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Incompressible surfaces of arbitrarily high genus in 3-manifolds

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Abstract

In this paper we shall show that given a compact, orientable 3-manifold M, then there is a link with at most three components whose complement contains separating, closed, incompressible surfaces of arbitrarily high genus.
Keywords Incompressible surface, Dehn surgery.

1 Introduction

Let M be a compact 3-manifold and F be a compact surface properly embedded in M. F is said to be compressible if either F bounds a 3-ball, or there is an essential, simple closed curve which bounds a disk in M; otherwise, F is said to be incompressible.

The Haken-Kneser finiteness theorem says that given M, there exist an integer $c(M)$, such that any collection of pairwise disjoint, non-parallel, closed, incompressible surfaces in M has at most $c(M)$ components. But it is possible that a compact 3-manifold contains closed, incompressible surfaces of arbitrarily high genus. W. Jaco has shown that a handlebody of genus at least two contains non-separating incompressible surfaces S of arbitrarily high genus such that $|\partial S| = 1$, and H. Howards and Ruifeng Qiu have independently shown that a handlebody of genus at least two contains separating incompressible surfaces S of arbitrarily high genus such that $|\partial S| = 1$, or 2. In this paper, we shall show that given a compact, orientable 3-manifold M, there exist a link in M such that the complement of L contains separating, closed, incompressible surfaces of arbitrarily high genus.

Let $L = k_1 \cup \ldots \cup k_m$ be a link in a compact 3-manifold M with m components. We denote by M_L the manifold $M - int(N(k_1) \cup \ldots \cup N(k_m))$ where $N(k_i)$ is a regular neighbourhood of k_i, and T_i the boundary of $N(k_i)$. Let r_i be a slope on T_i, $i = 1, \ldots, m$. We denote by $M_L(r_1, \ldots, r_m)$ the manifold obtained by attaching m solid tori $J_1, \ldots, J_m$ to M_L along $T_1, \ldots, T_m$ so that r_i bounds a disk in J_i, $i = 1, \ldots, m$.

The main result is the following.

Theorem 1 Let M be a compact, orientable 3-manifold. Then there exist a link $L = k_1 \cup \ldots \cup k_m$ in M with $m \leq 3$, such that M_L contains separating, closed, incompressible surfaces of arbitrarily high genus. Furthermore, there exist a slope r_i on T_i, $i = 1, \ldots, m$, such that $M_L(r_1, \ldots, r_m)$ does also contain separating, closed, inmcompressible surfaces of arbitrarily high genus.

2 The proof of Theorem 1

We first prove the following proposition.

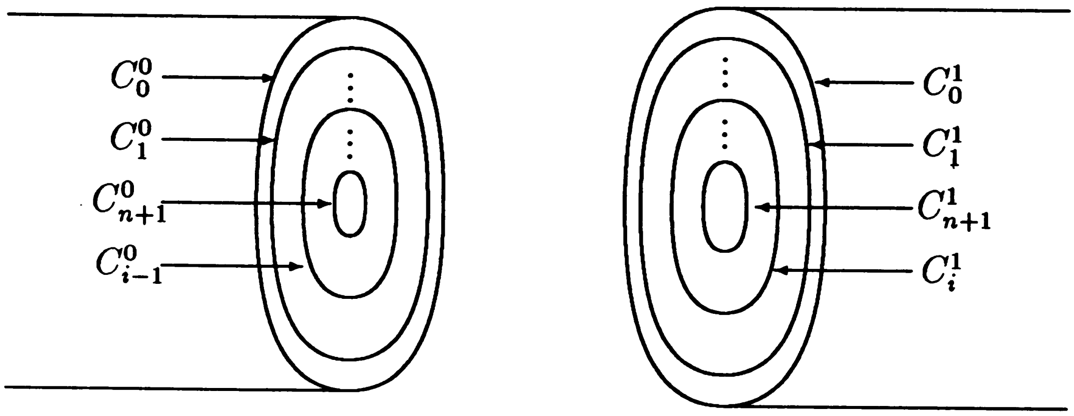

Figure 1

Proposition 1 Let F be an orientable, closed surface of genus at least two. Then there exist a link $L = k_1 \cup k_2$ in $F \times [0,1]$ such that $(F \times [0,1])_L$ contains separating, closed, incompressible surfaces of arbitrarily high genus. Furthermore, there is a slope r_i on T_i, $i = 1, 2$, such that $(F \times [0,1])_L(r_1, r_2)$ does also contain separating, closed, incompressible surfaces of arbitrarily high genus.

Proof Let c be a non-separating, simple closed curve on F, and $N(c)$ be a regular neighbourhood of c on F. Then $N(c)$ is an annulus. We denote by c^0 and c^1 the two boundary components of $N(c)$. Suppose that n is an integer at least two, and $x_0 = 0 < x_1 = 1/8 < \ldots < x_n = 7/8 < x_{n+1} = 1$. Then in $F \times [0,1]$, the surface $F \times \{x_i\}$ intersects the annulus $c^j \times [0,1]$ in the simple closed curve $c^j \times \{x_i\}$, where $j = 0, 1, i = 0, 1, \ldots, n+1$. We denote by c_i^j the simple closed curve $c^j \times \{x_i\}$.

It is easy to see that there are $n-1$ pairwise disjoint annuli $A_1', \ldots, A_{n-1}'$ properly embedded in $N(c) \times [0,1]$ such that $\partial A_i' = c_{i+1}^0 \cup c_i^1$, $i = 1, 2, \ldots, n-1$ (as in Figure 1). Now let $F_n = (\cup_{i=1}^n F \times \{x_i\} - int(N(c) \times [0,1])) \cup \cup_{l=1}^{n-1} A_l'$. Then $\partial F_n = c_1^0 \cup c_n^1$. Let $F_n \times [b_1, b_2]$ be a regular neighbourhood of F_n in $F \times [0,1]$. Then $F_n \times [b_1, b_2]$ intersects $c^j \times [0,1]$ in n annuli $A_1^j, \ldots, A_n^j$, where the core of A_i^j is c_i^j, $j = 0, 1, i = 1, \ldots, n$. Note that $A_1^0 \subset \partial(F_n \times [b_1, b_2])$, $A_n^1 \subset \partial(F_n \times [b_1, b_2])$, and for $2 \le i \le n$, A_i^0 is properly embedded in $F_n \times [b_1, b_2]$, for $1 \le i \le n-1$, A_i^1 is peoperly embedded in $F_n \times [b_1, b_2]$. We denote by $c_{i,1}^j$ the component of ∂A_i^j in $F_n \times \{b_1\}$, and $c_{i,2}^j$ the component of ∂A_i^j in $F_n \times \{b_2\}$ as in Figure 2.

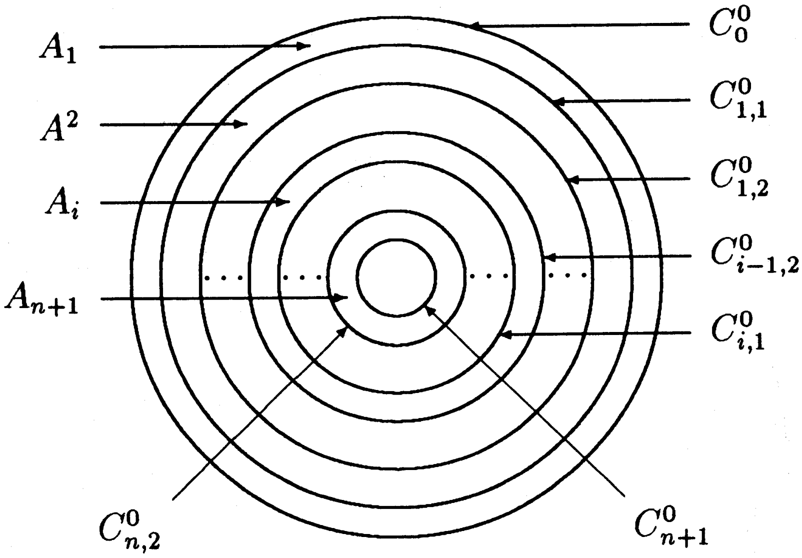

Figure 2

By construction, $\partial(F_n \times [b_1, b_2])$, denoted by S_n, is separating in $F \times I$, and $g(S_n) = 2n(g(F) - 1) + 1$.

Now let k_1^n be the knot in $F_n \times [b_1, b_2]$ obtained by pushing c_1^0 slightly into $int(F_n \times [b_1, b_2])$, and k_2^n be the knot obtained by pushing c_n^1 slightly into $int(F_n \times [b_1, b_2])$. Let $L_n = k_1^n \cup k_2^n$. Since $x_1 = 1/8, x_n = 7/8$ foy any integer

n, $k_1^{n_1} = k_1^{n_2}$ and $k_2^{n_1} = k_2^{n_2}$ even if $n_1 \neq n_2$. Thus we denote by k_1 the knot k_1^n, k_2 the knot k_2^n, and L the link L_n as in Figure 3.

Claim 1 S_n is incompressible in $(F_n \times [b_1, b_2])_L$.

Proof By construction, for any integer $n \geq 2$, c_1^0, together with the longitude slope on $T_1 = \partial N(k_1)$, say r', bounds an annulus A^1, and c_n^1, together with the longitude slope on $T_2 = \partial N(k_2)$, say r'', bounds an annulus A^2.

Now suppose that S_n is compressible in $(F_n \times [b_1, b_2])_L$. Let D be a compressing disk of S_n such that the number of components of $D \cap (A^1 \cup A^2)$, say $|D \cap (A^1 \cup A^2)|$, is minimal among all such disks. Note that $|D \cap (A^1 \cup A^2)| \neq 0$. Otherwise, one of $F_n \times \{b_1\}$ and $F_n \times \{b_2\}$ is compressible in $F_n \times [b_1, b_2]$.

If one component of $D \cap (A^1 \cup A^2)$ is a simple closed curve, then either $F \times [0,1]$ is boundary reducible, or there is a compressing disk D_0 of S_n such that $|D_0 \cap (A^1 \cup A^2)| < |D \cap (A^1 \cap A^2)|$. Thus we may assume that each component of $D \cap (A^1 \cup A^2)$ is an arc, the two end points of which lie in one of c_1^0 and c_n^1. Without loss of generality, we assume that $D \cap A^1 \neq \phi$. Let a_1 be an arc in $D \cap A^1$ which, together with an arc a_2 on c_1^0, bounds a disk D' in A^1 such that $intD'$ is disjoint from D. We denote by a_3 and a_4 the two components of $\partial D - \partial a_2$. Then each of $c_1 (= a_2 \cup a_3)$ and $c_2 (= a_2 \cup a_4)$ bounds a disk D_i in $(F_n \times [b_1, b_2])_L$. Since ∂D is essential in S, one of c_1 and c_2, say c_1, is essential. But $|D_1 \cap (A^1 \cup A^2)| < |D \cap (A^1 \cup A^2)|$, a contradiction. □ (Claim 1)

We denote by M the manifold $F \times [0,1] - int(F_n \times [b_1, b_2])$.

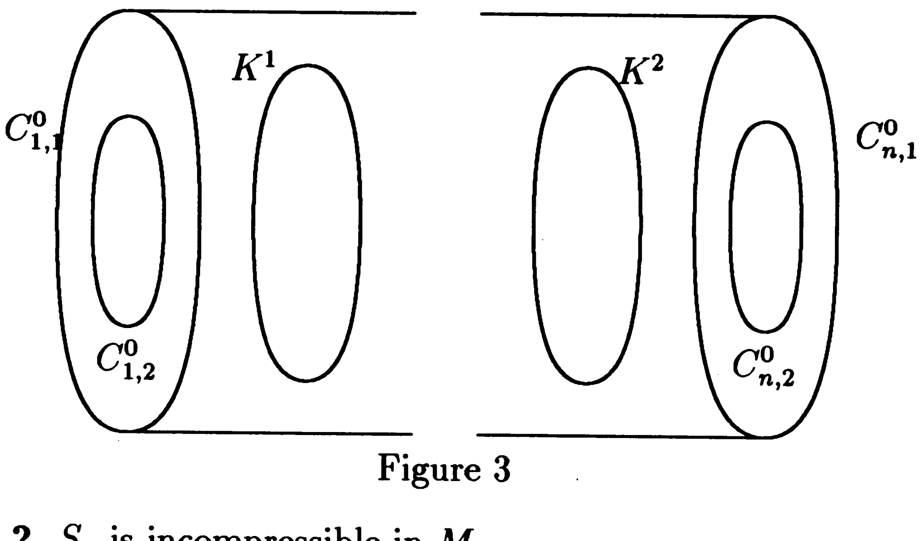

Figure 3

Claim 2 S_n is incompressible in M.

Proof By construction, M intersects $c^0 \times [0,1]$ in $n+1$ annuli $A_1, \ldots, A_{n+1}$, where A_1 is bounded by c_0^0 and $c_{1,1}^0$, A_{n+1} is bounded by c_{n+1}^0 and $c_{n,2}^0$, and for $2 \le i \le n$, A_i is bounded by $c_{i-1,2}^0$ and $c_{i,1}^0$ as in Figure 2. Similarly, M intersects $c^1 \times [0,1]$ in $n+1$ annuli, one of which, denoted by A_{n+2}, is bounded by c_{n+1}^1 and $c_{n,2}^1$ as in Figure 4.

Suppose that S_n is compressible in M. Let D be a compressing disk of S_n in M such that $|D \cap (\cup_{i=1}^{n+2} A_i)|$ is minimal among all such disks. Note that $|D \cap (\cup_{i=1}^{n+2} A_i)| \neq 0$. Otherwise, for some i, $F \times \{x_i\}$ is compressible in $F \times [0,1]$. By assumption, $D \cap (\cup_{i=1}^{n+2} A_i)$ contains no circle component. By the proof of Claim 1, if $a \in D \cap A_i$ then the two end points of a lie in distinct components of ∂A_i. That means that $D \cap (A_1 \cup A_{n+1} \cup A_{n+2}) = \phi$. Let a_1 be a component of $D \cap (\cup_{i=2}^{n}) A_i$ which, together with an arc a_2 on ∂D, bounds a disk D' in D such that $intD'$ is disjoint from $\cup_{i=2}^{n} A_i$. Without loss of generality, we assume that $a_1 \subset A_l$. Then one of the two end points of a_1 lies in $c_{l-1,2}^0$, and the other lies in $c_{l,1}^0$. Since $c_{l-1,2}^0 \subset F_n \times \{b_2\}$ and $c_{l,1}^0 \subset F_n \times \{b_1\}$, $a_2 \cap (c_{1,1}^0 \cup c_{n,2}^0) \neq \phi$. But $D \cap (A_1 \cup A_{n+2}) = \phi$, a contradiction. □

By Claim 1 and Claim 2, S_n is incompressible in $(F \times [0,1])_L$.

Note that c_i^0, together with the longitude slope r' on T_1, bounds an annulus, say A_i', and c_j^1, together with the longitude slope r'' on T_2, bounds an annulus, say A_j', where $i = 0, 1, j = n, n+1$.

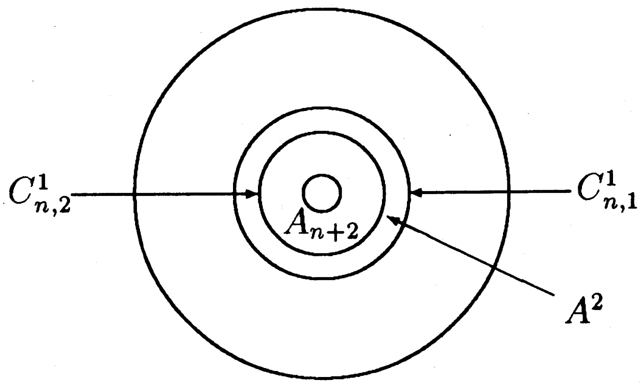

Figure 4

Claim 3 $(F \times [0,1])_L$ is irreducible. **Proof** Suppose that $(F \times [0,1])_L$ is reducible. Let P be a reducing 2-sphere in $(F \times [0,1])_L$ such that $|P \cap (A_0' \cup A_{n+1}')|$ is minimal among all such 2-spheres. Since $F \times [0,1]$ is reducible,

$|P\cap(A_0'\cup A_{n+1}')|\neq 0$. Without loss of generality, we assume that $P\cap A_0'\neq\phi$. There are two possibilities:

Case 1 One component of $P\cap A_0'$ bounds a disk D_1' in A_0'.

Now $\partial D_1'$ separates P into two disks D_2' and D_3'. Let $P_1 = D_1'\cup D_2'$, $P_2 = D_1\cup D_3'$. Then one of P_1 and P_2, say P_1, is a reducible 2-sphere. But $|P_1\cap(A_0'\cup A_{n+1}')| < |P\cap(A_0'\cup A_{n+1}'|$, a contradiction.

Case 2 Each component of $P\cap A_0'$ is essential on A_0'.

That means that c_0^0 bounds a disk in $F\times[0,1]$, a contradiction. □ (Claim 3)

Now let r_1 be a slope on T_1 such that $\Delta(r',r_1)\geq 2$, and r_2 be a slope on T_2 such that $\Delta(r'',r_2)\geq 2$. Then F_n, $F\times\{0\}$ and $F\times\{1\}$ are incompressible in $(F\times[0,1])_L(r_1,r_2)$ (see [CGLS][S][Wu]). □

The proof of Theorem 1

Let M be a compact, orientable 3-manifold.

Case 1 M contains a closed, incompressible surface F of genus at least two.

Let $F\times[0,1]$ be a regular neighbourhood of F in M. By Proposition 1, there is a link $L = k_1\cup k_2$ in $F\times[0,1]$ such that S_n constructed in Proposition 1 is incompressible in $(F\times[0,1])$, and there is a slope r_i on T_i, $i = 1,2$, such that S_n is incompressible in $(F\times[0,1])_L(r_1,r_2)$. Since $F\times\{0\}$ and $F\times\{1\}$ are incompressible in M and $(F\times[0,1])_L(r_1,r_2)$, S_n is incompressible in M_L and $M_L(r_1,r_2)$.

Case 2 M contains no closed, incompressible surface of genus at least 2.

We need only to prove that there is a knot k in M such that M_k contains a closed, incompressible surface of genus at least two.

Let $H_1\cup_S H_2$ be a Heegaard splitting of M with $g(S)\geq 1$, and a be a properly embedded arc in H_1 such that $H_1 - intN(a)$ is boundary irreducible. Then $H = H_2\cup N(a)$ is a compression body of genus at least 2. Let c be a simple closed curve on ∂H such that $\partial H - c$ is incompressible, and k be the knot in H obtained by pushing c slightly into $intH$.

Now we prove that ∂H is incompressible in $M' = H - intN(k)$.

Suppose that ∂H is compressible in M'. Now let D be a compressing disk of ∂H such that $|\partial D\cap c|$ is minimal among all such disks. Since $\partial H - c$ is incompressible, $|\partial D\cap c|\neq 0$. Since c, together with the longitude slope on $\partial N(k)$, bounds an annulus, by the proof of Claim 1, there is a compressing disk D' of ∂H, such that $|\partial D'\cap c| < |\partial D\cap c|$, a contradiction.

Since $H_1 - intN(a)$ is boundary irreducible, ∂H is incompressible in M_k. $\square$

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