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An expression of harmonic vector fields of hyperbolic 3-cone-manifolds, in terms of the hypergeometric functions

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§0. Introduction.

By a hyperbolic 3-cone-manifold, we will mean an orientable Riemannian 3-manifold C of constant sectional curvature -1 with cone-type singularity along simple closed geodesics Σ. To each component of the singularity Σ, is associated a cone angle α. The subset $N := C-\Sigma$ has a smooth, incomplete hyperbolic structure whose metric completion is identical to the singular hyperbolic structure on C. A sufficiently small tubular neighborhood U of each component of Σ in N has the metric $dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r d\phi^2$, where r is the distance from the singular locus, ϕ is the distance along the singular locus, θ is the angular measure around the singular locus defined *modulo* α. Let Δ be the Laplacian of N with this metric.

In this paper, we give an explicit expression of a harmonic vector field v in U, by means of the hypergeometric functions. This expression can be obtained, since a simultaneous ordinary differential equation with variable r, which is a consequence of separation of variables of the partial differential equation $(\Delta + 4)\tau = 0$ on U (this is equivalent to the equation $\Delta v = 0$ on U), can be solved exactly by means of Riemann's P-equations, where τ denotes a 1-form dual to v. In fact, single ordinary differential equations of higher order, which are consequences of an elimination of functions from the simultaneous equation, are transformed by the substitution $z = (\frac{\sinh r}{\cosh r})^2$ into linear differential equations of Fuchsian type with three singular points. The differential operators of these equations are factorized into operators which express Riemann's P-equations (see Theorem 3.1). Also it is seen that some relationships between differential operators of Riemann's P-equations holds (see Theorem 3.2). Then the single ordinary differential equations can be solved without integration of functions. Moreover, if a parameter, which is obtained at the procedure of the separation of the variables, satisfies a genericity condition (Assumption 5.1), then fundamental systems of solutions of the simultaneous differential equation can be concretely obtained (see §5.1 and §5.3). Then the functions consisting the fundamental systems are explicitly represented by means of the hypergeometric functions. In this paper, we will give the explicit expression of τ (hence v), with some condition on parameters (see the begining of §3). See [2] for the general case.

§1. Definition of hyperbolic 3-cone-manifolds.

In this section, we give the definition of hyperbolic 3-cone-manifolds (see [1]).

Consider an 3-dimensional manifold C which can be triangulated so that the link of each simplex is piecewise linear homeomorphic to the standard sphere and give a complete path metric on C such that the restriction of the metric to each simplex is isomorphic to a geodesic simplex of constant sectional curvature -1. The manifold together with the metric above is called a hyperbolic 3-cone-manifold and denote it again by C.

The singular locus Σ of a hyperbolic 3-cone-manifold C consists of the points with no neighborhood isometric to a ball in a Riemannian manifold. It is a union of totally geodesic closed simplices of dimension 1. At each point of Σ in an open 1-simplex, there is a cone angle which is the sum of dihedral angles of 3-simplices containing the point. The subset $C - \Sigma$ has a smooth Riemannian metric of constant curvature -1, but this metric is incomplete near Σ.

In this paper we consider hyperbolic 3-cone-manifolds of the following type. Let M be a closed orientable 3-manifold and Σ be a link in M of k components. Let us denote by Σ^j the j-th component of the link Σ. We assume that M is the underlying space of a hyperbolic 3-cone-manifold C with singular locus Σ. The subset $N := C - \Sigma$ has a smooth Riemannian metric g with constant sectional curvature -1 which is incomplete near each component Σ^j of Σ. The metric completion of the hyperbolic structure on N gives rise to C. Each component Σ^j of Σ is a totally geodesic submanifold, and in cylindrical coordinates around Σ^j, the metric g has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r d\phi^2,$$

where r is the distance from the singular locus, ϕ is the distance along the singular locus, θ is the angular measure around the singular locus defined *modulo* α^j for some $\alpha^j \in (0, \infty)$. The number α^j is a cone angle at Σ^j.

§2. Laplacian of hyperbolic 3-cone-manifolds.

In this section, we state a situation which we will argue in this paper, and make a preparation on differential geometry. (See Rosenberg [4] for general reference on Riemannian geometry and Hodgson-Kerckhoff [3] for a special setting on hyperbolic 3-cone-manifolds.)

Let C be an orientable hyperbolic 3-cone-manifold with singularity Σ. We assume that the singular set Σ forms a link $\Sigma = \Sigma^1 \cup \ldots \cup \Sigma^k$ as in §1. The subset $N = C - \Sigma$ has a smooth Riemannian metric g with constant sectional curvature -1.

Let $\Omega^p(N)$ denote the space of smooth, real-valued p-forms of N. Let d be the usual exterior derivative of smooth real-valued forms on N:

$$d \;:\; \Omega^p(N) \;\to\; \Omega^{p+1}(N).$$

Let $*$ be the Hodge star operator defined by using the Riemannian metric g on N:

$$g(\phi, *\,\psi)\; dN = \phi \wedge \psi,$$

for any real-valued p-form ϕ and $(3-p)$-form ψ, where dN is the volume form of N. Let δ be the adjoint of d:

$$\delta \;:\; \Omega^p(N) \;\to\; \Omega^{p-1}(N).$$

Let Δ be the Laplacian on smooth real-valued forms for the Riemannian manifold N:

$$\Delta = d\delta + \delta d.$$

Let U be a sufficiently small neighborhood of a component of Σ. Let α be the cone angle in U along the component of Σ. If we use cylindrical coordinates, (r, θ, ϕ), the metric g in U is $dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r d\phi^2$ as described in §1. We assume that the boundary of U consists of the points whose distances from the component of Σ are same. We adapt $(\omega_1, \omega_2, \omega_3) := (dr, \sinh r d\theta, \cosh r d\phi)$ for the co-frame in U. We denote by (e_1, e_2, e_3) the orthonormal frame in U dual to $(\omega_1, \omega_2, \omega_3)$. Then $e_1 = \frac{\partial}{\partial r}$, $e_2 = \frac{1}{\sinh r}\frac{\partial}{\partial \theta}$ and $e_3 = \frac{1}{\cosh r}\frac{\partial}{\partial \phi}$. For notational convenience, we set $r = x^1$, $\theta = x^2$, and $\phi = x^3$. We express the metric g on U as $\sum_{i,j} g_{i,j} dx^i \otimes dx^j$. Then $g_{1,1} = 1$, $g_{2,2} = \sinh^2 x^1$, $g_{3,3} = \cosh^2 x^1$ and $g_{i,j} = 0$ $(i \neq j)$. The Christoffel symbol $\Gamma^i_{j,k}$ can be calculated by using the formula

$$\Gamma^i_{j,k} = \frac{1}{2}\sum_l g^{i,l}\left(\frac{\partial g_{j,l}}{\partial x^k} + \frac{\partial g_{k,l}}{\partial x^j} - \frac{\partial g_{j,k}}{\partial x^l}\right),$$

where $(g^{k,l}) = (g_{i,j})^{-1}$. The Levi-Civita connection ∇ can be calculated by

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \sum_i \Gamma^i_{j,k} \frac{\partial}{\partial x^i}.$$

A direct calculation shows that $\left(\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}\right)$ is equal to

$$\begin{pmatrix} 0 & \frac{\cosh r}{\sinh r}\frac{\partial}{\partial \theta} & \frac{\sinh r}{\cosh r}\frac{\partial}{\partial \phi} \\ \frac{\cosh r}{\sinh r}\frac{\partial}{\partial \theta} & -\sinh r \cosh r \frac{\partial}{\partial r} & 0 \\ \frac{\sinh r}{\cosh r}\frac{\partial}{\partial \phi} & 0 & -\sinh r \cosh r \frac{\partial}{\partial r} \end{pmatrix},$$

and moreover, that the connection 1-form (ω_λ^μ) is equal to

$$\begin{pmatrix} 0 & -\frac{\cosh r}{\sinh r}\omega_2 & -\frac{\sinh r}{\cosh r}\omega_3 \\ \frac{\cosh r}{\sinh r}\omega_2 & 0 & 0 \\ \frac{\sinh r}{\cosh r}\omega_3 & 0 & 0 \end{pmatrix}.$$

Now let v be a vector field in N which satisfies the differential equation $\Delta v = 0$ in U. Namely, v is harmonic in U. Let τ be the 1-form dual to v. Then, by Weitzenböck formula and the fact that the Ricci curvature of N is -2, τ satisfies $(\Delta + 4)\tau = 0$ in U.

If we express τ as

$$\tau = f(r, \theta, \phi)\omega_1 + g(r, \theta, \phi)\omega_2 + h(r, \theta, \phi)\omega_3$$

in U, then, by explicit calculation, we obtain the following (see [3] pp.26-27):

$$\begin{aligned}(\Delta + 4)\tau &= \left(-f_{rr} - \left(\frac{s}{c} + \frac{c}{s}\right) f_r + \left(\frac{s^2}{c^2} + \frac{c^2}{s^2} - 2\right) f - \frac{1}{s^2} f_{\theta\theta} - \frac{1}{c^2} f_{\phi\phi} + \frac{2c}{s^2} g_\theta + \frac{2s}{c^2} h_\phi\right) \omega_1 \\ &+ \left(-g_{rr} - \left(\frac{s}{c} + \frac{c}{s}\right) g_r + \left(\frac{c^2}{s^2} - 2\right) g - \frac{1}{s^2} g_{\theta\theta} - \frac{1}{c^2} g_{\phi\phi} - \frac{2c}{s^2} f_\theta\right) \omega_2 \\ &+ \left(-h_{rr} - \left(\frac{s}{c} + \frac{c}{s}\right) h_r + \left(\frac{s^2}{c^2} - 2\right) h - \frac{1}{s^2} h_{\theta\theta} - \frac{1}{c^2} h_{\phi\phi} - \frac{2s}{c^2} f_\phi\right) \omega_3, \qquad (1)\end{aligned}$$

where subscripts denote derivatives with respect to variables and $s := \sinh r$, $c := \cosh r$.

The 1-form τ in U satifies equivariance properties depending on the shape of the neighborhood U. Since the cone angle is equal to α, $\tau(r, \theta + \alpha, \phi) = \tau(r, \theta, \phi)$. If the component of Σ has length l, it further satisfies $\tau(r, \theta, \phi + l) = \tau(r, \theta + t, \phi)$, where t measures the twist in the normal direction along the component of Σ. The complex number $l + t\sqrt{-1}$ is so called the complex length of the component of the singular locus Σ.

Because of the decomposition of the Laplacian in U, we can use separation of variables, assuming that $f(r, \theta, \phi)$ equals a function $f(r)$ times a function on the torus $(= \partial U)$. Similarly for the other functions $g(r, \theta, \phi)$ and $h(r, \theta, \phi)$. It suffices further to decompose the functions on the torus into eigenfunctions of the Laplacian on the torus, which are of the forms $\cos(a\theta + b\phi)$ and $\sin(a\theta + b\phi)$, where $a := \frac{2\pi n}{\alpha}$ and $b := \frac{(2\pi m + \alpha t)}{l}$ $(n, m \in \mathbf{Z})$. We say such a 1-form τ is an eigenform of the Laplacian. Then, from the expression (1), we see that such a 1-form τ must be of the following type:

$$\begin{aligned} \tau &= f(r)\cos(a\theta+b\phi)\omega_1 + g(r)\sin(a\theta+b\phi)\omega_2 + h(r)\sin(a\theta+b\phi)\omega_3, \qquad (2)\\ \text{or } \tau &= f(r)\sin(a\theta+b\phi)\omega_1 + g(r)\cos(a\theta+b\phi)\omega_2 + h(r)\cos(a\theta+b\phi)\omega_3. \end{aligned}$$

Then, we can verify the following (see the equation (21) in [3]):

$$\begin{aligned} &(\Delta+4)\tau = 0 \\ \Leftrightarrow &\begin{cases} \bullet\ \ f''(r) + (\frac{s}{c}+\frac{c}{s})f'(r) - (2+\frac{s^2}{c^2}+\frac{c^2}{s^2}+\frac{a^2}{s^2}+\frac{b^2}{c^2})f(r) - \frac{2ac}{s^2}g(r) - \frac{2bs}{c^2}h(r) = 0, \\ \bullet\ \ g''(r) + (\frac{s}{c}+\frac{c}{s})g'(r) - (2+\frac{c^2}{s^2}+\frac{a^2}{s^2}+\frac{b^2}{c^2})g(r) - \frac{2ac}{s^2}f(r) = 0, \\ \bullet\ \ h''(r) + (\frac{s}{c}+\frac{c}{s})h'(r) - (2+\frac{s^2}{c^2}+\frac{a^2}{s^2}+\frac{b^2}{c^2})h(r) - \frac{2bs}{c^2}f(r) = 0. \end{cases} \end{aligned}$$

Put

$$z^{1/2} := \frac{\sinh r}{\cosh r} \qquad \text{and} \qquad (1-z)^{1/2} := \frac{1}{\cosh r},$$

then we have that $z = (\frac{\sinh r}{\cosh r})^2$ and that

$$\begin{aligned} &(\Delta+4)\tau = 0 \\ \Leftrightarrow &\begin{cases} \bullet\ \ 4z^2f''(z) + 4zf'(z) - (\frac{2z}{(1-z)^2}+\frac{1}{(1-z)^2}+\frac{z^2}{(1-z)^2}+\frac{a^2}{1-z}+\frac{b^2z}{1-z})f(z) \\ \quad -\frac{2a}{(1-z)^{3/2}}g(z) - \frac{2bz^{3/2}}{(1-z)^{3/2}}h(z) = 0, \\ \bullet\ \ 4z^2g''(z) + 4zg'(z) - (\frac{2z}{(1-z)^2}+\frac{1}{(1-z)^2}+\frac{a^2}{1-z}+\frac{b^2z}{1-z})g(z) - \frac{2a}{(1-z)^{3/2}}f(z) = 0, \\ \bullet\ \ 4z^2h''(z) + 4zh'(z) - (\frac{2z}{(1-z)^2}+\frac{z^2}{(1-z)^2}+\frac{a^2}{1-z}+\frac{b^2z}{1-z})h(z) - \frac{2bz^{3/2}}{(1-z)^{3/2}}f(z) = 0. \end{cases} \end{aligned} \qquad (3)$$

§3. How to solve a single differential equation.

In this paper, we only consider the case where $a \neq 0$ and $b \neq 0$ (see [2] for all the other cases). Then we can transform the system of the simultaneous equation (3) to a single differential equation of the 6-th order with respect to the function $h(z)$.

Let Λ be a subset of $\mathbf{C}$ defined by

$$\Lambda := \{z \in \mathbf{R}\ ;\ z < 0,\ 1 < z\ \}.$$

In the rest of this paper, let us regard the variable z in the equations in (3) as a complex number in the domain $\mathbf{C} - \Lambda$. Then $z^{1/2}$ and $(1-z)^{1/2}$ are single-valued functions on the domain $\mathbf{C} - \Lambda$.

By the third equation of (3), we have

$$f(z) = \frac{2}{b}z^{\frac{1}{2}}(1-z)^{\frac{3}{2}}R_1(z, \frac{d}{dz}, a, b)h(z), \qquad (4)$$

where we put

$$R_1(z, \frac{d}{dz}, a, b) := \frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + \left(\frac{a^2}{4z^2(z-1)} + \frac{b^2-1}{4z(z-1)} + \frac{-3}{4z(z-1)^2}\right).$$

By eliminating the function $f(z)$ from the first equation in (3) and the relation (4) between $f(z)$ and $h(z)$, we obtain a relation between $g(z)$ and $h(z)$ as follows:

$$g(z) = \frac{1}{ab} z^{\frac{-1}{2}} R_2(z, \frac{d}{dz}, a, b) h(z), \qquad (5)$$

where

$$R_2(z, \frac{d}{dz}, a, b) := -4z^3(z-1)^3 \frac{d^4}{dz^4} + 12z^2(z-1)^2(1-2z)\frac{d^3}{dz^3}$$
$$+2z(z-1)(a^2 + 15z - a^2z + b^2z - 13z^2 - b^2z^2)\frac{d^2}{dz^2}$$
$$+(a^2 - 4z - a^2z - b^2z + 3z^2 + 3b^2z^2 - 2z^3 - 2b^2z^3)\frac{d}{dz}$$
$$+\frac{1}{4z(z-1)}(8a^2 - a^4 - 26a^2z + 2a^4z - 2a^2b^2z - 8z^2 + 20a^2z^2 - a^4z^2 - 6b^2z^2 + 4a^2b^2z^2$$
$$-b^4z^2 + 6z^3 - 2a^2z^3 + 8b^2z^3 - 2a^2b^2z^3 + 2b^4z^3 - z^4 - 2b^2z^4 - b^4z^4).$$

By eliminating $f(z)$ and $g(z)$ from the second equation of (3) and the relations (4) and (5), we obtain the following equation which $h(z)$ should satisfy:

$$h^{(6)}(z) + \frac{9(-1+2z)}{z(z-1)} h^{(5)}(z) + \frac{72 - 3a^2 - 394z + 3a^2z - 3b^2z + 387z^2 + 3b^2z^2}{4z^2(z-1)^2} h^{(4)}(z)$$
$$+ \quad \frac{1}{2z^3(z-1)^3}(-12 + 3a^2 + 212z - 12a^2z + 6b^2z - 543z^2 + 9a^2z^2 - 18b^2z^2 + 348z^3$$
$$+ \quad 12b^2z^3)h^{(3)}(z) + \frac{1}{16z^4(z-1)^4}(-12a^2 + 3a^4 - 272z + 92a^2z - 6a^4z - 24b^2z + 6a^2b^2z$$
$$+ \quad 1792z^2 - 158a^2z^2 + 3a^4z^2 + 212b^2z^2 - 12a^2b^2z^2 + 3b^4z^2 - 2828z^3 + 78a^2z^3 - 362b^2z^3$$
$$+ \quad 6a^2b^2z^3 - 6a^4z^3 + 1323z^4 + 174b^2z^4 + 3b^4z^4)h''(z) + \frac{1}{16z^5(z-1)^5}(-12a^2 + 3a^4 + 24a^2z$$
$$- \quad 6a^4z - 56z^2 - 6a^2z^2 + 3a^4z^2 - 44b^2z^2 + 6a^2b^2z^2 - 3b^4z^2 + 136z^3 - 12a^2z^3 + 148b^2z^3$$
$$- \quad 12a^2b^2z^3 + 12b^4z^3 - 149z^4 + 6a^2z^4 - 164b^2z^4 + 6a^2b^2z^4 - 15b^4z^4 + 54z^5 + 60b^2z^5$$
$$+ \quad 6b^4z^5)h'(z) + \frac{1}{64z^6(z-1)^6}(-64a^2 + 20a^4 - a^6 + 232a^2z - 70a^4z + 3a^6z + 12a^2b^2z$$
$$- \quad 3a^4b^2z - 316a^2z^2 + 83a^4z^2 - 3a^6z^2 - 56a^2b^2z^2 + 9a^4b^2z^2 - 3a^2b^4z^2 - 16z^3 + 180a^2z^3$$
$$- \quad 36a^4z^3 + a^6z^3 - 48b^2z^3 + 82a^2b^2z^3 - 9a^4b^2z^3 - 18b^4z^3 + 9a^2b^4z^3 - b^6z^3 + 56z^4 - 35a^2z^4$$
$$+ \quad 3a^4z^4 + 100b^2z^4 - 44a^2b^2z^4 + 3a^4b^2z^4 + 47b^4z^4 - 9a^2b^4z^4 + 3b^6z^4 - 34z^5 + 3a^2z^5$$
$$- \quad 71b^2z^5 + 6a^2b^2z^5 - 40b^4z^5 + 3a^2b^4z^5 - 3b^6z^5 + 9z^6 + 19b^2z^6 + 11b^4z^6 + b^6z^6)h(z) = 0.$$

This is a differential equation of Fuchsian type with regular singularities at $z=0$, $z=1$ and $z=\infty$. The characteristic exponents are

$$\pm\tfrac{a}{2},\ \tfrac{2\pm a}{2},\ \tfrac{4\pm a}{2}\ (z=0);\ \tfrac{-1}{2},\ \tfrac{3}{2},\ \tfrac{1}{2},\ \tfrac{5}{2},\ \tfrac{2\pm\sqrt{5}}{2}\ (z=1);\ \tfrac{-1\pm b\sqrt{-1}}{2},\ \tfrac{1\pm b\sqrt{-1}}{2},\ \tfrac{3\pm b\sqrt{-1}}{2}\ (z=\infty).$$

Let $X(z,\frac{d}{dz},a,b)$ denote the differential operator which represents the equation above. Then the equation above is written as

$$X(z,\frac{d}{dz},a,b)h(z)=0. \qquad (6)$$

By direct computation, it can be verified that the theorem below holds:

Theorem 3.1. *The differential operator $X(z,\frac{d}{dz},a,b)$ is factorised as below:*

$$\begin{aligned} X(z,\frac{d}{dz},a,b) &= P_3(z,\frac{d}{dz},a,b)P_2(z,\frac{d}{dz},a,b)P_1(z,\frac{d}{dz},a,b) \\ &= P_3(z,\frac{d}{dz},-a,b)P_2(z,\frac{d}{dz},-a,b)P_1(z,\frac{d}{dz},a,b), \end{aligned}$$

where

$$\begin{aligned} P_1(z,\frac{d}{dz},a,b) &:= \frac{d^2}{dz^2}+\left(\frac{1}{z}-\frac{1}{z-1}\right)\frac{d}{dz}+\left(\frac{a^2}{4z^2(z-1)}+\frac{b^2+1}{4z(z-1)}+\frac{-1}{4z(z-1)^2}\right), \\ P_2(z,\frac{d}{dz},a,b) &:= \frac{d^2}{dz^2}+\left(\frac{2}{z}+\frac{4}{z-1}\right)\frac{d}{dz}+\left(\frac{a(a+2)}{4z^2(z-1)}+\frac{b^2+25}{4z(z-1)}+\frac{5}{4z(z-1)^2}\right), \\ P_3(z,\frac{d}{dz},a,b) &:= \frac{d^2}{dz^2}+\left(\frac{6}{z}+\frac{6}{z-1}\right)\frac{d}{dz}+\left(\frac{(a-6)(a+4)}{4z^2(z-1)}+\frac{b^2+121}{4z(z-1)}+\frac{21}{4z(z-1)^2}\right). \end{aligned}$$

The operators $P_i(z,\frac{d}{dz},a,b)$'s are ones which give Riemann's P-equations and the fundamental solutions are written by the Riemann P-function.

By direct computation, it can be checked that some relationship between $P_1(z,\frac{d}{dz},a,b)$ and $P_2(z,\frac{d}{dz},a,b)$ holds:

Theorem 3.2. *Put*

$$P_4(z,\frac{d}{dz},a,b):=\frac{d^2}{dz^2}+\left(\frac{3}{z}+\frac{3}{z-1}\right)\frac{d}{dz}+\left(\frac{a^2-4}{4z^2(z-1)}+\frac{b^2+25}{4z(z-1)}+\frac{-1}{4z(z-1)^2}\right).$$

Then, the following equation as operators holds:

$$P_1(z,\frac{d}{dz},a,b)z^2(z-1)^4P_4(z,\frac{d}{dz},a,b)-z^2(z-1)^4P_3(z,\frac{d}{dz},a,b)P_2(z,\frac{d}{dz},a,b)=1.$$

We obtain a corollary of Theorem 3.1 and Theorem 3.2:

Corollary 3.3. *Solutins of the equation*

$$X(z, \frac{d}{dz}, a, b)u(z) = 0$$

are written as follows:

$$u(z) = v(z) + z^2(z-1)^4 P_4(z, \frac{d}{dz}, a, b)(w^+(z) + w^-(z)),$$

where $v(z)$, $w^+(z)$ and $w^-(z)$ are solutions of the equations $P_1(z, \frac{d}{dz}, a, b)v(z) = 0$, $P_2(z, \frac{d}{dz}, a, b)w^+(z) = 0$ and $P_2(z, \frac{d}{dz}, -a, b)w^-(z) = 0$ respectively. To the contrary, if $v(z)$, $w^+(z)$ and $w^-(z)$ are solutions of the equations $P_1(z, \frac{d}{dz}, a, b)v(z) = 0$, $P_2(z, \frac{d}{dz}, a, b)w^+(z) = 0$ and $P_2(z, \frac{d}{dz}, -a, b)w^-(z) = 0$ respectively, then

$$u(z) := v(z) + z^2(z-1)^4 P_4(z, \frac{d}{dz}, a, b)(w^+(z) + w^-(z))$$

satisfies the equation $X(z, \frac{d}{dz}, a, b)u(z) = 0$.

§4. Fundamental systems of solutions of the simulataneous differential equation.

For a solution $h(z)$ which is given by Corollrary 3.3, the corresponding functions $f(z)$ and $g(z)$ are obtained by the relations (4) and (5) respectively. These relations are expressed by the operators $R_1(z, \frac{d}{dz}, a, b)$ and $R_2(z, \frac{d}{dz}, a, b)$, the order of which are 2 and 4 respectively. We will see that each of these operators can be reduced to an operator of lower order and then give a simple expression of solutions.

By direct computation, we can verify the following lemma on the operator $R_2(z, \frac{d}{dz}, a, b)$:

Lemma 4.1. *Put*

$$Q(z, \frac{d}{dz}, a, b) := \frac{d^2}{dz^2} + \left(\frac{2}{z} + \frac{4}{z-1}\right)\frac{d}{dz} + \left(\frac{a^2}{4z^2(z-1)} + \frac{b^2+25}{4z(z-1)} + \frac{5}{4z(z-1)^2}\right).$$

Then the following equation holds:

$$R_2(z, \frac{d}{dz}, a, b) - a^2 = -4z^3(z-1)^3 Q(z, \frac{d}{dz}, a, b)P_1(z, \frac{d}{dz}, a, b).$$

Let the operators $P_i(z, \frac{d}{dz}, a, b)$, $P_i(z, \frac{d}{dz}, -a, b)$ and $R_i(z, \frac{d}{dz}, a, b)$ be abbreviated as P_i. P_i^- and R_i respectively.

By (4) and (5), the components of each solution $(f(z), g(z), h(z))$ of the simultaneous equation (3) which corresponds to a solution $bv(z)$ of the equation $P_1 v(z) = 0$ are

$$\begin{aligned} f(z) &= \frac{2}{b} z^{\frac{1}{2}}(1-z)^{\frac{3}{2}} R_1 bv(z) = -2z^{\frac{1}{2}}(1-z)^{\frac{1}{2}} \left(\frac{d}{dz} - \frac{1}{2(z-1)}\right) v(z), \\ g(z) &= \frac{1}{ab} z^{\frac{-1}{2}} R_2 bv(z) = az^{\frac{-1}{2}} v(z), \qquad h(z) = bv(z). \end{aligned}$$

The second equation on $f(z)$ is verified by dividing the operator by P_1 from the right and the second equation on $g(z)$ is seen by Lemma 4.1.

Next, we will argue a representation of solutions which correspond to solutions of the equations $P_2 w(z) = 0$ and $P_2^- w^-(z) = 0$.

By Cor 3.3, the components $(f(z), g(z), h(z))$ of each solution of the simultaneous equation (3) which corresponds to a solution $bw^+(z)$ of the equation $P_2 w^+ = 0$ are

$$\begin{aligned} f(z) &= \frac{2}{b} z^{\frac{1}{2}}(1-z)^{\frac{3}{2}} R_1 z^2 (z-1)^4 P_4 bw^+(z) \\ &= az^{\frac{1}{2}}(1-z)^{\frac{7}{2}} \left(\frac{d}{dz} + \frac{a+2}{2z} + \frac{3}{2(z-1)} - \frac{a^2+b^2}{2a(z-1)}\right) w^+(z), \\ g(z) &= \frac{1}{ab} z^{\frac{-1}{2}} R_2 z^2 (z-1)^4 P_4 bw^+(z) \\ &= az^{\frac{1}{2}}(1-z)^3 \left(\frac{d}{dz} + \frac{a+2}{2z} + \frac{3}{2(z-1)} - \frac{2}{a(z-1)}\right) w^+(z), \\ h(z) &= z^2 (z-1)^4 P_4 bw^+(z) \\ &= bz(1-z)^3 \left(\frac{d}{dz} + \frac{a+2}{2z} + \frac{3}{2(z-1)}\right) w^+(z). \end{aligned}$$

All the second equations above can be verified by dividing the operators by the operator P_2 from the right. Let $T_1(a,b)$, $T_2(a)$ and $T_3(a,b)$ denote the operators which correspond to $h(z)$, $g(z)$ and $f(z)$ respectively;

$$\begin{aligned} T_1(a,b) &:= T_1(z, \frac{d}{dz}, a, b) := z^{\frac{1}{2}}(1-z)^{\frac{7}{2}} \left(a\frac{d}{dz} + \frac{a(a+2)}{2z} + \frac{3a}{2(z-1)} - \frac{a^2+b^2}{2(z-1)}\right), \\ T_2(a) &:= T_2(z, \frac{d}{dz}, a) := az^{\frac{1}{2}}(1-z)^3 \left(\frac{d}{dz} + \frac{a+2}{2z} + \frac{3}{2(z-1)} - \frac{2}{a(z-1)}\right), \\ T_3(a,b) &:= T_3(z, \frac{d}{dz}, a, b) := bz(1-z)^3 \left(\frac{d}{dz} + \frac{a+2}{2z} + \frac{3}{2(z-1)}\right). \end{aligned}$$

Then the equations above are written as

$$f(z) = T_1(a,b) w^+(z), \quad g(z) = T_2(a) w^+(z), \quad h(z) = T_3(a,b) w^+(z).$$

In the same manner, the components $(f(z), g(z), h(z))$ of each solution of the simultaneous equation (3) which corresponds to a solution $bw^-(z)$ of the equation $P_2^- w^-(z) = 0$ are represented as follows, by using the operators T_1, T_2 and T_3,

$$f(z) = T_1(-a,b)w^-(z), \quad g(z) = -T_2(-a)w^-(z), \quad h(z) = T_3(-a,b)w^-(z).$$

Summarizing above, we have the proposition below:

Proposition 4.2. *Let $\{v_1(z), v_2(z)\}$, $\{w_1^+(z), w_2^+(z)\}$ and $\{w_1^-(z), w_2^-(z)\}$ be fundamental systems of solutions of the equations $P_1v(z) = 0$, $P_2w^+(z) = 0$ and $P_2^-w^-(z) = 0$ respectively. For each $i \in \{1, 2\}$, put*

$$\begin{aligned}
(f_i(z), g_i(z), h_i(z)) &:= \left(-2z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\left(\tfrac{d}{dz} - \tfrac{1}{2(z-1)}\right)v_i(z), az^{-\frac{1}{2}}v_i(z), bv_i(z)\right), \\
(f_{i+2}(z), g_{i+2}(z), h_{i+2}(z)) &:= (T_1(a,b)w_i^+(z), T_2(a)w_i^+(z), T_3(a,b)w_i^+(z)), \\
(f_{i+4}(z), g_{i+4}(z), h_{i+4}(z)) &:= (T_1(-a,b)w_i^-(z), -T_2(-a)w_i^-(z), T_3(-a,b)w_i^-(z)),
\end{aligned}$$

Then the 6 triples $\{(f_j(z), g_j(z), h_j(z)); j = 1, \ldots, 6\}$ forms a fundamental system of solutions of the simultaneous equations (3) *on the domain* $\mathbf{C} - \Lambda$.

§5. Explicit expressions of the fundamental systems of solutions.

In this section, by imposing a genericity condition on the parameter a, we will give an explicit expression of the fundamental systems of solutions of the differential equation (3), by means of the hypergeometric functions.

The characteristic exponents of the equations $P_1v(z) = 0$, $P_2w^+(z) = 0$ and $P_2^-w^-(z) = 0$ are

- $\frac{a}{2}, \frac{-a}{2}$ $(z=0)$; $\frac{2+\sqrt{5}}{2}, \frac{2-\sqrt{5}}{2}$ $(z=1)$; $\frac{-1+b\sqrt{-1}}{2}, \frac{-1-b\sqrt{-1}}{2}$ $(z=\infty)$,
- $\frac{a}{2}, \frac{-a-2}{2}$ $(z=0)$; $\frac{-1}{2}, \frac{-5}{2}$ $(z=1)$; $\frac{5+b\sqrt{-1}}{2}, \frac{5-b\sqrt{-1}}{2}$ $(z=\infty)$

and

- $\frac{a-2}{2}, \frac{-a}{2}$ $(z=0)$; $\frac{-1}{2}, \frac{-5}{2}$ $(z=1)$; $\frac{5+b\sqrt{-1}}{2}, \frac{5-b\sqrt{-1}}{2}$ $(z=\infty)$

respectively. The differences of the exponents at $z = 0$ are a, $-a$; $a+1$, $-a-1$ and $a-1$, $-a+1$ respectively.

We will put the following assumption to impose the genericity condition on a:

Assumption 5.1. The parameter a is not an integer.

The condition of Assumption 5.1 is equivalent to that no one of a, $-a$, $a+1$, $-a-1$, $a-1$, $-a+1$ is a negative integer.

Then we can and will choose fundamental systems of solutions explicitly as follows:

$$
\begin{aligned}
v_1(z) &:= v_1(z,a,b) := z^{\frac{a}{2}}(1-z)^{\frac{2+\sqrt{5}}{2}} F\left(\tfrac{a+1+b\sqrt{-1}+\sqrt{5}}{2}, \tfrac{a+1-b\sqrt{-1}+\sqrt{5}}{2}; a+1; z\right),\\
v_2(z) &:= v_1(z,-a,b) = z^{\frac{-a}{2}}(1-z)^{\frac{2+\sqrt{5}}{2}} F\left(\tfrac{-a+1+b\sqrt{-1}+\sqrt{5}}{2}, \tfrac{-a+1-b\sqrt{-1}+\sqrt{5}}{2}; -a+1; z\right),\\
w_1^+(z) &:= w_1^+(z,a,b) := z^{\frac{a}{2}}(1-z)^{\frac{-1}{2}} F\left(\tfrac{a+4+b\sqrt{-1}}{2}, \tfrac{a+4-b\sqrt{-1}}{2}; a+2; z\right),\\
w_2^+(z) &:= w_1^+(z,-a-2,b) = z^{\frac{-a-2}{2}}(1-z)^{\frac{-1}{2}} F\left(\tfrac{-a+2+b\sqrt{-1}}{2}, \tfrac{-a+2-b\sqrt{-1}}{2}; -a; z\right),\\
w_1^-(z) &:= w_1^+(z,-a,b) = z^{\frac{-a}{2}}(1-z)^{\frac{-1}{2}} F\left(\tfrac{-a+4+b\sqrt{-1}}{2}, \tfrac{-a+4-b\sqrt{-1}}{2}; -a+2; z\right),\\
w_2^-(z) &:= w_1^+(z,a-2,b) = z^{\frac{a-2}{2}}(1-z)^{\frac{-1}{2}} F\left(\tfrac{a+2+b\sqrt{-1}}{2}, \tfrac{a+2-b\sqrt{-1}}{2}; a; z\right),
\end{aligned}
$$

where $F(\alpha,\beta;\gamma;z)$ is the hypergeometric function.

Remark. If we do not put Assumption 5.1, we may employ the standard procedure in the theory of hypergeometric functions, that is, we may have to take logarithmic terms to form the fundamental systems of solutions.

Then, by Proposition 4.2 with using the formula

$$\frac{d}{dz}F(\alpha,\beta;\gamma;z) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;z),$$

we obtain explicitly the fundamental system of solutions of the simultaneous equation (3):

$$
\begin{aligned}
f_1(z) &= -2z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\left(\tfrac{d}{dz} - \tfrac{1}{2(z-1)}\right)v_1(z) = -2z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\left(\tfrac{d}{dz} - \tfrac{1}{2(z-1)}\right)v_1(z,a,b)\\
&= z^{\frac{a-1}{2}}(1-z)^{\frac{1+\sqrt{5}}{2}}(z+az+\sqrt{5}z-a)F\left(\tfrac{a+1+b\sqrt{-1}+\sqrt{5}}{2}, \tfrac{a+1-b\sqrt{-1}+\sqrt{5}}{2}; a+1; z\right)\\
&\quad -\tfrac{(a+1+b\sqrt{-1}+\sqrt{5})(a+1-b\sqrt{-1}+\sqrt{5})}{2(a+1)} z^{\frac{a+1}{2}}(1-z)^{\frac{3+\sqrt{5}}{2}}\\
&\quad \times F\left(\tfrac{a+3+b\sqrt{-1}+\sqrt{5}}{2}, \tfrac{a+3-b\sqrt{-1}+\sqrt{5}}{2}; a+2; z\right)\\
&=: f_1(z,a,b),\\
g_1(z) &= az^{\frac{-1}{2}}v_1(z) = az^{\frac{-1}{2}}v_1(z,a,b)\\
&= az^{\frac{a-1}{2}}(1-z)^{\frac{2+\sqrt{5}}{2}}F\left(\tfrac{a+1+b\sqrt{-1}+\sqrt{5}}{2}, \tfrac{a+1-b\sqrt{-1}+\sqrt{5}}{2}; a+1; z\right)\\
&=: g_1(z,a,b),\\
h_1(z) &= bv_1(z) = bv_1(z,a,b)\\
&= bz^{\frac{a}{2}}(1-z)^{\frac{2+\sqrt{5}}{2}}F\left(\tfrac{a+1+b\sqrt{-1}+\sqrt{5}}{2}, \tfrac{a+1-b\sqrt{-1}+\sqrt{5}}{2}; a+1; z\right)\\
&=: h_1(z,a,b),
\end{aligned}
$$

$$
\begin{aligned}
f_2(z) &= -2z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\left(\tfrac{d}{dz} - \tfrac{1}{2(z-1)}\right)v_2(z) = -2z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}\left(\tfrac{d}{dz} - \tfrac{1}{2(z-1)}\right)v_2(z,-a,b)\\
&= f_1(z,-a,b),\\
g_2(z) &= az^{\frac{-1}{2}}v_2(z) = az^{\frac{-1}{2}}v_1(z,-a,b) = -g_1(z,-a,b),\\
h_2(z) &= bv_2(z) = bv_1(z,-a,b) = h_1(z,-a,b),
\end{aligned}
$$

$$
\begin{aligned}
f_3(z) &= T_1(a,b)w_1^+(z) = T_1(a,b)w_1^+(z,a,b) \\
&= \tfrac{1}{2}z^{\frac{a-1}{2}}(1-z)^2(-4az - a^2z + b^2z + 2a + 2a^2)F\left(\tfrac{a+4+b\sqrt{-1}}{2}, \tfrac{a+4-b\sqrt{-1}}{2}; a+2; z\right) \\
&\quad + \tfrac{a(a+4+b\sqrt{-1})(a+4-b\sqrt{-1})}{4(a+2)} z^{\frac{a+1}{2}}(1-z)^3 F\left(\tfrac{a+6+b\sqrt{-1}}{2}, \tfrac{a+6-b\sqrt{-1}}{2}; a+3; z\right) \\
&=: f_3(z,a,b), \\
g_3(z) &= T_2(a)w_1^+(z) = T_2(a)w_1^+(z,a,b) \\
&= \tfrac{1}{2}z^{\frac{a-1}{2}}(1-z)^{\frac{3}{2}}(4z - 4az - 2a^2z + 2a + 2a^2)F\left(\tfrac{a+4+b\sqrt{-1}}{2}, \tfrac{a+4-b\sqrt{-1}}{2}; a+2; z\right) \\
&\quad + \tfrac{a(a+4+b\sqrt{-1})(a+4-b\sqrt{-1})}{4(a+2)} z^{\frac{a+1}{2}}(1-z)^{\frac{5}{2}} F\left(\tfrac{a+6+b\sqrt{-1}}{2}, \tfrac{a+6-b\sqrt{-1}}{2}; a+3; z\right) \\
&=: g_3(z,a,b), \\
h_3(z) &= T_3(a,b)w_1^+(z) = T_3(a,b)w_1^+(z,a,b) \\
&= \tfrac{b}{2}z^{\frac{a}{2}}(1-z)^{\frac{3}{2}}(-4z - 2az + 2 + 2a)F\left(\tfrac{a+4+b\sqrt{-1}}{2}, \tfrac{a+4-b\sqrt{-1}}{2}; a+2; z\right) \\
&\quad + \tfrac{b(a+4+b\sqrt{-1})(a+4-b\sqrt{-1})}{4(a+2)} z^{\frac{a+2}{2}}(1-z)^{\frac{5}{2}} F\left(\tfrac{a+6+b\sqrt{-1}}{2}, \tfrac{a+6-b\sqrt{-1}}{2}; a+3; z\right) \\
&=: h_3(z,a,b),
\end{aligned}
$$

$$
\begin{aligned}
f_4(z) &= T_1(a,b)w_2^+(z) = T_1(a,b)w_1^+(z,-a-2,b) \\
&= \tfrac{(a^2+b^2-2a)}{2} z^{\frac{-a-1}{2}}(1-z)^2 F\left(\tfrac{-a+2+b\sqrt{-1}}{2}, \tfrac{-a+2-b\sqrt{-1}}{2}; -a; z\right) \\
&\quad - \tfrac{(-a+2+b\sqrt{-1})(-a+2-b\sqrt{-1})}{4} z^{\frac{-a-1}{2}}(1-z)^3 F\left(\tfrac{-a+4+b\sqrt{-1}}{2}, \tfrac{-a+4-b\sqrt{-1}}{2}; -a+1; z\right) \\
&=: f_4(z,a,b), \\
g_4(z) &= T_2(a)w_2^+(z) = T_2(a)w_1^+(z,-a-2,b) \\
&= (2-a)z^{\frac{-a-1}{2}}(1-z)^{\frac{3}{2}} F\left(\tfrac{-a+2+b\sqrt{-1}}{2}, \tfrac{-a+2-b\sqrt{-1}}{2}; -a; z\right) \\
&\quad - \tfrac{(-a+2+b\sqrt{-1})(-a+2-b\sqrt{-1})}{4} z^{\frac{-a-1}{2}}(1-z)^{\frac{5}{2}} F\left(\tfrac{-a+4+b\sqrt{-1}}{2}, \tfrac{-a+4-b\sqrt{-1}}{2}; -a+1; z\right) \\
&=: g_4(z,a,b), \\
h_4(z) &= T_3(a,b)w_2^+(z) = T_3(a,b)w_1^+(z,-a-2,b) \\
&= -bz^{\frac{-a}{2}}(1-z)^{\frac{3}{2}} F\left(\tfrac{-a+2+b\sqrt{-1}}{2}, \tfrac{-a+2-b\sqrt{-1}}{2}; -a; z\right) \\
&\quad - \tfrac{b(-a+2+b\sqrt{-1})(-a+2-b\sqrt{-1})}{4a} z^{\frac{-a}{2}}(1-z)^{\frac{5}{2}} F\left(\tfrac{-a+4+b\sqrt{-1}}{2}, \tfrac{-a+4-b\sqrt{-1}}{2}; -a+1; z\right) \\
&=: h_4(z,a,b),
\end{aligned}
$$

$$
\begin{aligned}
f_5(z) &= T_1(-a,b)w_1^-(z) = T_1(-a,b)w_1^+(z,-a,b) = f_3(z,-a,b), \\
g_5(z) &= -T_2(-a)w_1^-(z) = -T_2(-a)w_1^+(z,-a,b) = -g_3(z,-a,b), \\
h_5(z) &= T_3(-a,b)w_1^-(z) = T_3(-a,b)w_1^+(z,-a,b) = h_3(z,-a,b),
\end{aligned}
$$

$$
\begin{aligned}
f_6(z) &= T_1(-a,b)w_2^-(z) = T_1(-a,b)w_1^+(z,a-2,b) = f_4(z,-a,b), \\
g_6(z) &= -T_2(-a)w_2^-(z) = -T_2(-a)w_1^+(z,a-2,b) = -g_4(z,-a,b), \\
h_6(z) &= T_3(-a,b)w_2^-(z) = T_3(-a,b)w_1^+(z,a-2,b) = h_4(z,-a,b).
\end{aligned}
$$

Recall that the parameters a and b are real numbers and that $z \in \mathbf{C} - \Lambda$. Then, it is easy to see that the following proposition holds:

Proposition 5.2. *For each* $i \in \{1, \dots, 6\}$,

$$f_i(\overline{z}) = \overline{f_i(z)}, \quad g_i(\overline{z}) = \overline{g_i(z)}, \quad h_i(\overline{z}) = \overline{h_i(z)}.$$

Especially, if $0 < z < 1$, *then for each* $i \in \{1, \dots, 6\}$, $f_i(z), \ g_i(z), \ h_i(z) \in \mathbf{R}$.

6. Eigenforms of the Laplacian.

Let $\{(f_j(z), g_j(z), h_j(z)); j = 1, \dots, 6\}$ be the fundamental system of solutions of the simultaneous equation (3) on the domain $\mathbf{C} - \Lambda$, which is given in Proposition 4.2.

Let $f_j(r)$, $g_j(r)$ and $h_j(r)$ be functions of $r (> 0)$ obtained by the substitution $z = (\frac{\sinh r}{\cosh r})^2$ into the functions $f_j(z)$, $g_j(z)$ and $h_j(z)$ respectively.

Then, by summarizing all the argument and calculations in the previous sections, we have:

Theorem 6.1. *Let* $(f_j(r), g_j(r), h_j(r))$*'s be the functions given as above. Then any harmonic vector field* v *on* U, *whose dual 1-form* τ *is an eigenform* (2) *of the Laplacian with the condition that* $a \notin \mathbf{Z}$ *and* $b \neq 0$, *is given by a linear combination as follows (or the same form with* sin *and* cos *interchanged):*

$$v = \Sigma_{j=1}^{6} \{p_j f_j(r) \cos(a\theta + b\phi)\, e_1 + q_j g_j(r) \sin(a\theta + b\phi)\, e_2 + r_j h_j(r) \sin(a\theta + b\phi)\, e_3\},$$

where $p_j, q_j, r_j \in \mathbf{R}$.

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