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Notes on discrete subgroups of $PU(1,2;\mathbf{C})$ with Heisenberg translations IV

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1. Introduction

In the study of discrete groups it is important to find out conditions for a group to be discrete. We concern ourselves with subgroups of $PU(1,2;\mathbf{C})$. By using the stable basin theorem, Basmajian and Miner have shown

Theorem 1.1 ([1; Theorem 9.11]). *Fix a stable basin point (r,ε). Let g be a Heisenberg translation of $PU(1,2;\mathbf{C})$ with the form*

$$g=\begin{pmatrix}1&0&0\\ s&1&\overline{a}\\ a&0&1\end{pmatrix},$$

where $Re(s)=\frac{1}{2}|a|^2$. If f is a loxodromic element of $PU(1,2;\mathbf{C})$ with fixed points 0 and q, satisfying $|\lambda(f)-1|<\varepsilon$ and

$$(*)\ \ \delta(0,q)>\frac{\delta(0,g(0))}{r^2}(1+r^2+\sqrt{1+r^2}),$$

then the group $<f,g>$ generated by f and g is not discrete.

Parker has independently proved the following theorem in a different manner from Basmajian and Miner's.

Theorem 1.2 ([10; Theorem 2.1]). *Let g be the same Heisenberg translation as in Theorem 1.1. Let f be any element of $PU(1,2;\mathbf{C})$ with isometric sphere of radius R_f. If*

$$R_f^2>\delta(gf^{-1}(\infty),f^{-1}(\infty))\delta(gf(\infty),f(\infty))+2|a|^2,$$

then the group $<f,g>$ generated by f and g is not discrete.

At first sight it is not clear what the relation between these results is. In our previous papers [8] and [9] we have proved that Theorem 1.1 follows from Theorem 1.2. The assumption $(*)$ in Theorem 1.1 is rather strong and we would like to be able to replace it with a weaker and more geometrical condition. So far we have not been able to do this for all stable basin points. However, by placing additional restriction on (r,ε) we show that $(*)$ may be replaced with a weaker condition. The assumption $(*)$ in Theorem 1.1 is

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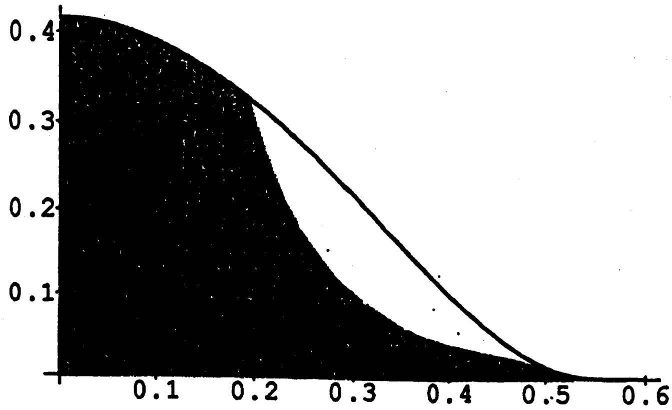
closely related to a condition on the cross ratio as shown in section 4. Let D be the set of stable basin points (r, ε) such that

$$\frac{1-r}{r} > (2\varepsilon)^{\frac{1}{2}} \left\{ 2 + \left(8 + \frac{M(\varepsilon)}{2} \right)^{\frac{1}{2}} \right\},$$

where $M(\varepsilon) = (1+\varepsilon)^{\frac{1}{2}} + (1+\varepsilon)^{-\frac{1}{2}}$.

The shading in the following figure indicates the set D.



We have

Theorem 1.3. *Fix a stable basin point (r, ε) in D. Let g be the Heisenberg translation as in Theorem 1.1. If f is a loxodromic element of $PU(1,2;\mathbf{C})$ with fixed points 0 and q, satisfying $|\lambda(f) - 1| < \varepsilon$ and $|[0, q, g(0), g(q)]| < r^4$, then the group $< f, g >$ generated by f and g is not discrete.*

2. Preliminaries

We recall some definitions and notation. Let $\mathbf{C}$ be the field of complex numbers. Let $V = V^{1,2}(\mathbf{C})$ denote the vector space $\mathbf{C}^3$, together with the unitary structure defined by the Hermitian form

$$\widetilde{\Phi}(z^*, w^*) = -(\overline{z_0^*} w_1^* + \overline{z_1^*} w_0^*) + \overline{z_2^*} w_2^*$$

for $z^* = (z_0^*, z_1^*, z_2^*), w^* = (w_0^*, w_1^*, w_2^*)$ in V. An automorphism g of V, that is a linear bijection such that $\widetilde{\Phi}(g(z^*), g(w^*)) = \widetilde{\Phi}(z^*, w^*)$ for z^*, w^* in V, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,2;\mathbf{C})$. Let $V_0 = \{w^* \in V | \quad \widetilde{\Phi}(w^*, w^*) = 0\}$ and $V_- = \{w^* \in V | \quad \widetilde{\Phi}(w^*, w^*) < 0\}$. It is clear that both V_0 and V_- are invariant under $U(1,2;\mathbf{C})$. We denote $U(1,2;\mathbf{C})/(center)$ by $PU(1,2;\mathbf{C})$. Set $V^* = V_- \cup V_0 - \{0\}$. Let $\pi : V^* \longrightarrow \pi(V^*)$ be the projection map defined by $\pi(w_0^*, w_1^*, w_2^*) = (w_1, w_2)$, where $w_1 = w_1^*/w_0^*$ and $w_2 = w_2^*/w_0^*$. We write ∞ for $\pi(0,1,0)$. We may identify $\pi(V_-)$ with the Siegel domain

$$H^2 = \{w = (w_1, w_2) \in \mathbf{C}^2 \mid \ Re(w_1) > \frac{1}{2}|w_2|^2\}.$$

We can regard an element of $PU(1,2;\mathbf{C})$ as a transformation acting on H^2 and its boundary ∂H^2 (see [6]). Denote $H^2 \cup \partial H^2$ by $\overline{H^2}$. We define a new coordinate system in $\overline{H^2} - \{\infty\}$. Our convention slightly differs from Basmajian-Miner [1] and Parker [8]. The *H − coordinates* of a point $(w_1, w_2) \in \overline{H^2} - \{\infty\}$ are defined by $(k, t, w_2)_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C}$ such that $k = Re(w_1) - \frac{1}{2}|w_2|^2$ and $t = Im(w_1)$. For simplicity, we write $(t_1, w')_H$ for $(0, t_1, w')_H$.

The *Cygan metric* $\rho(p,q)$ for $p = (k_1, t_1, w')_H$ and $q = (k_2, t_2, W')_H$ is given by

$$\rho(p,q) = |\{\frac{1}{2}|W' - w'|^2 + |k_2 - k_1|\} + i\{t_1 - t_2 + Im(\overline{w'}W')\}|^{\frac{1}{2}}.$$

We note that the Cygan metric ρ is a generalization of the *Heisenberg metric* δ in ∂H^2 and that ρ is invariant under Heisenberg translations (see [7]).

Let $f = (a_{ij})_{1 \le i,j \le 3}$ be an element of $PU(1,2;\mathbf{C})$ with $f(\infty) \neq \infty$. We define the *isometric sphere I_f* of f by

$$I_f = \{w = (w_1, w_2) \in \overline{H}^2 \mid \ |\tilde{\Phi}(W, Q)| = |\tilde{\Phi}(W, f^{-1}(Q))|\},$$

where $Q = (0,1,0)$, $W = (1, w_1, w_2)$ in V^* (see [4]). It follows that the isometric sphere I_f is the sphere in the Cygan metric with center $f^{-1}(\infty)$ and radius $R_f = \sqrt{1/|a_{12}|}$, that is,

$$I_f = \left\{ z = (k, t, w')_H \in (\mathbf{R}^+ \cup \{0\}) \times \mathbf{R} \times \mathbf{C} \mid \rho(z, f^{-1}(\infty)) = \sqrt{\frac{1}{|a_{12}|}} \right\}.$$

Given four distinct points q_1, q_2, q_3, q_4 of ∂H^2, we define the *cross ratio* of these points as

$$|[q_1, q_2, q_3, , q_4]| = \frac{\delta(q_3, q_1)^2 \delta(q_4, q_2)^2}{\delta(q_4, q_1)^2 \delta(q_3, q_2)^2}.$$

We note that the cross ratio is invariant under $PU(1,2;\mathbf{C})$. The definition is extended by continuity to the case when one of the q_i is ∞ so, for example,

$$|[q_1, q_2, \infty, q_4]| = \frac{\delta(q_4, q_2)^2}{\delta(q_4, q_1)^2}.$$

Using the cross ratio, one can formulate in an invariant way what it means for pairs of fixed points to be close.

Proposition 2.1 ([1; Proposition 7.1]). *Let f and g be loxodromic elements with fixed points $\{q_1, q_2\}, \{q_3, q_4\}$, respectively. If the cross ratio $|[q_1, q_2, q_3, , q_4]| = r^4 < 1$, then there exists an element $h \in PU(1,2;\mathbf{C})$ such that*

(1) hfh^{-1} has fixed points at 0 and ∞, and

(2) hgh^{-1} has fixed points at Cygan distance r and $1/r$ from 0.

3. Stable basin region

We recall the stable basin region (see [1], [8] and [9]). Let

$$B_r = \{z \in \partial H^2 \mid \delta(z, 0) < r\},$$

and let $\overline{B}_s^c = \partial H^2 - \overline{B}_s$. Given r and s with $r < s$, the pair of open sets $(B_r, \overline{B}_s^c)$ is said to be ***stable*** with respect to a set S of elements in $PU(1,2;\mathbf{C})$ if for any element $g \in S$,

$$g(0) \in B_r \quad g(\infty) \in \overline{B}_s^c.$$

Let $S(r, \varepsilon)$ denote the family of loxodromic elements f with fixed points in B_r and $\overline{B}_{1/r}^c$, and satisfying $|\lambda(f) - 1| < \varepsilon$. For positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, we define $\varepsilon(r, r')$ by

$$\varepsilon(r, r') = \sup\{|\lambda(f) - 1|\}, \tag{3.1}$$

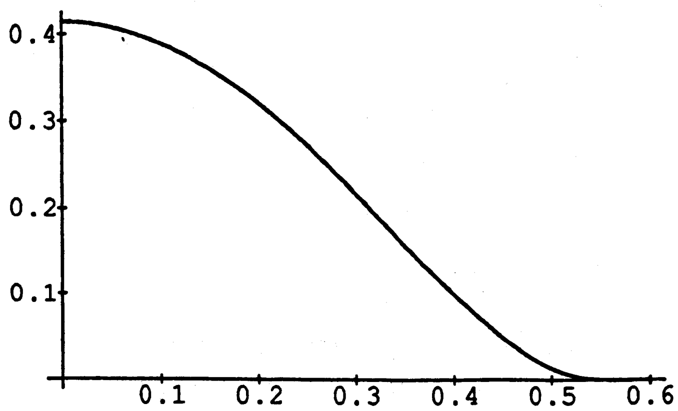
where $|\lambda(f) - 1|$ satisfies the inequality

$$|\lambda(f) - 1| < \sqrt{1 + \left(\frac{1 - (3 + |\lambda(f) - 1|)r^2)}{1 - 2r^2}\right)^2 \left(\frac{1 - 3r^2}{1 - r^2}\right)^2 \left(\frac{r'}{r}\right)^2} - 1. \tag{3.2}$$

A triple of non-negative numbers (r, r', ε) is said to be a ***basin point*** provided that $r < 1/\sqrt{3}$, $r' < 1$ and $\varepsilon < \varepsilon(r, r')$. In particular, if $r' \le r$, we call (r, r', ε) a ***stable basin point***. Call the set of all such points the ***stable basin region***.

Theorem 3.1 ([9; Theorem 2.2], Stable Basin Theorem). *Given positive real numbers r and r' with $r < 1/\sqrt{3}$ and $r' < 1$, the pair of open sets $(B_{r'}, \overline{B}_{1/r'}^c)$ is stable with respect to the family $S(r, \varepsilon(r, r'))$, where $\varepsilon(r, r')$ is given by (3.1).*

The following figure shows the stable basin region.



4. Groups with Heisenberg translations

In this section we show that Theorem 1.3 follows from Theorem 1.2. To prove Theorem 1.3, we need two lemmas.

Lemma 4.1. *Suppose that* $\delta(0, g(0)) < \delta(q, g(q))$. *If* $|[0, q, g(0), g(q)]| < r^4$, *then*

$$\delta(0, q) > \left(\frac{1-r}{r}\right)\delta(0, g(0)).$$

Proof. Using the triangle inequality and the invariance of δ under Heisenberg translations, we have

$$\delta(q, g(0)) \leq \delta(0, g(0)) + \delta(0, q)$$

and

$$\delta(0, g(q)) \leq \delta(0, g(0)) + \delta(g(0), g(q)) = \delta(0, g(0)) + \delta(0, q).$$

It follows that

$$\begin{aligned} r^4 &> |[0, q, g(0), g(q)]| \\ &= \left(\frac{\delta(0, g(0))\delta(q, g(q))}{\delta(0, g(q))\delta(q, g(0))}\right)^2 \\ &> \left(\frac{\delta(0, g(0))}{\delta(0, g(0)) + \delta(0, q)}\right)^4, \end{aligned}$$

which implies

$$\delta(0, q) > \left(\frac{1-r}{r}\right)\delta(0, g(0)).$$

Lemma 4.2 ([9; Lemma 3.3]). *Let f be a loxodromic element with fixed points* 0 *and* q, *satisfying* $|\lambda(f) - 1| < \varepsilon$. *Then*

$$\left(\frac{\delta(0, q)}{R_f}\right)^2 \leq 2\varepsilon.$$

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Without loss of genarality, we may assume that $\delta(0, g(0)) < \delta(q, g(q))$, because Theorem 1.2 is invariant under Heisenberg translations. Let (r, ε) be a stable basin point in D. By Lemmas 4.1 and 4.2,

$$\begin{aligned}
R_f &> \left(\frac{1}{2\varepsilon}\right)^{\frac{1}{2}} \delta(0,q) \\
&> \left(\frac{1}{2\varepsilon}\right)^{\frac{1}{2}} \left(\frac{1-r}{r}\right) \delta(0,g(0)) \\
&= \left(\frac{1}{2\varepsilon}\right)^{\frac{1}{2}} \left(\frac{1-r}{r}\right) |s|^{\frac{1}{2}} \\
&> \{2 + \left(8 + \frac{M(\varepsilon)}{2}\right)^{\frac{1}{2}} \}|s|^{\frac{1}{2}} \\
&> \left\{2 + \left(8 + \frac{L}{2}\right)^{\frac{1}{2}}\right\} |s|^{\frac{1}{2}} \\
&= 2|s|^{\frac{1}{2}} + \left(8|s| + \frac{L|s|}{2}\right)^{\frac{1}{2}} \\
&> \sqrt{2}|a| + \left(4|a|^2 + \frac{L|s|}{2}\right)^{\frac{1}{2}}.
\end{aligned}$$

In the same manner as in the proof Theorem 4.5 in [8] we have

$$\begin{aligned}
R_f^2 &> \frac{|s|L}{2} + 2\sqrt{2}|a|R_f + 2|a|^2 \\
&> \delta(gf(\infty), f(\infty))\delta(gf^{-1}(\infty), f^{-1}(\infty)) + 2|a|^2.
\end{aligned}$$

We conclude from Theorem 1.3 that the group $< f, g >$ generated by f and g is not discrete.

Collorary 4.3. *Fix a stable basin point (r, ε) in D. Let g be the same Heisenberg translation as in Theorem 1.1. If f is a loxodromic element with fixed points 0 and q, satisfying $|\lambda(f) - 1| < \varepsilon$ and $\delta(0,q) > \frac{\delta(0,g(0))}{r^2}(1 + r^2 + \sqrt{1+r^2})$, then the group $< f, g >$ generated by f and g is not discrete.*

We need a lemma to prove Collorary 4.3.

Lemma 4.4 ([1; Lemma 7.3]). *If $\delta(0,q) > \delta(0,g(0))$, then*

$$|[0,q,g(0),g(q)]|^{\frac{1}{2}} \leq \left(1 + \frac{\delta(0,q)}{\delta(0,q) - \delta(0,g(0))}\right)\left(\frac{\delta(0,g(0))}{\delta(0,q) - \delta(0,g(0))}\right).$$

Proof of Collorary 4.3. We see that our assumption

$$\delta(0,q) > \frac{\delta(0,g(0))}{r^2}(1 + r^2 + \sqrt{1+r^2})$$

is equivalent to

$$\left(1+\frac{\delta(0,q)}{\delta(0,q)-\delta(0,g(0))}\right)\left(\frac{\delta(0,g(0))}{\delta(0,q)-\delta(0,g(0))}\right)<r^2.$$

It follows from Lemma 4.4 that $|[0,q,g(0),g(q)]|<r^4$. By Theorem 1.3, the group $<f,g>$ generated by f and g is not discrete.

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