

Uniform discreteness and Heisenberg screw motions

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Abstract

We give a version of Shimizu's lemma for groups of complex hyperbolic isometries one of whose generators is a Heisenberg screw motion. Our main result interpolates between known results for groups with a generator that is a vertical translation or a Heisenberg rotation. We also give an interpretation of our result in terms of the relation between radii of isometric spheres and their distance from the axis of the Heisenberg screw motion.

1 Introduction

Shimizu's lemma [12] gives a necessary condition for a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ containing a parabolic element fixing ∞ to be discrete. It was generalised for discrete groups of higher dimensional real hyperbolic isometries containing a parabolic element by Leutbecher [8], Wielenberg [14], Ohtake [9] and Waterman [13]. The hyperbolic plane is not only real hyperbolic 2-space $\mathbf{H}_{\mathbb{R}}^2$, but also complex hyperbolic 1-space $\mathbf{H}_{\mathbb{C}}^1$. Therefore it is natural to generalise Shimizu's lemma to discrete groups of isometries of higher dimensional complex hyperbolic $\mathbf{H}_{\mathbb{C}}^n$ space containing a parabolic element. This is part of a wider project to give generalisations of Jørgensen's inequality

for complex hyperbolic space, see [1, 4]. Parabolic isometries of $\mathbf{H}_{\mathbb{C}}^n$ are either Heisenberg translations or *screw-parabolic* transformations (also called *ellipto-parabolic* transformations). The latter are the composition of an elliptic transformation and a Heisenberg translation preserving the fixed point set of the elliptic map. For screw-parabolic maps in (complex) dimension $n = 2$ the Heisenberg translation involved is necessarily vertical, although in higher dimensions non vertical translations are also possible. The elliptic transformation and its fixed point set are called the *holonomy* and the *axis* of the screw parabolic transformation respectively.

In [6], Kamiya generalised Shimizu's lemma for vertical translations and in [7] he gave a geometric version which says that there must be a precisely invariant horoball whose height depends only on the length of the translation. In [10], Parker observed that Kamiya's result may be extended to groups containing a screw-parabolic map whose holonomy has finite order. On the other hand, also in [10], he showed that, for groups containing a non-vertical Heisenberg translation (Proposition 7.3) or a screw-parabolic map with infinite order holonomy (Proposition 6.4), there is no precisely invariant horoball (compare [9] for the corresponding real hyperbolic result). In [11], Parker showed that for non-vertical Heisenberg translations there is a version of Shimizu's lemma where the radius of an isometric sphere is bounded in terms of the translation length of the Heisenberg translation at its centre (see [13] for an analogous result in real hyperbolic space).

This paper summarises the results of [5] which gives a generalisation of Shimizu's lemma to groups of isometries of $\mathbf{H}_{\mathbb{C}}^2$ a screw-parabolic element. This will give analogues of Shimizu's lemma for all parabolic isometries of complex hyperbolic 2-space. However, our result does not lead to a nice expression for a precisely invariant sub-horospherical region.

Screw motions are the most complicated of all parabolic maps. They combine features of boundary elliptic maps as well as pure parabolic maps. Indeed, by letting the holonomy tend to the identity a screw motion tends to a vertical translation and, similarly, letting the translation length tend to zero a screw motion tends to a boundary elliptic map. Below we will show that our main theorem interpolates between similar results for these types of isometry. Specifically, we show that as the holonomy tends to the identity, our result will tend to Kamiya's result [6]. On the other hand we will show that as the translation length goes to zero our result will tend toward the version of Jørgensen's inequality for boundary elliptic maps proved in [4].

Our main result depends on a normalisation our screw parabolic map as a particular Heisenberg screw motion. In the last section of the paper we

will restate the result to give a bound on the radii of isometric spheres in terms of the distance of their centres from the axis of the screw-parabolic. As indicated above, in Proposition 6.4 of [10] it is shown that there is no precisely invariant horoball for groups containing a screw-parabolic map. In this construction, the isometric spheres of very large radius are a long way away from the axis of the screw-parabolic (in particular, it has a very large translation length at the centre). Our result will indicate that this condition is necessary. Specifically, we show that if A is a screw parabolic map and B is an element of $\mathrm{PU}(2, 1)$ with isometric sphere of very large radius r_B and if $\langle A, B \rangle$ is discrete, then the centre of the isometric sphere of either B or B^{-1} must be very far from the axis of A .

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2 Complex hyperbolic space

We give the necessary background material in this section. More extensive details can be found in [3].

Let $\mathbb{C}^{2,1}$ denote the complex vector space of dimension 3, equipped with a non-degenerate Hermitian form of signature $(2, 1)$. There are several such forms. We will use the following form, called the *second Hermitian form* in [2]

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1\bar{w}_3 + z_2\bar{w}_2 + z_3\bar{w}_1,$$

where \mathbf{z} and \mathbf{w} in $\mathbb{C}^{2,1}$ are the column vectors with entries (z_1, z_2, z_3) and (w_1, w_2, w_3) respectively.

Consider the following subspace in $\mathbb{C}^{2,1}$:

$$\begin{aligned} V_0 &= \{\mathbf{z} \in \mathbb{C}^{2,1} - \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\} \\ V_- &= \{\mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}. \end{aligned}$$

Let $\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \rightarrow \mathbb{P}\mathbb{C}^{2,1}$ be the canonical projection onto complex projective space. Then $\mathbf{H}_{\mathbb{C}}^2 = \mathbb{P}(V_-)$ associated with the Bergman metric is *complex hyperbolic space*. The biholomorphic isometry group of $\mathbf{H}_{\mathbb{C}}^2$ is $\mathrm{PU}(2, 1)$ acting by linear projective transformations. Here $\mathrm{PU}(2, 1)$ is the projective unitary group with respect to the Hermitian form defining $\mathbb{C}^{2,1}$. This means

that the inverse of $B \in \text{PU}(2, 1)$ has the following form

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \bar{j} & \bar{f} & \bar{c} \\ \bar{h} & \bar{e} & \bar{b} \\ \bar{g} & \bar{d} & \bar{a} \end{bmatrix}. \quad (1)$$

The boundary of $\mathbf{H}_{\mathbb{C}}^2$ is $\partial\mathbf{H}_{\mathbb{C}}^2 = \mathbb{P}(V_0)$. It may be identified with the one point compactification of 3-dimensional Heisenberg group \mathfrak{N} , which is $\mathbb{C} \times \mathbb{R}$ with the group law

$$(\zeta_1, v_1) \diamond (\zeta_2, v_2) = (\zeta_1 + \zeta_2, v_1 + v_2 + 2\Im(\zeta_1 \bar{\zeta}_2)).$$

The identification between $\mathfrak{N} \cup \{\infty\}$ and $\mathbb{P}(V_0)$ is given by

$$(\zeta, v) \mapsto \begin{bmatrix} -|\zeta|^2 + iv \\ \sqrt{2}\zeta \\ 1 \end{bmatrix} \in \mathbb{P}(V_0), \quad \infty \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{P}(V_0)$$

where $(\zeta, v) \in \mathfrak{N}$.

The Heisenberg group has several natural metrics. We choose to work with the *Cygan metric*. This is defined by

$$\rho_0((\zeta_1, v_1), (\zeta_2, v_2)) = \left| |\zeta_1 - \zeta_2|^2 + i(v_1 - v_2 + 2\Im(\zeta_1 \bar{\zeta}_2)) \right|^{1/2}.$$

The Heisenberg group acts on itself by (left) *Heisenberg translation*: translation $T_{(\tau, t)}$ by $(\tau, t) \in \mathfrak{N}$ is given by

$$T_{(\tau, t)} : (z, v) \mapsto (\tau, t) \diamond (z, v) = (z + \tau, v + t + 2\Im(\tau \bar{z})).$$

Heisenberg translation by $(0, t)$ for any given $t \in \mathbb{R}$ is called *vertical translation* by t . The unitary group $\text{U}(1)$ acts on the Heisenberg group by *Heisenberg rotation*: rotation $R_{\lambda, 0}$ with *holonomy* $\lambda \in \text{U}(1)$ and axis the chain $(0, v) \subset \mathfrak{N}$ is given by

$$R_{\lambda, 0} : (z, v) \mapsto (\lambda z, v).$$

For any $\tau \in \mathbb{C}$, the Heisenberg rotation $R_{\lambda, \tau}$ by $\lambda \in \text{U}(1)$ with axis $(\tau, v) \subset \mathfrak{N}$ is given by conjugating $R_{\lambda, 0}$ by $T_{(\tau, t)}$.

The product of a Heisenberg translation and a Heisenberg rotation is a *Heisenberg screw motion*. The easiest example would be the product of vertical translation $T_{(0, t)}$ with $R_{\lambda, 0}$. This is $S_{\lambda, 0, t}$ given by

$$S_{\lambda, 0, t} : (z, v) \mapsto (\lambda z, v + t).$$

It has axis the chain $(0, v) \subset \mathfrak{N}$, rotates about the axis with holonomy $\lambda \in U(1)$ and translates along the axis by a Cygan distance $\sqrt{|t|} \in \mathbb{R}$, its translation length. Other Heisenberg screw motions may be obtained by conjugating this one by a Heisenberg translation or by composing other Heisenberg translations with other Heisenberg rotations.

Heisenberg translations, rotations and screw motions are all isometries of the Cygan metric, indeed the group of Heisenberg isometries is generated by $T_{(\tau,t)}$, $R_{\lambda,0}$ where (τ, t) and λ vary over \mathfrak{N} and $U(1)$ respectively.

The action of Heisenberg isometries can be extended to complex hyperbolic space. The Heisenberg translation $T_{(\tau,t)}$, Heisenberg rotation $R_{(\lambda,0)}$ and Heisenberg screw motion $S_{\lambda,0,t}$ correspond to the following matrices in $PU(2, 1)$

$$\begin{aligned} T_{(\tau,t)} &= \begin{bmatrix} 1 & -\sqrt{2}\bar{\tau} & -|\tau|^2 + it \\ 0 & 1 & \sqrt{2}\tau \\ 0 & 0 & 1 \end{bmatrix}, \\ R_{\lambda,0} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ S_{\lambda,0,t} &= \begin{bmatrix} 1 & 0 & it \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Elements of $PU(2, 1)$ are classified as parabolic, loxodromic or elliptic just as for Möbius transformations. An element is called parabolic if and only if it has a unique fixed point which is in $\partial \mathbf{H}_{\mathbb{C}}^2$.

- (i) A parabolic element is called *pure parabolic* if it is conjugate to a Heisenberg translation.
- (ii) A parabolic element is called *screw parabolic* if it is conjugate to a Heisenberg screw motion.

Suppose that $B \in PU(2, 1)$ does not fix ∞ , which is equivalent to $g \neq 0$ when B has the form (1). Then the *isometric sphere* (see [10]) of B is the sphere in the Cygan metric with centre $B^{-1}(\infty)$ and radius $r_B = 1/\sqrt{|g|}$. Likewise the isometric sphere of B is the Cygan sphere of radius $1/\sqrt{|g|}$ with centre $B(\infty)$.

3 The main results

In this section we summarise the main results of [5].

Theorem 1 Suppose that $A, B \in \text{PU}(2, 1)$ have the form

$$A = \begin{bmatrix} 1 & 0 & it \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

where $t \geq 0$, $\lambda \in \text{U}(1)$ with $|\lambda - 1| < 1$ and $g \neq 0$. For any real $k > \frac{2t|\lambda-1|}{1-|\lambda-1|^2}$ let $N(k)$ denote $\max\{|e|^2 - 1, |d|^2k, |h|^2k, |g|^2k^2\}$. If

$$(N(k) + 1)\left(|\lambda - 1| + \frac{t}{k}\right)^2 < 1 + \frac{t^2}{k^2}, \quad (2)$$

then the group $\langle A, B \rangle$ is not discrete.

Remark. If we set $A = S_{\lambda, \xi, t}$ with axis the chain $(\xi, v) \subset \mathfrak{N}$ then we need to define $N(k)$ to be

$$\max\left\{|e + \bar{\xi}d\sqrt{2} + \xi h\sqrt{2} + 2|\xi|^2g|^2 - 1, |d + \xi g\sqrt{2}|^2k, |h + \bar{\xi}g\sqrt{2}|^2k, |g|^2k^2\right\}.$$

We now give two easy corollaries from our theorem. If $\lambda = 1$, then A is a vertical translation. In this case, the following corollary shows that Theorem 1 reduces to Shimizu's lemma for vertical Heisenberg translation.

Corollary 2 (Theorem 3.2 of [6], Proposition 5.2 of [10]) Suppose A and B in $\text{PU}(2, 1)$ are as in Theorem 1 with $\lambda = 1$. If

$$0 < |g|t < 1,$$

then the group $\langle A, B \rangle$ is not discrete.

Proof: Choose k sufficiently large so that $N(k) = |g|^2k^2$. Since $|g|t < 1$, we have

$$(N(k) + 1)\frac{t^2}{k^2} < 1 + \frac{t^2}{k^2}.$$

This is the inequality (2) with $|\lambda - 1| = 0$. So Theorem 1 implies that the group $\langle A, B \rangle$ is not discrete. \square

If $t = 0$, then A is a boundary elliptic element. In this case, under the condition $|e| > 1$, our theorem reduces to the generalisation of Jørgensen's inequality for boundary elliptic element of $\text{PU}(2, 1)$ given by the authors.

Corollary 3 (Theorem 5.2(3) of [4]) Suppose A and B in $\text{PU}(2, 1)$ are as in Theorem 1 with $t = 0$ and $|e| > 1$. If

$$0 < |e||\lambda - 1| < 1,$$

then the group $\langle A, B \rangle$ is not discrete.

Proof: Since $t = 0$ and $|e| > 1$, we can choose a sufficiently small k so that $N(k) = |e|^2 - 1$. Now from $|e||\lambda - 1| < 1$, we have

$$(N(k) + 1)|\lambda - 1|^2 < 1.$$

This is the inequality (2) with $t = 0$. So Theorem 1 implies that the group $\langle A, B \rangle$ is not discrete. \square

Finally we restate our main theorem as in an invariant way. Namely, we give a bound on the radii of the isometric spheres of B and B^{-1} in terms of the Cygan distance between their centres and the axis of A .

Given real numbers $x \in [0, 1)$, and $k > 2x/(1 - x^2)$ consider the following function $\Phi_x(k)$ defined by

$$\Phi_x(k) = \frac{k(xk + 1)}{\sqrt{k^2 + 1} - (xk + 1)}.$$

The condition $k > 2x/(1 - x^2)$ ensures the denominator is positive and so $\Phi_x(k)$ is positive. For a fixed value of x , the function $\Phi_x(k)$ is convex on the interval $k \in (2x/(1 - x^2), \infty)$. Let $k_0(x)$ be the value of k in this interval where $\Phi_x(k)$ attains its minimum value. Using elementary calculus, we see that $k_0(x)$ is a root of the polynomial

$$(x^4 - x^2)k^4 + 4x^3k^3 + (2x^2 + x^4)k^2 + 2(x + 2x^3)k + (1 + 2x^2).$$

Theorem 4 Let A be a screw parabolic element of $\text{PU}(2, 1)$ with fixed point $q_A \in \partial \mathbf{H}_{\mathbb{C}}^2$, axis L_A , holonomy $\lambda \in \mathbf{U}(1)$ and translation length \sqrt{t} for some choice of Cygan metric ρ_0 on $\overline{\mathbf{H}_{\mathbb{C}}^2} - \{q_A\}$. Let B be any element of $\text{PU}(2, 1)$ not projectively fixing q_A and let r_B denote the radius (with respect to ρ_0) of the isometric sphere of B . Let

$$R = \max \left\{ \rho_0(L_A, B(\infty)), \rho_0(L_A, B^{-1}(\infty)), \sqrt{tk_0(|\lambda - 1|)/2} \right\}.$$

Then

$$r_B^2 \leq t \Phi_{|\lambda-1|}(2R^2/t) = \frac{2R^2(2|\lambda-1|R^2 + t)}{\sqrt{4R^4 + t^2 - (2|\lambda-1|R^2 + t)}}.$$

References

- [1] A. Basmajian & R. Miner, *Discrete groups of complex hyperbolic motions*, Invent. Math. **131** (1998), 85–136.
- [2] D.B.A. Epstein, Complex hyperbolic geometry, in *Analytical and Geometric Aspects of Hyperbolic Space*, L.M.S. Lecture Notes **111**, 1987, 93–111.
- [3] W.M. Goldman, Complex Hyperbolic Geometry, Oxford University Press, 1999.
- [4] Y. Jiang, S. Kamiya & J.R. Parker, *Jørgensen's inequality for complex hyperbolic space*, to appear in Geometriae Dedicata.
- [5] Y. Jiang & J.R. Parker, *Uniform discreteness and Heisenberg screw motions*, to appear
- [6] S. Kamiya, *Notes on non-discrete subgroups of $\hat{U}(1,n;F)$* , Hiroshima Math. J. **13** (1983), 501–506.
- [7] S. Kamiya, *Notes on elements of $U(1,n;C)$* , Hiroshima Math. J. **21** (1991), 23–45.
- [8] A. Leutbecher, *Über Spitzen diskontinuierlicher Gruppen von lineargebrochenen Transformationen*, Math. Zeit. **100**, 183–200.
- [9] H. Otake, *On discontinuous subgroups with parabolic transformations of the Möbius groups*, J. Math. Kyoto Univ. **25** (1985), 807–816.
- [10] J.R. Parker, *Shimizu's lemma for complex hyperbolic space*, International Journal of Mathematics. **3** (1992), 291–308.
- [11] J.R. Parker, *Uniform discreteness and Heisenberg translations*, Math. Zeit. **225** (1997), 485–505.
- [12] H. Shimizu, *On discontinuous subgroups operating on the product of the upper half planes*, Ann. of Math. **77** (1963), 33–71.
- [13] P.L. Waterman, *Möbius transformations in all dimensions*, Advances in Maths. **101** (1993), 87–113.
- [14] N. Wielenberg, *Discrete Möbius groups: Fundamental polyhedra and convergence*, American J. of Maths. **99** (1977), 861–877.