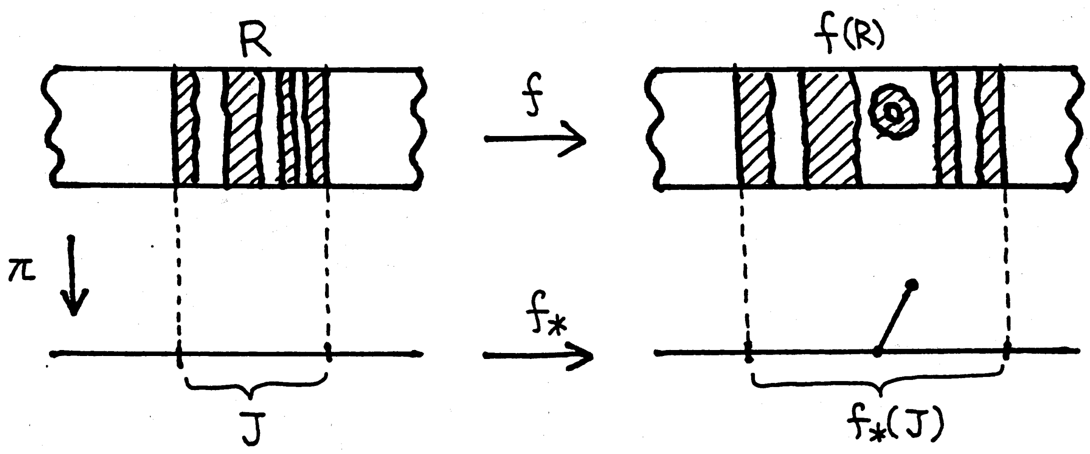


Figure 4

Let us introduce some notations, which are used in the following sections.

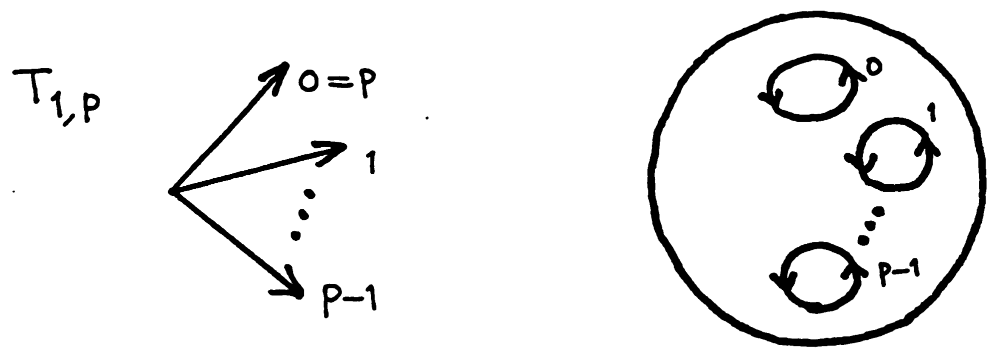
DEFINITION Let $\mathcal{B}_x$ = {components of $T - \{x\}$ }. Each $b \in \mathcal{B}_x$ are called a *branch* at x. For any $b \in \mathcal{B}_x$, there exists a unique branch $b' \in \mathcal{B}_{F(x)}$ such that if $y \in b$ is sufficiently near x, then $F(x) \in b'$. Then we write $F_\#(b) = b'$. Also define $DF(b) = \lim_{b \ni y \to x} DF(y)$.

3. EXAMPLES

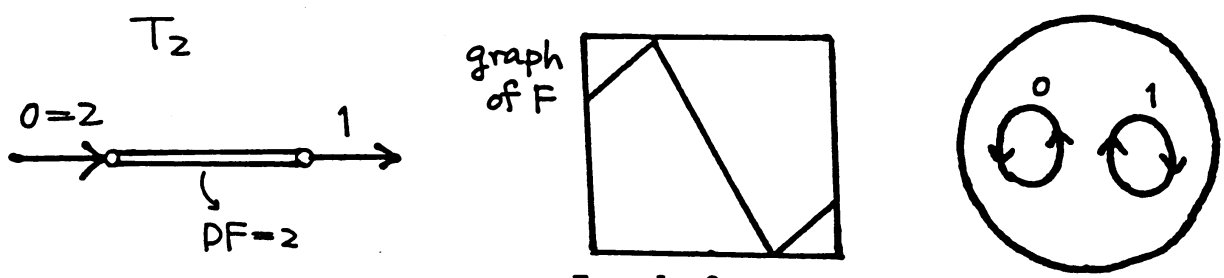
Here are some examples of trees which satisfy (a)-(f) of Theorem 1. Theorem 2 in §4 assures that there exist rational functions which realize these trees. The degree of such rational functions are denoted by "deg f".

Periodic arcs I_j are indicated by arrowed segments with numerals j $(j=0,1,\dots,p-1)$. The map F on each tree is supposed to be the simplest one which sends I_j to I_{j+1}. The doubled line denotes the part of the tree on which $DF \geq 2$.

EXAMPLE 1(p). Let p be a positive integer. The tree $T_{1,p}$ consists of only periodic arcs I_j of period p. This is the simplest tree and realized by a rational function of deg $f = 3$. On the right, the corresponding configuration is shown. If $p = 2$, this is Configuration A in Introduction.



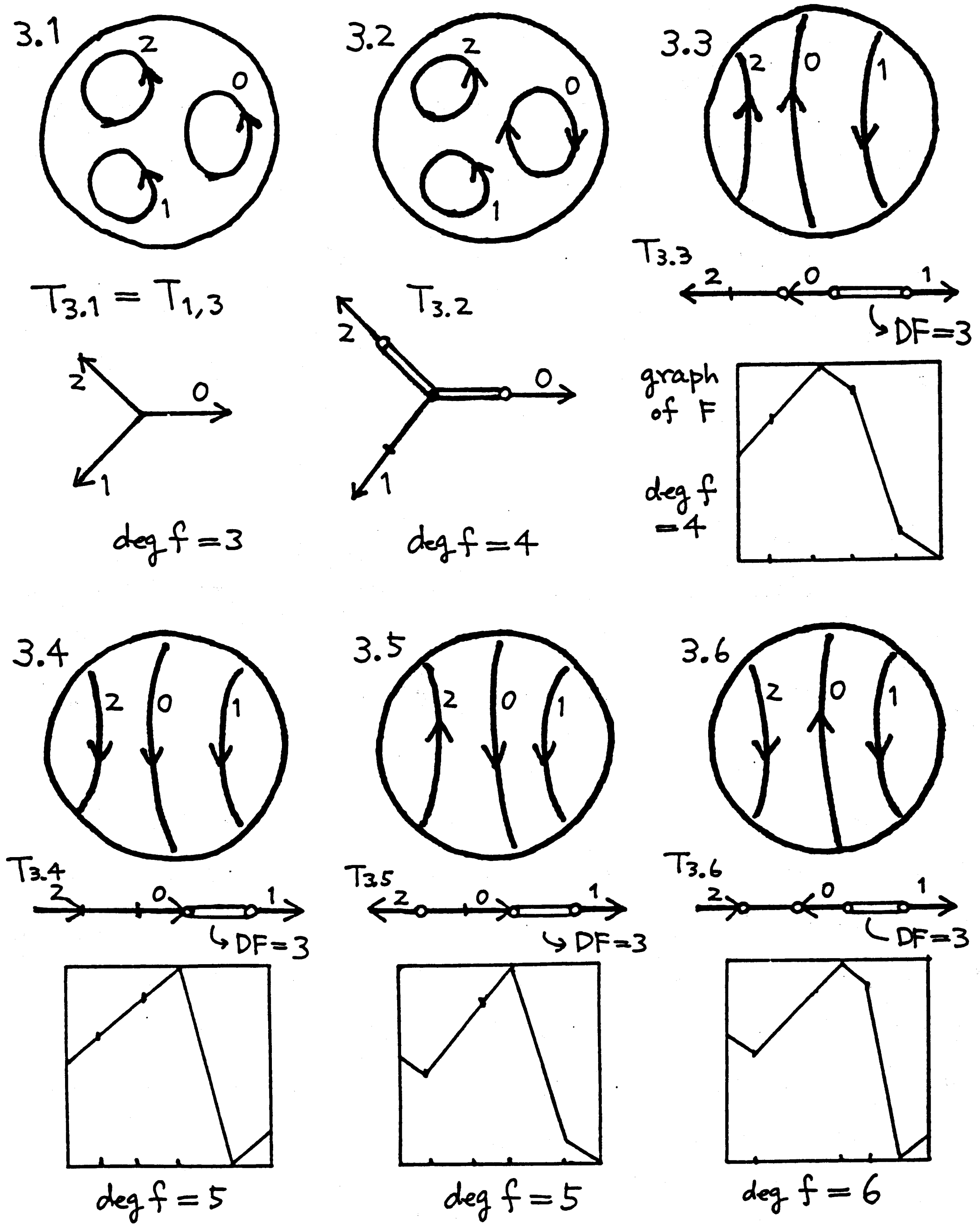
Example 1(p)



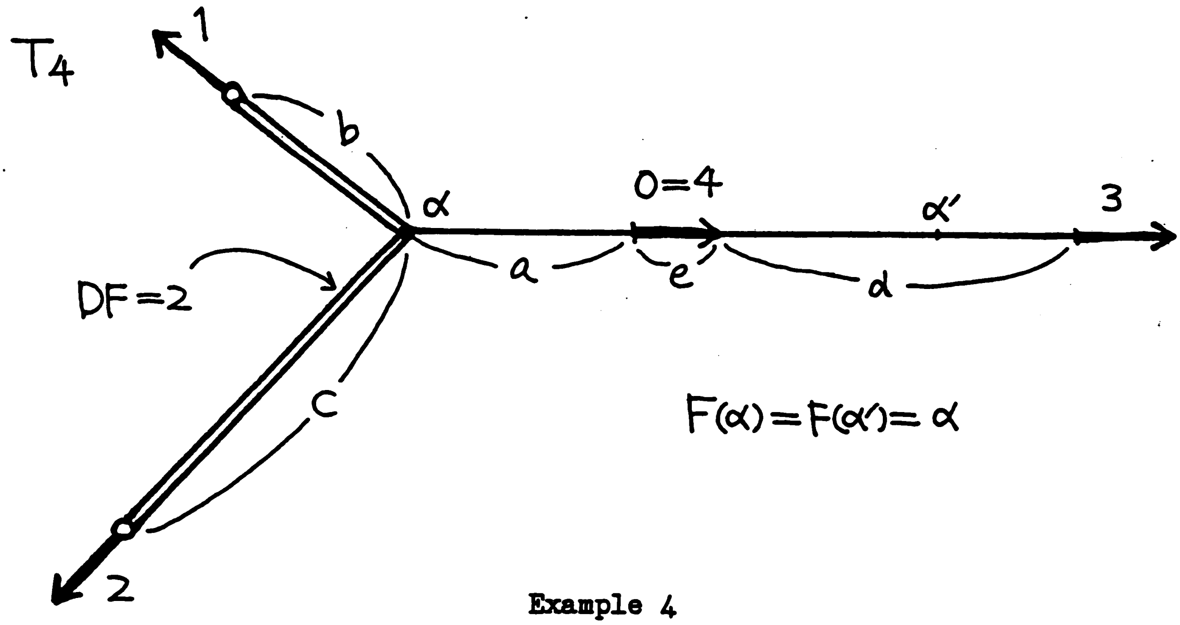
Example 2

EXAMPLE 2. The tree T_2 is (isometric to) an interval. Configuration B corresponds to this tree. deg $f = 4$.

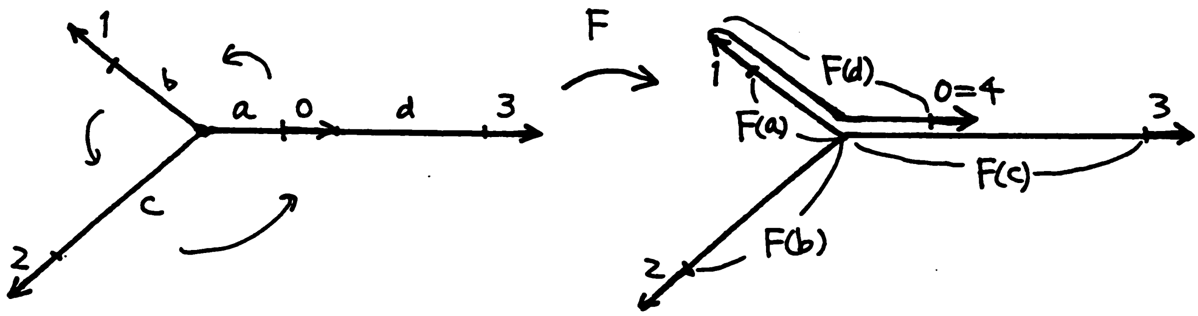
XAMPLE 3.1-3.6. Listed here are *all* possible configurations of period 3 elow each configuration, an example of tree which realizes it is shown. ote that deg f > 3 except Example 3.1.



EXAMPLE 4. The tree T_4 is of period 4 and $\deg f = 3$.



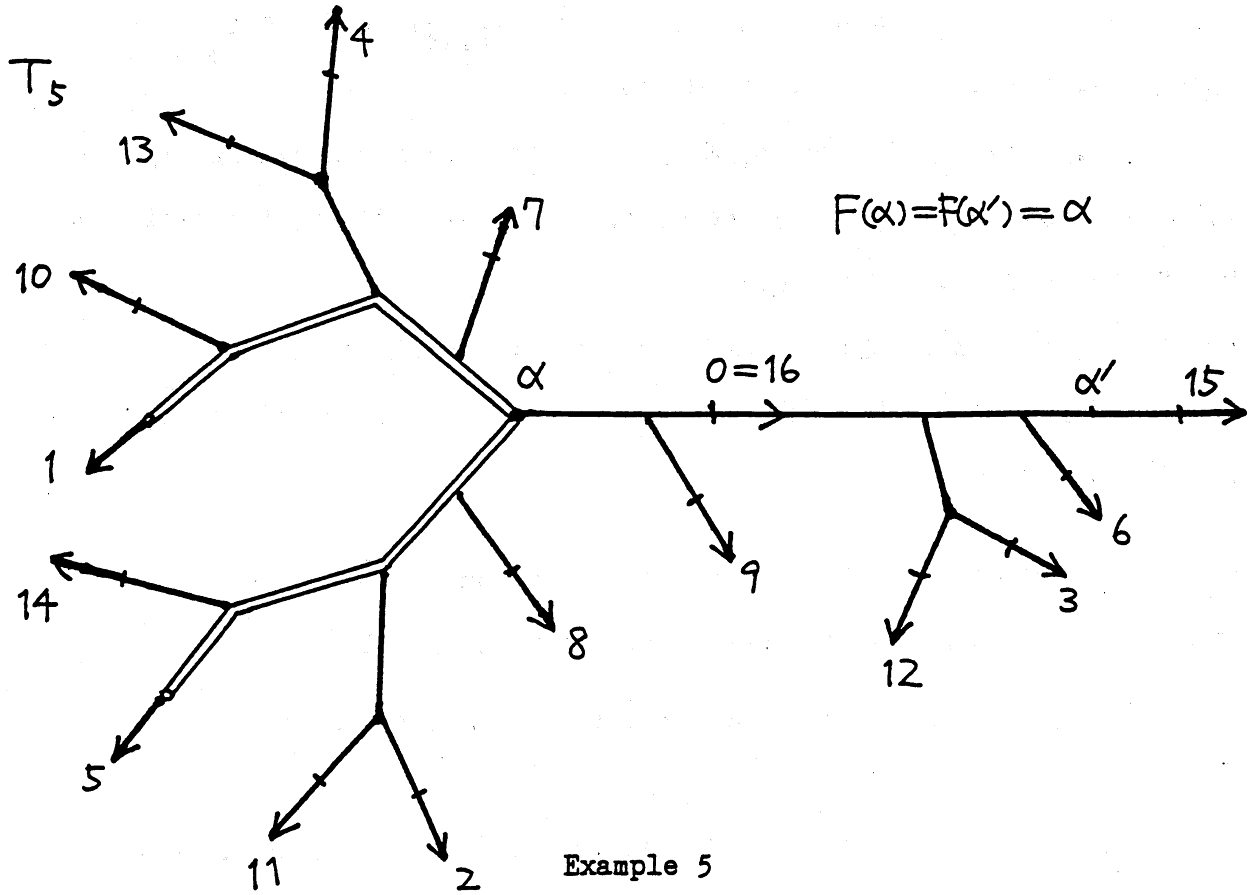
Example 4



Let e = the length of the periodic arc I_j, which is independent of j. It is possible to determine the lengths of segments a,b,c,d, which we also denote by the same letters a,b,c,d. First, F maps segment a (resp. d) onto b (resp. $I_1 \cup b \cup a$) with $DF = 1$, i.e., isometrically, hence we have $a = b$ (resp. $d = e+b+a$). Similarly, F maps segment b (resp. c) onto c (resp. $a \cup I_0 \cup d$) with $DF = 2$, hence $2b = c$ (resp. $2c = a+e+d$). Solving these equations, we get

$$a = 2e, \quad b = 2e, \quad c = 4e \quad \text{and} \quad d = 5e.$$

EXAMPLE 5. This tree is of period 16 and looks complicated, however it is realizable with $\deg f = 3$!



Example 5

4. REALIZATION PROBLEM

Given a tree (T,d,F) satisfying (a)-(f), it is natural to ask whether there exists a rational function realizing the tree or not. Under certain conditions, one can construct such a function, by means of surgery (cf. [3] and also [1]).

DEFINITION A *model* for (T,d,F) is a set $(X,\{p_b\},g)$ satisfying:

— X is a finite subset of T containing $\mathrm{Sing}(T,F)$, end points, branch points of F and ∂I_j;

— $\{p_b\} = \{p_b \mid b\in\mathcal{B}_x,\ x\in X\cup F(X)\}$ is a set of points of $(X\cup F(X))\times\overline{\mathbf{C}}$, such that P_b are distinct and $p_b \in \bar{\mathbb{C}}_x = \{x\} \times \bar{\mathbb{C}}$ if $b \in \mathcal{B}_x$;

— g is an analytic mapping from $X \times \bar{\mathbb{C}}$ to $F(X) \times \bar{\mathbb{C}}$ such that for $x \in X$, $b \in \mathcal{B}_x$, $g(\bar{\mathbb{C}}_x) = \bar{\mathbb{C}}_{F(x)}$, $g(p_b) = p_{F_*(b)}$ and $\deg_{p_b} g = DF(b)$ ($\deg_x g$ is the local degree of g at x);

— If $\partial I_0 = \{x_1, x_2\}$ and $b_i \in \mathcal{B}_{x_i}$ is the branch containing I_0 ($i=1,2$), then p_{b_1}, p_{b_2} are the centers of Siegel disks of rotation number θ, $-\theta$ respectively ($\theta \in R-Q$);

— Let $X_* = \{x \in X \mid$ the forward orbit of x is finite and contained in $X\}$. All stable regions (see Remark below) of $g|_{X_* \times \bar{\mathbb{C}}}$ are simply connected.

REMARK. For any $n \geq 0$, g^n can be defined on $X_* \times \bar{\mathbb{C}}$. So stable regions (maximal domains where g^n are equicontinuous) and Siegel disks are defined for $g|_{X_* \times \bar{\mathbb{C}}}$, like those for a single rational function.

THEOREM 2 *Suppose that* (T,d,F) *satisfies* (a)-(f) *and has a model* $(X,\{p_b\},g)$ *such that* $X = X_*$ *(hence all points of* $\mathrm{Sing}(T,F)$ *must be preperiodic). Then* (T,d,F) *is realizable by a rational function, i.e., there exists a rational function* f *having Herman rings* $A_0, A_1, \ldots, A_{p-1}$ *such that* (T_f,d,f_*) *defined as in* §1 *for* A_j *is conjugate to* (T,d,F) *by an isometry.*

It is expected that the condition $X = X_*$ can be eliminated.

A sphere can be constructed, by gluing together cylinders instead of components of $T-X$ and spheres instead of points of X. See Figure 5. One can construct the rational function f, appropriately defining a mapping and a conformal structure.

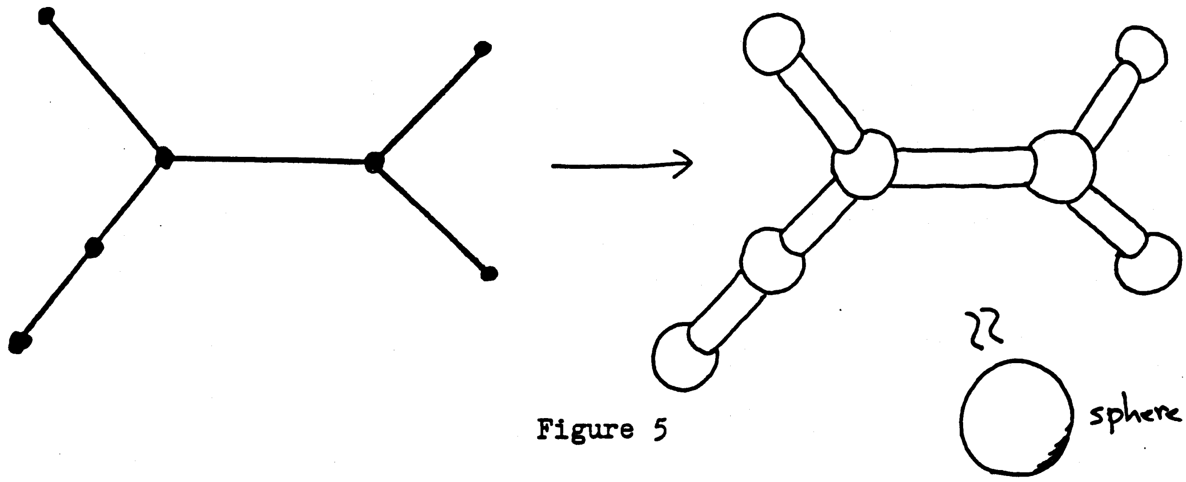


Figure 5

REMARK. It is easy to show that all the trees in §3 have models, hence they are realizable. The degree of f, $\deg f$, can be computed from the number of critical points of the model.

5. LOWER BOUND ON THE DEGREE OF RATIONAL FUNCTIONS REALIZING THE TREES

In this section, we give a lower bound on the degree of rational functions realizing trees or configurations. The degree measures, in a sense, the complexity of the tree. We suppose that (T,d,F) satisfy (a)-(f) and $\mathrm{Sing}(T,F)$ contains all the end points and the branch points of T.

DEFINITION For $x \in T$,

$$m_1(x) = \max\left\{ \sum_{b\in\mathcal{B}_x \cap F_\#^{-1}(b')} DF(b) \;\middle|\; b' \in \mathcal{B}_{F_\#(x)} \right\}, \quad m_2(x) = \sum_{b\in\mathcal{B}_x} (DF(b)-1),$$

$$n_c^*(x) = 2m_1(x)-2-m_2(x),$$

$$n_c(x) = \begin{cases} n_c^*(x) & \text{if } n_c^*(x) \geq 0 \\ 0 & \text{if } n_c^*(x) < 0 \text{ and is even} \\ 1 & \text{if } n_c^*(x) < 0 \text{ and is odd.} \end{cases}$$

$$n_c(X) = \sum_{x\in X} n_c(x) \quad \text{for any subset } X \text{ of } T.$$

$$\deg F = 1 + \tfrac{1}{2}n_c(T).$$

Note here that $n_c(x) = 0$ for $x \notin \mathrm{Sing}(T,F)$, hence the summation in $n_c(X)$ is actually finite.

THEOREM 3 *Suppose that* f *is a rational function with Herman rings and* T_f *and* $F = f_*$ *are constructed as in* §1. *Then*

$$\deg f \geq \deg F.$$

As for Examples in §3, computations show that $\deg F = \deg f$, in other words, they cannot be realized by rational functions of lower degree. On the other hand, there exists a tree for which the equality in the above estimate cannot hold.

STARTING FROM CONFIGURATIONS.

Given a configuration (of Herman rings), it is also possible to derive a lower bound on the degree of rational function realizing it. We will see it for Configuration B and Example 3.3. For this purpose, we need

LEMMA 4 (i) *Suppose* $\mathcal{B}_x = \{b_1, b_2\}$, $b_1 \neq b_2$. *Then*

$$n_c(x) = \begin{cases} |DF(b_1)-DF(b_2)| & \text{if } F_\#(b_1) \neq F_\#(b_2) \\ DF(b_1)+DF(b_2) & \text{if } F_\#(b_1) = F_\#(b_2). \end{cases}$$

(ii) *Suppose* L *is an open arc in* T *containing no branching points. Let* $\partial L = \{x_1, x_2\}$ *and* $b_i \in \mathcal{B}_{x_i}$ *be the branch at* x_i *containing* L

$(i=1,2)$. *Then*

$$n_c(L) \geq |DF(b_1)-DF(b_2)|.$$

Moreover, if $F|_L$ *is not injective, then*

$$n_c(L) \geq DF(b_1)+DF(b_2).$$

The proof is easy. There are more general estimates on $n_c(X)$ for several kinds of subsets X of T.

CONFIGURATION B. Let us consider a tree T corresponding to Configuration B. It follows from (f) that T is (isometric to) an interval. Furthermore, I_0 and I_1 do not intersect, since F is continuous and respects the orientation of I_j. See the figure below.

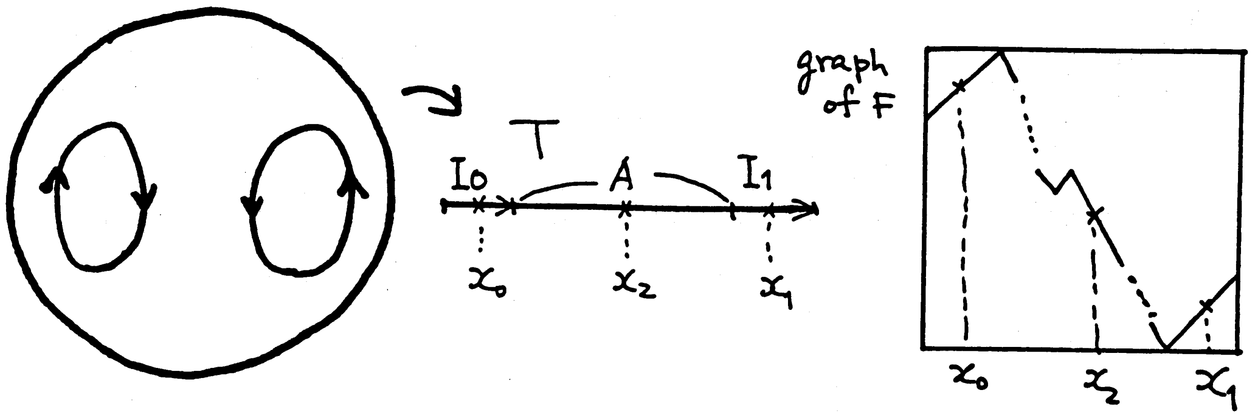


Figure 6

Let $A = T-I_0 \cup I_1$ and $\lambda = \max\{DF(x) \mid x \in A\text{-Sing}(T,F)\}$. Since F maps A onto the whole T, λ must be greater than one. Take $x_0, x_1, x_2 \in T$ so that $x_j \in \text{int } I_j$ $(j=0,1)$ and $x_2 \in A\text{-Sing}(T,F)$, $DF(x_2) = \lambda$. Applying Lemma 4 (ii) to arcs $L_0 = (x_0,x_2)$ and $L_1 = (x_2,x_1)$, we have

$$n_c(L_j) \geq 1+\lambda \geq 3,$$

since F is not injective at $L_j \cap \partial I_j$. Therefore,

$$\deg F \geq 1 + \frac{1}{2}(n_c(L_0)+n_c(L_1)) \geq 4.$$

Thus by Theorem 3, Configuration B cannot be realized by any rational function of degree less than 4.

CONFIGURATION 3.3. Let T be a tree corresponding to Configuration 3.3. It is similarly proved that T is an interval and I_j do not intersect.

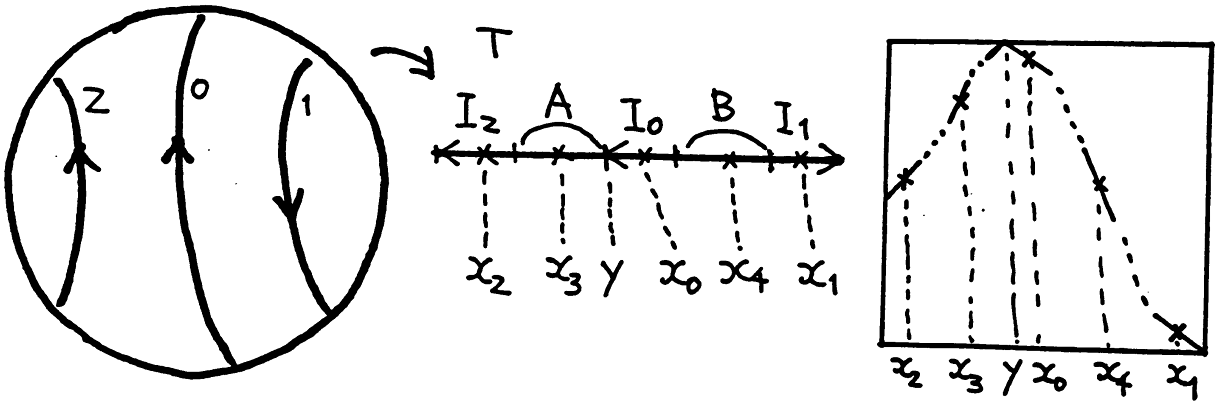


Figure 7

Define A and B as in the above figure and let $a = |A|$, $b = |B|$, $e = |I_j|$ and $\lambda = \max\{DF(x) \mid x \in A\text{-Sing}(T,F)\}$, $\mu = \max\{DF(x) \mid x \in B\text{-Sing}(T,F)\}$. Since $F(A) \supset B \cup I_1$ and $F(B) \supset A \cup I_0 \cup B$, we have

$$\lambda a \geq b+e \quad \text{and} \quad \mu b \geq a+e+b.$$

It follows that

$$\lambda \geq 1, \ \mu > 1 \quad \text{and} \quad \lambda+\mu \geq 4,$$

since $a,b,e > 0$. (If $\lambda = 1$ and $\mu = 2$, then e must be zero.)
Take $x_i \in T$ so that $x_j \in \operatorname{int} I_j$ $(j=0,1,2)$, $x_3 \in A\text{-Sing}(T,F)$, $DF(x_3) =$

λ and $x_4 \in$ B-Sing(T,F), $DF(x_4) = \mu$. Then by Lemma 4 (ii), we have

$$n_c((x_2,x_3)) \geq \lambda-1, \quad n_c((x_3,x_0)) \geq \lambda+1,$$
$$n_c((x_0,x_4)) \geq \mu-1 \quad \text{and} \quad n_c((x_4,x_1)) \geq \mu-1.$$

(Note that F is not injective at y.) Therefore,

$$\deg F \geq 1 + \frac{1}{2}\{(\lambda-1)+(\lambda+1)+2(\mu-1)\} \geq \lambda+\mu \geq 4.$$

So Configuration 3.3 cannot be realized by a rational function of degree less than 4.

It can be similarly checked that each "deg f" in Example 3.2 or 3.4-3.6 is *the lowest degree of rational functions realizing the configuration.*

6. GENERALIZATION

Changing suitably the choice of $\mathcal{A}_0$ and B, one can define a tree associated with (super)attractive basins or Siegel disks. For example, consider

$$f(z) = c\left(\frac{z}{1+z^2}\right)^3, \quad \text{with} \quad |c| > 67.$$

Then f has a superattractive fixed point 0. There exists an analytic local coordinate at 0 by which f is conjugate to $g: z \to z^3$. In this coordinate, one can choose a collection $\mathcal{A}_0$ of disjoint annuli of the form $\{z\in\mathbf{C} |\ r<|z|<r'\}$ $(0<r<r')$ such that if $A \in \mathcal{A}_0$, then $g(A) \in \mathcal{A}_0$ and A does not intersect the orbit of critical points. Let $B = \{0,\infty\}$ As in §1, $\mathcal{A}'$, $\mathcal{A}$ and the tree T are defined.

In this case, the tree T is an infinite interval $[-\infty,+\infty] = \mathbf{R}\cup\{\pm\infty\}$, where $\pi(0) = -\infty$, $\pi(\infty) = +\infty$ and $\pi(\pm 1) = 0$. The induced map f_* is the following: $f_*(x) = a - 3|x|$ on $\mathbb{R}$ $(a > 0)$, $f_*(\pm\infty) = -\infty$. It is not difficult to see that $\pi(J_f) = \{x \in \mathbb{R} \mid \{F^n(x) \mid n \geq 0\}$ is bounded$\}$, which is a Cantor set and J_f itself is homeomorphic to $\pi(J_f) \times S^1$.

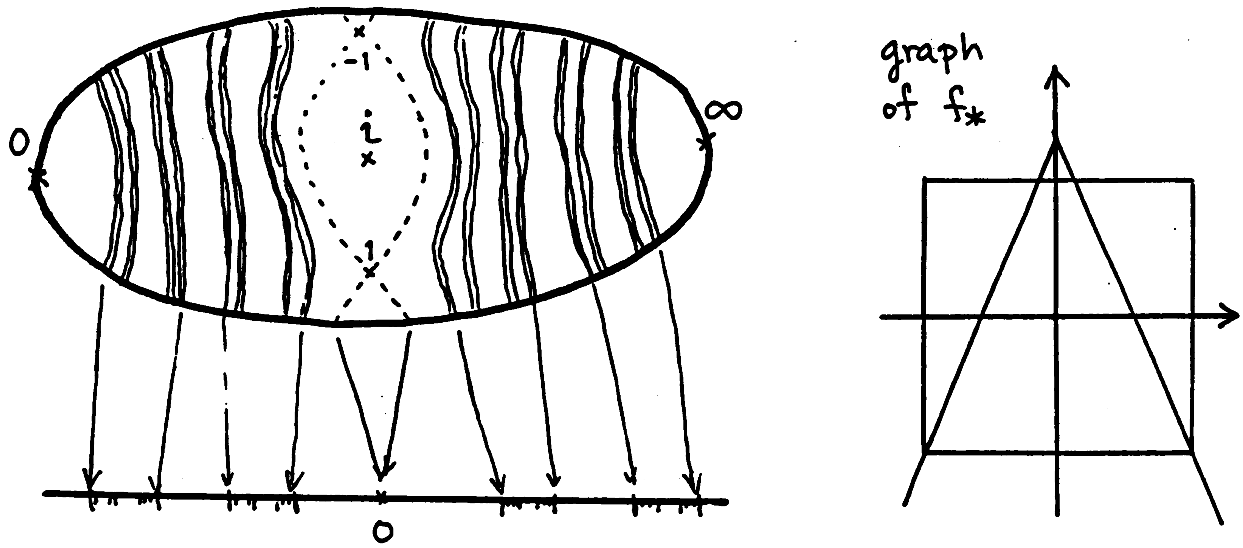


Figure 8

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