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ON DOUBLE TORUS FIBERED KNOTS (SURVEY)

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ABSTRACT. In this note, we review recent works of fibered knots embedded in a standard closed surface of genus 2. In particular, we discuss fiberedness of some class of double torus knots which are band-sum of two torus knots. The problem includes: how to calculate the Alexander polynomials, how to construct minimal genus Seifert surfaces and how to show the constructed candidates are actually fiber surfaces. As an application, we give an easy quick formula for the Alexander polynomials of 2-bridge knots. Detailed proof will be in the forthcoming paper.

1. INTRODUCTION

A knot (or link) K in S^3 is called a *double torus knot (or link)* if K can be embedded in the Heegaard surface of genus 2 (i.e., a standardly embedded closed surface of genus 2). In [7] and [8], such knots are extensively studied and we follow the notations invented in them.

Double torus knots form a large family of knots that contains torus knots, 2-bridge knots, knots with $(1,1)$-decomposition (i.e., genus one bridge one knots) and tunnel number one knots. However, the class of doudle torus knots is not hopelessly large, with some 3-bridge knots outside the category. For example the knots 8_{16} and 8_{17} are not double torus knots. The former is a hyperbolic knot with an 2-string essential tangle decomposition, which is never the case for hyperbolic double torus knots by [14]. The knot 8_{17} is not invertible (cf. [9, p131]), while a double torus knot is either of period 2 or strongly invertible.

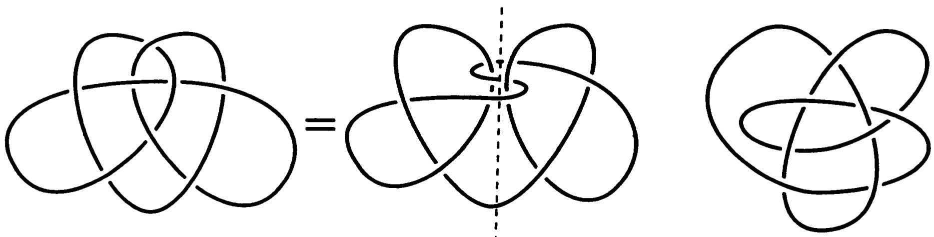


Figure 1.1: non double torus knots 8_{16} and 8_{17}. They are fibered knots.

As another criterion, double torus knots have tunnel number at most 2, and hence the connected sum of three trefoils is not a double torus knot.

Other interesting examples of double torus knots are Berge's doubly primitive knots [1], and Dean's twisted torus knots [3] [13]. The class of Berge's knots is conjectured (c.f. [5]) to cover all knots which yield lens spaces via Dehn surgery. They are known to be fibered knots [16], [8]. Some twisted torus knots yield small Seifert fibered spaces via Dehn surgery, and known examples which yield those with finite fundamental groups are known to be fibered. Encouraged by these facts, the following is set up.

Conjecture 1.1. *If a knot K yields a manifold M with a finite fundamental group via Dehn surgery, then K is a fibered knot of tunnel number one.*

If K is a tunnel number one knot, K is a double torus knot, because $K \subset \partial N(K \cup \tau)$, where τ is an unknotting tunnel, and $N(K \cup \tau)$ by definition is a standardly embedded handlebody of genus 2.

Study of fibered double torus knots is interesting partially because of that conjecture, and partially because algebraic and geometric methods interact nicely. Torus knots are fibered, and all fibered 2-bridge knots are already classified. Sufficient conditions for some other double torus knots to be fibered are obtained in [8].

In [8], several methods are introduced to find fibered knots among double torus knots. The examples contain: some ribbon knots obtained as a band-sum of two torus knots, *joins of* positive braids, knots with *coherently oriented connection diagrams*, some knots with $(1,1)$-decomposition, and Berge's knots. In [6], some conditions are given for a class of knots with $(1,1)$-decomposition to be fibered knots.

In this paper, introduce a recently obtained result: determination of double torus knots of type $(1,1)$, which are band-sum of torus knots. In the following section, we first review notations for double torus knots intruduced in [7].

2. NOTATIONS

Let K be a knot embedded in a standard double torus H. We regard H as being obtained by glueing two once-punctured tori T_L, T_R along the circle $\mathcal{O}$. Conversely, K is cut by $\mathcal{O}$ into parallel classes of arcs properly embedded in T_L, T_R. If K misses one of the tori, K is a torus knot and we do not consider the case.

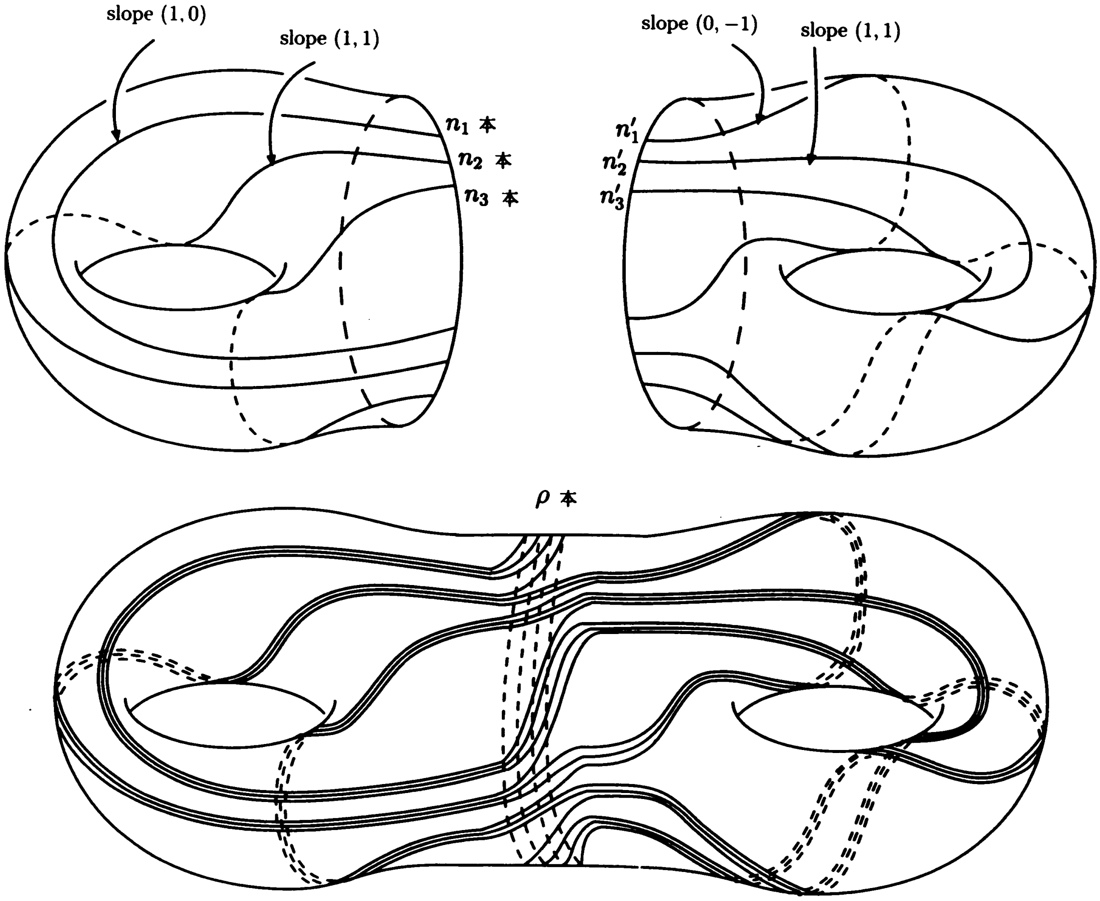


Figure 2.1: $K\{(3,3,3;3,3,3|4)(1,0,1,1)(0,-1,1,1)\}$

On each tori, $K \setminus \mathcal{O}$ consists of at most three parallel classes. Then as in Figure 2.1, we denote by $(n_1, n_2, n_3, n'_1, n'_2, n'_3)$ the numbers of constituent arcs. Off course we have the equality $n_1 + n_2 + n_3 = n'_1 + n'_2 + n'_3 := n$. Denote by $(p, q), (r, s)$ the slopes of the first and second parallel classes of arcs in T_L, and the slope of the third is automatically $(-p + r, -q + s)$. Also denote by $(p', q'), (r', s')$ the slopes of the two of the parallel classes in T_R. The convention of ordering the arcs and that of the slope should be inferred from the figure. Finally, in gluing the arcs along $\mathcal{O}$, we have a choice, which is denoted by $-n < \rho \leq n$. Then by arranging the above numbers as in $K = \{(n_1, n_2, n_3; n'_1, n'_2, n'_3|\rho)(p, q, r, s)(p', q', r', s')\}$ we can express a double torus knot K.

When K has only one parallel classes of arcs on T_L and T_R, K can be denoted by $K = \{(n, 0, 0; n, 0, 0|\rho)(p, q, -, -)(p', q', -, -)\}$ and we say that K is *of type* $(1, 1)$. As other types, we have $(1,2)$-, $(1,3)$-, $(2,2)$-, $(2,3)$- and $(3,3)$-types. We say that K is *separating* if $H \setminus K$ is disconnected, and otherwise, K is *non-separating.*

3. Study of double torus knots of type $(1,1)$

To study the fiberedness of double torus knots of type $(1,1)$, we state some properties of them.

Proposition 3.1. [8, Proposition 4.5] *Let K be a double torus knot of type $(1,1)$. Then; K is a non-separating knot if and only if $\gcd(n,p)=1$ and n is odd.*

If K is a separating knot, The separated piece of the double torus is a Seifert surface of genus 1. Fibered knots of genus 1 are only the trefoil and the figure-eight knot. Then we are only interested in a non-separating double torus knots of type $(1,1)$ and we may assume the following;

$$n \text{ is odd and } \gcd(n,p)=1. \tag{3.1}$$

As found in [8] our knot is obtained by band-connected sum of two torus knots of type (r,s) and $(r',-s')$. This can be convinced by drawing a figure. Moreover we have the following;

Proposition 3.2. [8, Theorem 4.4] *Let $K=\{(n,0,0;n,0,0|p)(r,s,-,-)(r',s',-,-)\}$ be a double torus knot of type $(1,1)$. Assume $|r|,|s|,|r'|,|s'|\geq 2$. Then K is a satellite knot.*

First, we observe that K is a satellite knot with companion a torus knot of type (r,s) and its pattern knot K' is another double torus knot of type $(1,1)$ of the form:

$$K'=\{(n,0,0;n,0,0|p)(1,rs,-,-)(r',s',-,-)\} \tag{3.2}$$

Now K' is again a satellite knot and its companion is a torus knot of type $(r',-s')$ and its pattern knot K'' is a double torus knot of type $(1,1)$ of the form:

$$K'=\{(n,0,0;n,0,0|p)(1,rs,-,-)(1,r's',-,-)\} \tag{3.3}$$

These observations lead to the following;

Proposition 3.3. *Let K be a double torus knot of type $(1,1)$, and K'' its final pattern knot given in (3.3). Then we have;*
(1) K is fibered if and only if K'' is fibered.
(2) K has a monic Alexander polynomial if and only if so does K''.

Proof. Since n is odd, the linking number between the pattern knot and the core of the companion torus is not 0. Therefore, the conclusion follows from Propositions 4.15 and 8.23 of [2]. □

By Proposition 3.3, it suffices to study the fiberedness for the knots of the form;

(3.4) $$K = \{(n, 0, 0; n, 0, 0|p)(1, \alpha, -, -)(1, \beta, -, -)\},$$

where α and β are non-zero integers. (If $\alpha = 0$ or $\beta = 0$, then K is a trivial knot.)

Remark 3.4. For example, the following knot K' is different from K in (3.4). $K' = \{(n, 0, 0; n, 0, 0|p)(\alpha, 1, -, -)(1, \beta, -, -)\}$. However, it is easy to show that K' is equivalent to the double torus knot $\tilde{K}$ of the form given in (3.4).

In the following, for simplicity, we denote (3.4) by $K(n, p|\alpha, \beta)$.

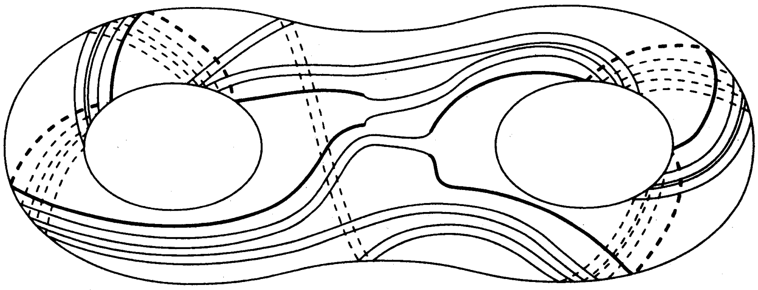
By rotating, twisting the right side torus T_R, mirroring, and isotopies, we have the following;

Proposition 3.5. *We have the following equivalences, where $-K$ means the mirror image of K.* $K(n, p|\alpha, \beta) \cong K(n, -p|\alpha, \beta) \cong K(n, n-p|\alpha, \beta) \cong K(n, p| -\beta, -\alpha) \cong -K(n, p|\beta, \alpha)$

As a corollary, we have and from now on assume the following;

Corollary 3.6. *Suppose $K(n, p|\alpha, \beta)$ is non-trivial and non-separating. Then $n > 3$, n is odd, $\gcd(n, p) = 1$ and $\alpha\beta \neq 0$. Without loss of generality, we may assume $n > p > 0$ and $\alpha \geq |\beta| > 0$. We may further restrict p to be even.*

Note that $K(n, p|\alpha, \beta)$ is obtained from the split union of two unknots by banding. As in the following figures, we can express $K(n, p|\alpha, \beta)$ by a schematic figure, where the core of the band is depicted by an arc. Moreover, we can prove that the full-twists of the arcs can be removed without affecting the fiberedness, while preserving the Alexander polynomials. For simplicity, we assume $\beta > 0$, but the other case is similarly understood. Remark the similarity to the Schubert's diagram of the 2-bridge knots (c.f. Figure 3.3).



$K(5,2|2,2)$

A double torus knot $K(n,p|\alpha,\beta)$ is a band-sum of two unknots.

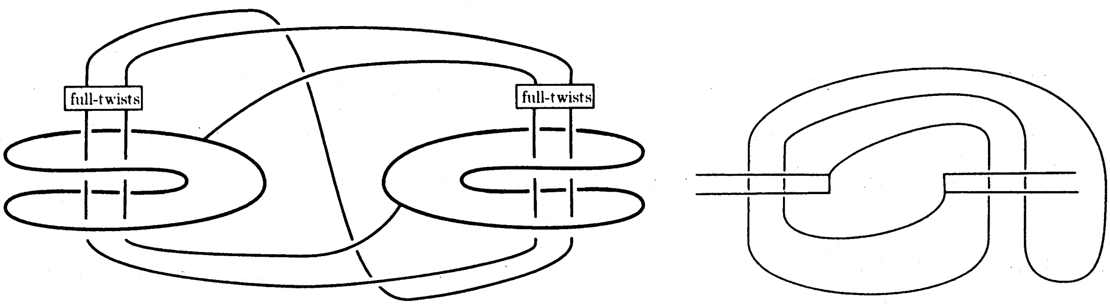


Figure 3.1: $K(5,2|2,2)$

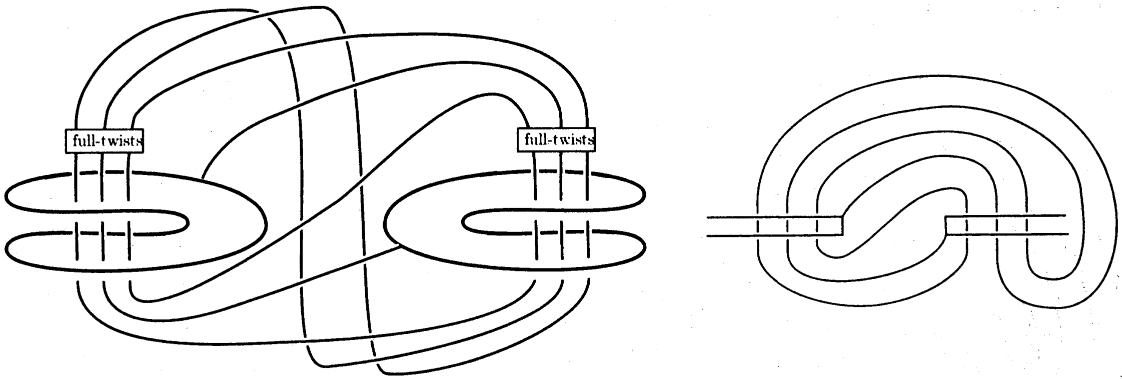


Figure 3.2: $K(7,4|2,2)$

Concerning the Alexander polynomial of double torus knots of type $(1,1)$, the following has been obtained. We denote by $B(n,p)$ the 2-bridge knot of type (n,p) using Schubert's notation.

Proposition 3.7. [8, Theorem 4.7] *Let* $K = K\{(n,0,0;n,0,0|p)(r,s,-,-)(r',s',-,-)\}$. *Then* K *is a band some of two torus knots* $T(r,s)$ *and* $T(r',-s')$, *and the Alexander* *polynomial of K is of the form;*

$$\Delta_K(t) = \Delta_{T(r,s)}(t)\Delta_{T(r',-s')}(t)f(t)f(t^{-1})$$

for some $f(t)$. Moreover, if $rs = r's' = \alpha$, then $f(t) = \Delta_{B(n,p)}(t^\alpha)$.

This is the first place where we can see an algebraic relationship between the 2-bridge knot $B(n,p)$ and $K\{(n,0,0;n,0,0|p)(r,s,-,-)(r',s',-,-)\}$. Inspired by this, T. Nakamura found a geometric relationship of them [12]. The most important observation in [12] concerning double torus knots contains the following.

Proposition 3.8. *The double torus knot $K = K(n,p|1,1)$ can be deformed by 'twistings of bands'into the connected sum of $B(n,p)$ and its orientation-reversed mirror image. Moreover the twistings preserve the Alexander polynomial.*

Figure 3.3 illustrates Proposition 3.8. Note that by constructing Seifert surfaces, we can see that the twisting of bands are realized by Stallings twists on minimal genus Seifert surfaces. However, in general, the Seifert surface obtained by smoothing the ribbon singularities of the ribbon disk is not of minimal genus.

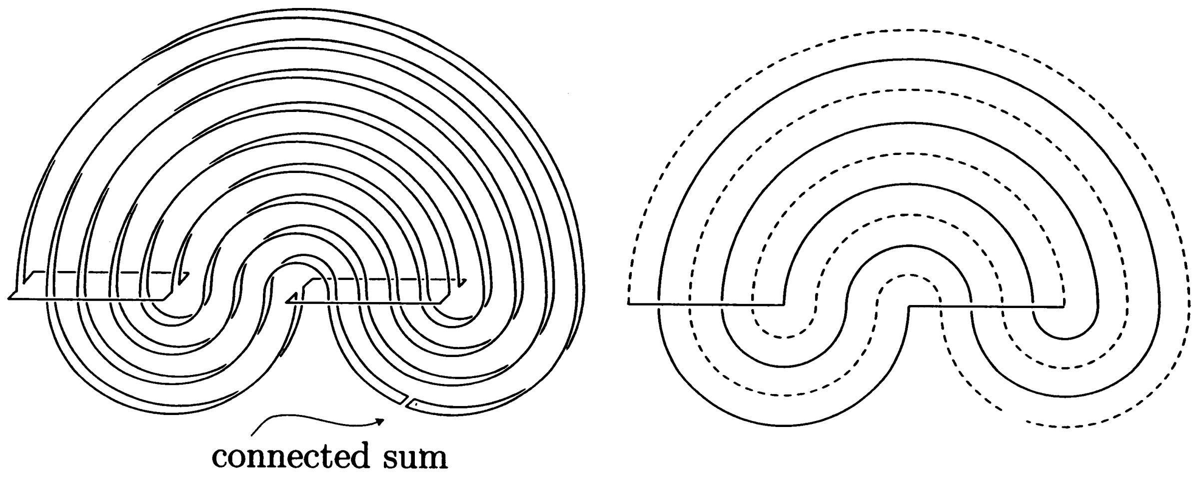


Figure 3.3: The connected sum of $B(5,4)$ and $-B(5,4)$.

Put $B(5,4)$ on this side of the sheet and $-B(5,4)$ on the other side. Applying the connected sum cuts 'the band' and we obtain a ribbon knot; two unknots connected by a band along 'the half' of Schubert's diagram.

4. Statements of results

In the next section we define the polynomial $h(t)$ for $K = K(n,p|\alpha,\beta)$ and give a formula to calculate the Alexander polynomial. We also define the graph $H(K)$ for K to calculate $h(t)$.

The main results we report in this note are as follows;

Theorem 4.1. *Let K be a non-separating double torus knot of type $(1,1)$. Then K is fibered if and only if the Alexander polynomial $\Delta_K(t)$ is monic.*

Though we do not go in detail in this note, the proof is given by the following two theorems. It is well known that fibered knots have monic Alexander polynomials.

Theorem 4.2. *If the Alexander $\Delta_K(t)$ is monic, then the graph $H(K)$ is admissible.*

Theorem 4.3. *If the graph $H(K)$ is admissible, then K is fibered.*

5. Fundamental tools and calculation of the Alexander polynomials

We intruduce our fundamental tool to calculate the Alexander polynomials of $K(n,p|\alpha,\beta)$, and 2-bridge knots $B(n,p)$. By Corollary 3.6, we assume;

(*) $n > 3$ is odd, p is even, $n > p > 0$, $\gcd(n,p) = 1$ and $\alpha \geq |\beta| > 0$.

In the following, we define a polynomial $h(t)$ and then we can calculate the Alexander polynomials as follows;

Theorem 5.1. *For $K = K(n,p|\alpha,\beta)$ satisfying (*), we have $\Delta_K(t) \doteq h(t)h(t^{-1})$.*

Recall that for a 2-bridge knot $B(n,p)$, we may assume n is odd and $n > p$ is even. Now by Theorem 5.1 and Proposition 3.8, we can calculate the Alexander polynomial of $B(n,p)$ as follows;

Corollary 5.2. *For a 2-bridge knot $K = B(n,p)$ with p even, we have $\Delta_K(t) \doteq h_{K'}(t)$, where $K' = K(n,p|1,1)$.*

Now we introduce basic tools. Given a pair of co-prime integers (n,p), let $S = \{p, 2p, \ldots, (n-1)p\}$. Then choose the representative $\overline{kp}, (1 \leq k \leq n-1)$ mod $2n$ so that $-n < \overline{kp} < n$, and define a new sequence of integers $\overline{S} = \{\overline{p}, \overline{2p}, \ldots, \overline{(n-1)p}\}$.

The following is an important fact which relates $K(n,p|\alpha,\beta)$ and the 2-bridge knot

Fact 5.3. *The sequence $\overline{S}$ for the pair (n,p) recovers the Schubert normal form of the diagram for the 2-bridge knot $B(n,p)$.*

Let ε_k be the sign of $\overline{kp}$, i.e., $\varepsilon_k = \overline{kp}/|\overline{kp}|$. *The sequence of signs for the pair* (n,p) is defined to be $\widetilde{S} = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}\}$. Let $Q = \{q_i\} = \{-\beta\varepsilon_1, \alpha\varepsilon_2, -\beta\varepsilon_3, \dots, -\beta\varepsilon_{n-2}, \alpha\varepsilon_{n-1}\}$, and $R = \{r_i\} = \{\sum_{k=1}^{i} q_k\}$. Define $h(t) = 1 + \Sigma_{i=1}^{n-1}(-1)^i t^{r_i}$.

By introducing the *graph* $H(K)$ of K, we can easily obtain $h(t)$ from (a half of) Schubert's diagram without manipulating the above sequences;

Define the graph $H(K)$ of $K(n,p|\alpha,\beta)$ as obtained by plotting $(0,0)$, $(1,r_1)$, $(2,r_2)$, $\dots, (n-1, r_{n-1})$ in the xy-plane and connecting adjacent vertices. Bi-color the vertices of $H(K)$ black and white alternatingly so that the first and the last (the n-th) are black. Then we can read off $h(t)$ from $H(K)$: the coefficient for t^j equals the number of black vertices at the j-level minus that of white vertices at the j-level. Now, the graph $H(K)$ can be directly obtained from (the half) of Schubert's diagram for the 2-bridge knot $B(n,p)$ by following the underpath and record from which way it goes through the overpath, where each time the y-coordinate of the vertex goes up or down by α or β.

We say that the graph $H(K)$ is *admissible* if $H(K)$ has exactly one absolute maximal vertex and one absolute minimal vertex.

See Figure 5.1 for example at $K(n,p|\alpha,\beta) = K(11,8|2,1)$.
$\overline{S} = \{8,-6,2,10,-4,4,-10,-2,6,-8\}$, $\widetilde{S} = \{1,-1,1,1,-1,1,-1,-1,1,-1\}$,
$Q = \{-1,-2,-1,2,1,2,1,-2,-1,-2\}$, $R = \{-1,-3,-4,-2,-1,1,2,0,-1,-2\}$.
$h(t) = -t^{-4} + 2t^{-3} + t^{-2} - 3t^{-1} + 2 + t - t^2$.

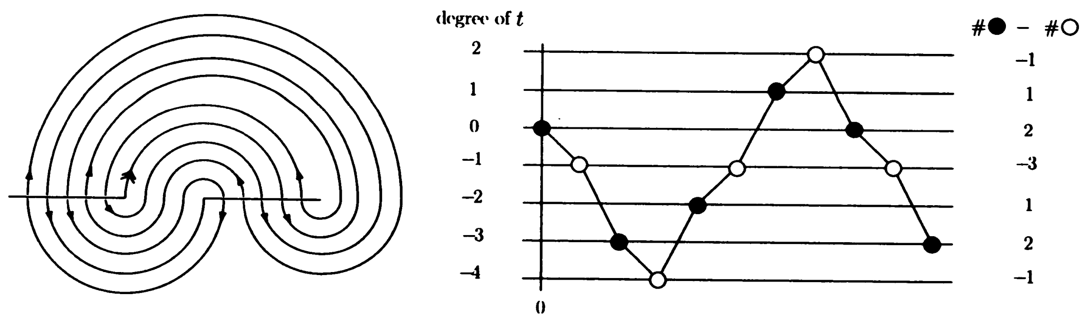


Figure 5.1: The half of the Schubert's diagram of $B(11,8)$
to construct the graph $H(K)$ to obtain $h(t)$.

Example 5.4. See Figures 5.2, 5.3 for the cases of $(n,p) = (7,2)$ and $(7,4)$ for various (α,β).

Remark 5.5. If $(\alpha,\beta) = (1,1)$, cancellations of the terms in $h(t)$ never occur, because the vertices of the graph H lying in the same y-coordinate have the same color. However in general, as seen in $K(7,2|2,1)$, some terms of $h(t)$ may cancel each other. Moreover as in $K(11,2|2,1)$, cancellation among terms of local maximum may happen. Cancellations among terms of the highest degree may yield a 'non-fibered knot with a monic Alexander polynomial'. However, Theorem 4.2 asserts it never happens.

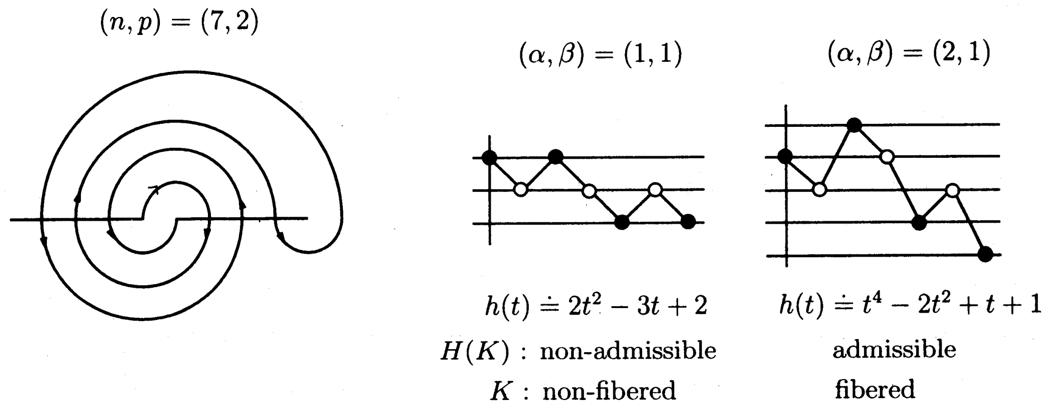


Figure 5.2

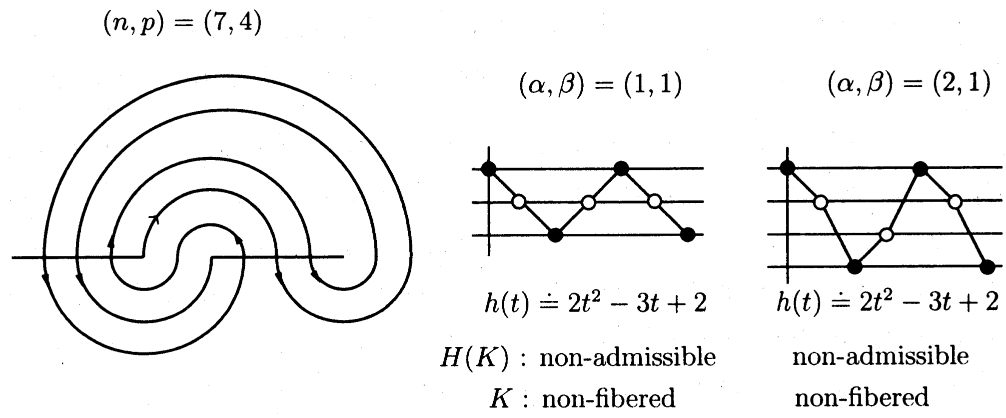


Figure 5.3

Remark 5.6. Note that as 2-bridge knots, $B(7,2) = B(7,4)$, and hence they have the same Alexander polynomial. However, as seen in the above example, the ways terms appear are different. This difference causes the following interesting fact. Note that

$B(7,2), B(7,4)$ are non-fibered, and hence $K(7,2|1,1), K(7,4|1,1)$ are non-fibered. However, $K(7,2|2,1)$ is fibered while $K(7,4|2,1)$ is non-fibered.

5.1. **Application of diagrammatic calculations of $\Delta_K(t)$.** In this subsection, we use the diagrammatic calculations of $\Delta_K(t)$ to have straight forward explanation to the facts found in [8]. The latter half of Theorem 3.7 is understood as follows;
For $K = K(n,p|\alpha,\alpha)$ with p even, the graph $H(K)$ is obtained by similarly expanding by α the graph for $K' = K(n,p|1,1)$. Therefore $h_K(t) \doteq h_{k'}(t^\alpha)$. Meanwhile, by Proposition 3.8, we have $h_{K'}(t) \doteq h_{B(n,p)}(t)$. We give one more application: The following was pointed out in [8, p. 636]. We understand this by seeing that the Alexander polynomial is not monic.

Proposition 5.7. *For any non-zero α, $K(n,p|\alpha,-\alpha)$ is not a fibered knot.*

Proof. Assume p is even. Then the y-coordinates of the vertices $v_0, \dots, v_{n-1}$ are as follows:

$$\{0, \alpha\varepsilon_1, \alpha\varepsilon_1 + \alpha\varepsilon_2, \alpha\varepsilon_1 + \alpha\varepsilon_2 + \alpha\varepsilon_3, \dots, \alpha\varepsilon_1 + \alpha\varepsilon_2 + \dots + \alpha\varepsilon_{n-1}\}.$$

By the skew-symmetry of $\{\varepsilon_1, \dots, \varepsilon_{n-1}\}$, we see that the above is equal to

$$\{0, \alpha\varepsilon_1, \alpha\varepsilon_1 + \alpha\varepsilon_2, \alpha\varepsilon_1 + \alpha\varepsilon_2 + \alpha\varepsilon_3, \dots, \alpha\varepsilon_1 + \alpha\varepsilon_2 + \alpha\varepsilon_3, \alpha\varepsilon_1 + \alpha\varepsilon_2, \alpha\varepsilon_1, 0\}.$$

Since the number of vertices of $H(K)$ is odd $(= n)$, this means the bi-colored graph $H(K)$ is symmetric with respect to a vertical line which goes through the center vertex $v_{\frac{n-1}{2}}$, i.e., each vertex other than $v_{\frac{n-1}{2}}$ has its counterpart of the same color at the same y-coordinate. Therefore, $h_K(t)$ is not monic, and hence by Theorem 5.1, neither is $\Delta_K(t)$.

5.2. **When $\Delta_K(t)$ is monic?** In this subsection, we consider when $\Delta_K(t)$ is monic for $K = K(n,p|\alpha,\beta)$, or equivalently, when K is fibered. In the previous subsection, we saw that for any $\alpha \neq 0$, $\Delta_{K(n,p|\alpha,-\alpha)}(t)$ is not monic, while $\Delta_{K(n,p|\alpha,\alpha)}(t)$ is monic if and only if $\Delta_{B(n,p)}(t)$ is monic, i.e., the 2-bridge knot $B(n,p)$ is fibered. However, even if $\Delta_{B(n,p)}(t)$ is not monic, it can happen that $\Delta_{K(n,p|\alpha,\beta)}(t)$ is monic for some α, β (see Remark 5.6). For more detailed discussion, we have the following propositions;

Proposition 5.8. *Suppose $\alpha > 0$ and $\beta > 0$. Then we have;*
(1) $\Delta_{K(n,p|\alpha,\beta)}(t)$ is monic if and only if $\Delta_{K(n,p|2,1)}(t)$ is monic.
(2) $\Delta_{K(n,p|\alpha,-\beta)}(t)$ is monic if and only if $\Delta_{K(n,p|2,-1)}(t)$ is monic.

Proposition 5.9. *If $B(n,p)$ is fibered, then for any $\alpha, \beta > 0$, $\Delta_{K(n,p|\alpha,\beta)}(t)$ is monic.*

There are six cases for the monic property of the Alexander polynomial of $K(n, p|\alpha, \beta)$. In the right column, we give an example in each case.

Case(A)	$B(n,p)$ is fibered.	
(1)	$\Delta_{K(n,p\|\alpha,\beta)}(t)$ is monic $\Leftrightarrow \alpha \neq -\beta$.	*e.g.*, $B(25,16)$
(2)	$\Delta_{K(n,p\|\alpha,\beta)}(t)$ is monic $\Leftrightarrow \alpha\beta > 0$.	*e.g.*, $B(5,2)$

Case(B)	$B(n,p)$ is not fibered.	
(1)	$\Delta_{K(n,p\|\alpha,\beta)}(t)$ is monic $\Leftrightarrow \alpha \neq \pm\beta$.	*e.g.*, $B(7,2)$
(2)	$\Delta_{K(n,p\|\alpha,\beta)}(t)$ is monic $\Leftrightarrow \alpha\beta > 0, \alpha \neq \beta$.	*e.g.*, $B(9,2)$
(3)	$\Delta_{K(n,p\|\alpha,\beta)}(t)$ is monic $\Leftrightarrow \alpha\beta < 0, \alpha \neq -\beta$.	*e.g.*, $B(17,10)$
(4)	$\Delta_{K(n,p\|\alpha,\beta)}(t)$ is not monic for any α, β	*e.g.*, $B(9,4)$.

Remark 5.10. In [8] it was proved that $K(n, n-1|\alpha, \beta)$ belongs to Case A1, by calculating the fundamental groups of explicitly constructed fiber surfaces and their complement.

Problem 5.11. An open problem is to determine the monic property with respect to (n, p).

6. TOOLS TO PROVE FIBEREDNESS

In this section, we recall two important reasoning to prove that a Seifert surface is a fiber surface. One is called *Stallings twists* and the other *Kobayashi's banding on pre-fiber surfaces.*

A *Stallings twist* is an operation to produce a new fiber surface from one with a certain condition; Let c be an unknotted oriented circle embedded in a surface F in S^3. Suppose the linking number $\mathrm{lk}(c, c') = 0$, where c' is a push off of c in a normal direction of F. Then apply ± 1-surgery along c. Briefly, the operation is to cut F by a disk spanned by c' and then glue back after a twisting. Obviously, the new ambient manifold is S^3, but we have a new Seifert surface for a (different) link. J. Stallings [15] showed the following;

Proposition 6.1. *Suppose a seifert surface F' is obtained from F by a Stallings twist. Then F' is a fiber surface if and only if so is F.*

In [10], T. Kobayashi introduced the notion of *pre-fiber (Seifert) surface* for links and, using that notion, determined in [11] when a band connected sum of links is a fibered link. In this section, we recall his results. For the notion of *sutured manifold*, we refer to [10] or [4].

Let L be a link with a Seifert surface F. Denote by $F_E = F \cap E(L)$ the restriction of F in the link exterior $E(L) = \mathrm{cl}(S^3 - N(L))$. The sutured manifold $(N, \delta) = (F_E \times I, \partial F_E \times I)$ is a *product sutured manifold*, where $R_+(\delta)$ and $R_-(\delta)$ are respectively $F_E \times \{1\}$ and $F_E \times \{0\}$. The sutured manifold $(N^c, \delta^c) = (\mathrm{cl}(E(L) - N), \mathrm{cl}(\partial E(L) - \delta))$, where $R_\pm(\delta^c) = R_\mp(\delta)$ is called the *complementary sutured manifold* for F.

Definition. A Seifert surface S is a *pre-fiber surface* if there exist pairwise disjoint compressing disks D^+, D^- in N^c for $R_+(\delta^c) = R_-(\delta^c)$ respectively such that $(\bar{N}, \delta^c)$ is homeomorphic to a (not necessarily connected) product sutured manifold, where $\bar{N}$ denotes the manifold obtained from N^c by cutting along $D^+ \cup D^-$. Then there is a pair of compressing disks $\bar{D}^+, \bar{D}^-$ for S such that $\bar{D}^\pm \cap N^c = D^\pm$, which we call *a pair of canonical compressing disks* for S.

To determine when a band connected sum of two links are fibered, the following notion is essential. Kobayashi called the following banding a *band of type F*, now after Kobayashi, we call it a *K-band*.

Let S be a pre-fiber surface with a pair of canonical compressing disks $D^+ \cup D^-$. Let p_+ and p_- be properly embedded arcs in S sharing exactly one end point $e \in \partial S$. Their interiors may intersect each other in S. Push p_+ (resp. p_-) in the positive (resp. negative) normal direction of S, and then push $e = p_+ \cap p_-$ off S so that we obtain an arc α in S^3 such that $\alpha \cap S = \partial\alpha \subset \partial S$. Suppose α intersects each of D^+ and D^- in exactly in one point.

Definition. Let S be a pre-fiber surface and β a band whose ends are attached to ∂S and whose interior misses S. We call β a *K-band* if its core γ (fixing its end points) is isotopic to an arc α obtained by the above construction.

Kobayashi obtained the following; (Theorem 3 and Proposition A in [11]).

Proposition 6.2. *Let $L = L_1 \cup L_2$ be a split link with a 2-sphere separating L_1 and L_2 in S^3. Then L bounds a pre-fiber surface S if and only if both L_1 and L_2 are fibered.*

Proposition 6.3. *Let F be a Seifert surface obtained from a pre-fiber surface S by adding a band β. Then F is a fiber surface if and only if β is a K-band.*

Remark 6.4. Note that the twisting of β is irrelevant because that can be generated by Stallings twists using D^+.

Example. Following sequence of Seifert surfaces $\Sigma_1, \Sigma_2, \cdots$ in Figure 6.1 are examples of pre-fiber surfaces. First, Σ_1 is an annulus, which is obtained by tubing two disks. Second, Σ_2 is obtained from Σ_1 by another tubing, where the new tube goes through the first tube. Next, Σ_3 is obtained from Σ_2 again by adding a tube which goes though the innermost tube of Σ_2. Inductively, we can construct Σ_i's. By [11, Theorem 3], any pre-fiber surface for the 2-component trivial link is isotopic to Σ_i for some i, where the pair of canonical compressing disk comes from the innermost disk among $Q - (Q \cap \Sigma_i)$, where Q is the separating 2-sphere for the trivial link.

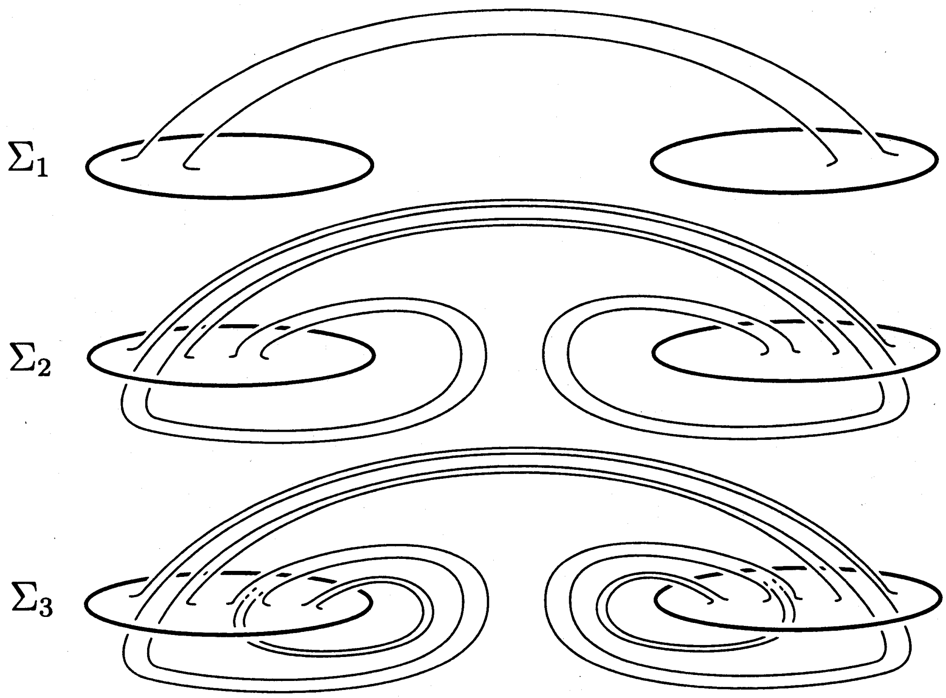


Figure 6.1: Pre-fiber surfaces for the 2-component trivial link.

7. Construction of fiber surfaces

In this section we show an example of a fiber surface. Let us take $K = K(5, 2|2, 1)$ as an example. With the band represented by an arc, K appears as in Figure 6.1 (a), where each blank box contains a full-twist. Slide the ends of the arcs to obtain Figure 6.1 (b). As we will see briefly, we can eliminate the full-twists by Stallings twists. Then isotope the bands as in Figure 6.1 (c). Then we see that K spans a Seifert surface which is a banding of a pre-fiber surface in Figure 6.1 (d). Moreover the band is a K-band, and hence by Proposition 6.3, we have a fiber surface. Actually, for general $K(n, p|\alpha, \beta)$ we can systematically and explicitly read off from the graph $H(K)$ how we should isotope the bands so that we obtain a Seifert surface F, where $g(F)$ is the half of the degree of the Alexander polynomial and hence F is a minimal genus Seifert surface.

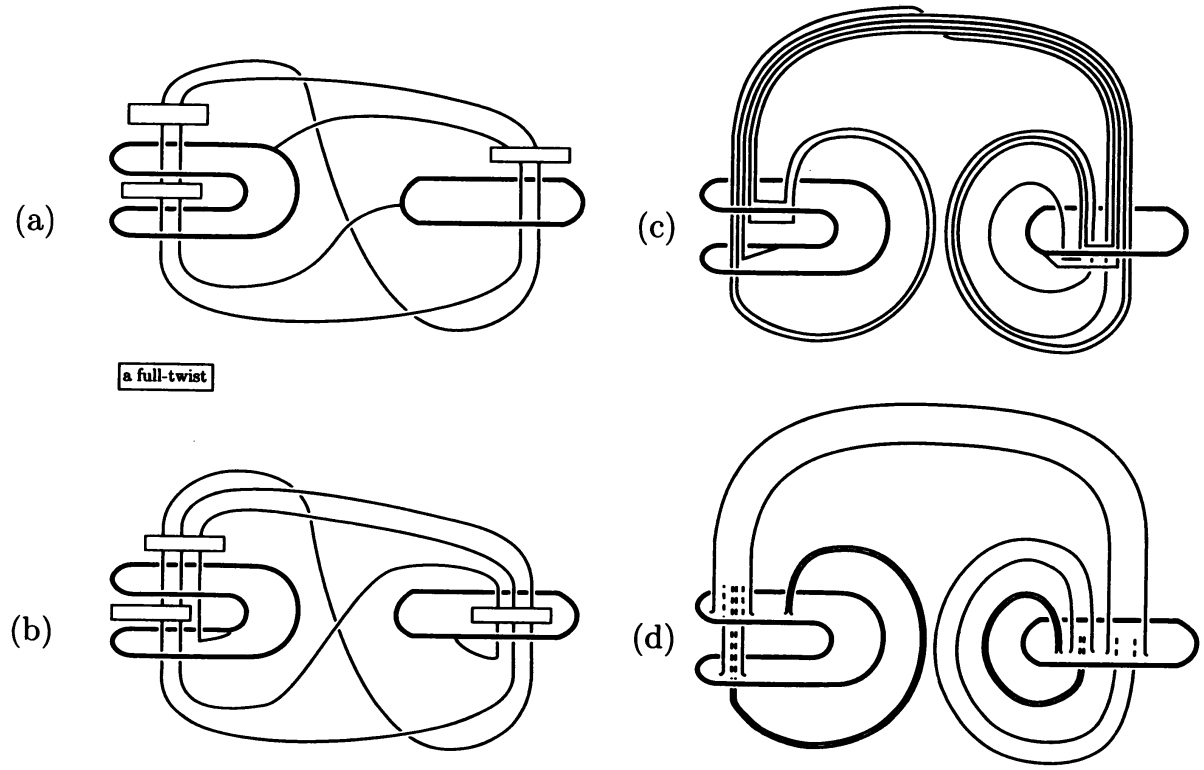


Figure 7.1

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