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Transcendence of certain reciprocal sums of linear recurrences

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1 Introduction

Let C be a field of characteristic 0 and d an integer greater than 1. We consider the function $f(z)$ defined by

$$f(z)=\sum_{k\geq 0}\frac{a^k z^{d^k}}{H(z^{d^k})}, \tag{1}$$

where $H(z)\in C[z]$ with $H(0)=1$ and $\deg H(z)\geq 1$, and $a\in C$ with $a\neq 0$. Then the function $f(z)$ satisfies the functional equation

$$af(z^d)=f(z)-\frac{z}{H(z)}. \tag{2}$$

It is known that $f(z)$ represents a rational function in the following four cases:

(i) If $d=2, a=4$, and $H(z)=(1+z)^2$, then

$$f(z)=\sum_{k\geq 0}\frac{4^k z^{2^k}}{\left(1+z^{2^k}\right)^2}=\frac{z}{(1-z)^2}.$$

(ii) If $d=2, a=-2$, and $H(z)=1-z+z^2$, then

$$f(z)=\sum_{k\geq 0}\frac{(-2)^k z^{2^k}}{1-z^{2^k}+z^{2^{k+1}}}=\frac{z}{1+z+z^2}.$$

(iii) If $d=2, a=2$, and $H(z)=1+z$, then

$$f(z)=\sum_{k\geq 0}\frac{2^k z^{2^k}}{1+z^{2^k}}=\frac{z}{1-z}.$$

(iv) If $d = 2, a = 1$, and $H(z) = 1 - z^2$, then

$$f(z) = \sum_{k \geq 0} \frac{z^{2^k}}{1 - z^{2^{k+1}}} = \frac{z}{1-z}.$$

It is natural to ask whether there exist rational functions of the form (1) other than these four cases. The purpose of this paper is to answer this question.

Theorem 1.1. *Let $f(z)$ be the function defined by* (1). *Suppose that* $\deg H \leq 3$. *Then $f(z)$ is a transcendental function over $C(z)$ except in the four cases stated above.*

In the case of $a \neq 1$, we can dispense with the assumption $\deg H \leq 3$.

Theorem 1.2. *Let $f(z)$ be the function defined by* (1). *Suppose that $a \neq 1$. Then $f(z)$ is a transcendental function over $C(z)$ except in the three cases stated above.*

We shall apply Theorem 1.1 to establish the transcendence of new type of reciprocal sums of binary linear recurrences.

Let $\{F_n\}_{n \geq 0}$ be the sequence of the Fibonacci numbers defined by

$$F_0 = 0, \ F_1 = 1, \ F_{n+2} = F_{n+1} + F_n \quad (n \geq 0),$$

and $\{L_n\}_{n \geq 0}$ be the sequence of the Lucas numbers defined by

$$L_0 = 2, \ L_1 = 1, \ L_{n+2} = L_{n+1} + L_n \quad (n \geq 0).$$

Lucas [6] proved that

$$\theta_1 = \sum_{k \geq 0} \frac{1}{F_{2^k}} = \frac{7 - \sqrt{5}}{2}.$$

Erdös and Graham [5] asked for arithmetic character of the related sums

$$\theta_2 = \sum_{k \geq 0} \frac{1}{L_{2^k}}, \quad \theta_3 = \sum_{k \geq 0} \frac{1}{F_{2^k+1}}.$$

Transcendence of θ_2 and that of θ_3 were proved by Bundschuh and Pethö [2] and by Becker and Töpfer [1], respectively.

Let $\{R_n\}_{n \geq 0}$ be a sequence of integers satisfying the binary linear recurrence relation

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n \quad (n \geq 0), \tag{3}$$

where $A_1 \neq 0$, A_2 are integers, $\Delta = A_1^2+4A_2 > 0$ is not a perfect square, and R_0, R_1 are integers not both zero. We can express $\{R_n\}_{n\geq 0}$ as follows:

$$R_n = g_1\alpha^n + g_2\beta^n \quad (n \geq 0),$$

where $g_1 = (R_1 - \beta R_0)/(\alpha - \beta)$, $g_2 = (\alpha R_0 - R_1)/(\alpha - \beta)$, and α, β are the roots of

$$X^2 - A_1X - A_2 = 0.$$

Then we define R_l for any $l \in \boldsymbol{Z}$ by $R_l = g_1\alpha^l + g_2\beta^l$.

Becker and Töpfer [1] proved a more general theorem.

Theorem A (Becker and Töpfer [1]). *Let $\{R_n\}_{n\geq 0}$ be a sequence of integers satisfying (3), $\{a_k\}_{k\geq 0}$ be a periodic sequence of algebraic numbers which is not identically zero, and d, c, and l be integers with $d \geq 2$ and $c \geq 1$. Then the number*

$$\theta = \sum_{k\geq 0}{}' \frac{a_k}{R_{cd^k+l}},$$

where the sum $\sum'_{k\geq 0}$ is taken over those k with $cd^k + l \geq 0$ and $R_{cd^k+l} + b \neq 0$, is algebraic if and only if $\{a_k\}_{k\geq 0}$ is a constant sequence, $d = 2, |A_2| = 1$, and $R_l = 0$.

Their result was much more improved by Nishioka, Tanaka, and Toshimitu [10]. Indeed they established the algebraic independence of the numbers

$$\sum_{k\geq 0}{}' \frac{a_k}{(R_{cd^k+l})^m} \quad (d \geq 2, m \geq 1, l \in \boldsymbol{Z})$$

even under a weaker condition on $\{R_n\}_{n\geq 0}$.

Duverney [3] showed that

$$\sum_{k\geq 1} \frac{4^k}{L_{2^k} + 2} = 4, \quad \sum_{k\geq 1} \frac{(-2)^k}{L_{2^k} - 1} = -\frac{1}{2}. \tag{4}$$

These numbers are special cases of the following reciprocal sums

$$\phi = \sum_{k\geq 0}{}' \frac{a_k}{R_{cd^k+l} + b}, \tag{5}$$

where the sum $\sum'_{k\geq 0}$ is taken over those k with $cd^k + l \geq 0$ and $R_{cd^k+l} + b \neq 0$, $\{a_k\}_{k\geq 0}$ is a linear recurrence of algebraic numbers which is not identically zero, and b, c, d, and l are integers with $c \geq 1$ and $d \geq 2$. Using Theorem 1.1 and applying a method developed in [9], we can show that these numbers are transcendental except some few cases including the numbers given by (4).

Theorem 1.3. *Let* $\{R_n\}_{n\geq 0}$ *be a sequence of integers satisfying* (3). *Then the number* ϕ *defined by* (5) *is transcendental except in the following three cases:*

(i) $|A_2| = 1, d = 2, b = 0, R_l = 0$, *and* $\{a_k\}_{k\geq 0}$ *is a constant sequence.*

(ii) $|A_2| = 1, d = 2, A_1 R_l = 2R_{l+1}, R_l = b$, *and* $a_k = c4^k$ $(k \geq 0)$ *for some nonzero* $c \in \overline{Q}$.

(iii) $|A_2| = 1, d = 2, A_1 R_l = 2R_{l+1}, R_l = -2b$, *and* $a_k = c(-2)^k$ $(k \geq 0)$ *for some nonzero* $c \in \overline{Q}$.

Remark 1.1. Becker and Töpfer's result stated above can be deduced from Theorem 1.3.

2 Proof of Theorems

2.1 Proof of Theorem 1.1

The function $f(z)$ is transcendental over $C(z)$ if $f(z) \notin C(z)$ (cf. [7]). Suppose on the contrary that $f(z) = P(z)/Q(z)$ with $P(z), Q(z) \in C[z]$ prime to each other. As $f(0) = 0$, we have $P(0) = 0$ and $Q(0) \neq 0$, so that we may assume $Q(0) = 1$. By (2) we have

$$a\frac{P\left(z^d\right)}{Q\left(z^d\right)} = \frac{P\left(z\right)}{Q\left(z\right)} - \frac{z}{H(z)},$$

and so

$$aP(z^d)Q(z)H(z) = P(z)Q(z^d)H(z) - zQ(z)Q(z^d). \tag{6}$$

As $P(z^d)/Q(z^d)$ is irreducible, $Q(z^d)$ divides $Q(z)H(z)$. Therefore there exist $A(z) \in C[z]$ such that

$$A(z)Q(z^d) = Q(z)H(z). \tag{7}$$

As $P(0) = 0$, we put $P(z) = zR(z)$. We have from (6)

$$Q(z)^2 = A(z)\{R(z)Q(z^d) - az^{d-1}R(z^d)Q(z)\}. \tag{8}$$

In what follows, let h, p, q, and r be the degrees of H, P, Q, and R, respectively. Then we have by (7) and (8)

$$\deg A = h - (d-1)q \leq 2q. \tag{9}$$

We shall prove

$$1 \leq 1 + r = p \leq q. \tag{10}$$

As $P(0) = 0$, we have $p \geq 1$. If $p > q$, we get $\deg PH > \deg zQ$, since $1 \leq h$ (≤ 3) by (2) and (7). Then (6) yield $dp + q + h = dq + p + h$, a contradiction and (10) follows.

The proof will be done in three cases; Case I. $p < q$, Case II. $p = q$ and $a \neq 1$, Case III. $p = q$ and $a = 1$.

Case I. Let $p < q$. We have $q \geq 2$ by (10) and $2q = \deg A + r + dq$ by (8). This with (10) implies $\deg A = 0 = r$, and $d = 2$. Hence $A(z) = 1$ and $R(z) = 1$, since $A(0) = 1$ by (7) and $R(0) = 1$ by (8). Then we have by (8)

$$Q(z)^2 = Q(z^2) - azQ(z). \tag{11}$$

Writing $Q(z) = a_q z^q + a_{q-s} z^{q-s} + \cdots$, where $a_q \neq 0$, $a_{q-s} \neq 0$ $(1 \leq s \leq q)$, we have from (11)

$$\begin{aligned} {a_q}^2 z^{2q} &+ 2a_q a_{q-s} z^{2q-s} + \cdots \\ &= a_q z^{2q} + a_{q-s} z^{2q-2s} + \cdots - az(a_q z^q + a_{q-s} z^{q-s} + \cdots). \end{aligned}$$

We see that $a_q = 1$. First we consider the case of $q \geq 3$. If $1 \leq s \leq q-2$, then $2q - s > 2q - 2s$ and $2q - s > q + 1$, so we have $a_{q-s} = 0$, which is a contradiction. Therefore we have $s = q - 1$ or $s = q$. Thus we have $Q(z) = z^q + a_1 z + 1$, where $a_1 \neq 0$ if $s = q - 1$, $= 0$ if $s = q$. We have from (11)

$$\begin{aligned} z^{2q} + 2a_1 z^{q+1} &+ 2z^q + {a_1}^2 z^2 + 2a_1 z + 1 \\ &= z^{2q} + a_1 z^2 + 1 - az(z^q + a_1 z + 1). \end{aligned}$$

Noting that $q \geq 3$ and comparing the coefficients of z^q in the both sides, we have a contradiction.

Therefore we have $q = 2$, and so $Q(z) = z^2 + a_1 z + 1$. It follows from (11) that

$$\begin{aligned} z^4 + 2a_1 z^3 &+ 2z^2 + {a_1}^2 z^2 + 2a_1 z + 1 \\ &= z^4 + a_1 z^2 + 1 - az(z^2 + a_1 z + 1). \end{aligned}$$

Comparing the coefficients of the both sides, we have

$$2a_1 = -a, \quad 2 + {a_1}^2 = a_1 - aa_1.$$

Hence we have $(a, a_1) = (4, -2)$ or $(-2, 1)$, and so we get

$$f(z) = \sum_{k \geq 0} \frac{4^k z^{2^k}}{(1 + z^{2^k})^2} = \frac{z}{(1-z)^2}$$

$$f(z) = \sum_{k \geq 0} \frac{(-2)^k z^{2^k}}{1 - z^{2^k} + z^{2^{k+1}}} = \frac{z}{1 + z + z^2},$$

which are the rational functions given in the case (i) and (ii), respectively.

Case II. Let $p = q$ and $a \neq 1$. We have from (8) $2q = \deg A + r + dq$. This with (10) implies $\deg A = 0 = r, q = 1$, and $d = 2$. Hence $A(z) = 1$ and $R(z) = 1$, since $A(0) = 1$ by (7) and $R(0) = 1$ by (8). Writing $Q(z) = 1 - bz$ with $b \neq 0$, we have from (8)

$$1 - 2bz + b^2z^2 = 1 - bz^2 - az(1 - bz).$$

Comparing the coefficients of both sides, we have

$$b^2 = -b + ab, \quad 2b = a.$$

Hence we have $a = 2, b = 1$, and so we get

$$f(z) = \sum_{k \geq 0} \frac{2^k z^{2^k}}{1 + z^{2^k}} = \frac{z}{1 - z}.$$

which is the rational function given in the case (iii).

Case III. Let $p = q$ and $a = 1$. From (8) we have

$$Q(z)^2 = A(z)\{R(z)Q(z^d) - z^{d-1}R(z^d)Q(z)\}. \tag{12}$$

Lemma 2.1. *We can express $Q(z)$ as*

$$Q(z) = \Pi_{i=1}^{d-1}(1 - \gamma_i{}^{-1}z)^{n_i}Q_1(z),$$

where γ_i $(1 \leq i \leq d-1)$ are the $(d-1)$-th roots of unity, $n_i \geq 1$ $(1 \leq i \leq d-1)$, and $Q_1(z) \in C[z]$ such that $Q_1(\gamma_i) \neq 0$ for any i. Furthermore

$$A(z) = \Pi_{i=1}^{d-1}(1 - \gamma_i{}^{-1}z)^{n_i}A_1(z),$$

where $A_1(z) \in C[z]$ such that $A_1(\gamma_i) \neq 0$ for any i. In paticular,

$$d - 1 \leq \deg A \quad \textit{and} \quad d - 1 \leq q. \tag{13}$$

Proof. Letting $z = \gamma_i$ in (12) we have $Q(\gamma_i) = 0$ for any i. We may put

$$Q(z) = \Pi_{i=1}^{d-1}(1 - \gamma_i{}^{-1}z)^{n_i}Q_1(z),$$

where $n_i \geq 1$ $(1 \leq i \leq d-1)$ and $Q_1(z) \in C[z]$ such that $Q_1(\gamma_i) \neq 0$ for any i. From (12) we have

$$\Pi_{i=1}^{d-1}(1-\gamma_i{}^{-1}z)^{n_i}Q_1(z)^2 = A(z)\{R(z)Q_1(z^d)\Pi_{i=1}^{d-1}\varphi(\gamma_i{}^{-1}z)^{n_i} - z^{d-1}R(z^d)Q_1(z)\},$$

where $\varphi(z) = (1-z^d)/(1-z)$. Letting $z = \gamma_j$ for fixed j, we have

$$0 = A(\gamma_j)R(\gamma_j)Q_1(\gamma_j)(\Pi_{i=1}^{d-1}\varphi(\gamma_i{}^{-1}\gamma_j)^{n_i} - 1).$$

We note that $\varphi(\gamma_i{}^{-1}\gamma_j) = 1$ if $i \neq j$ and $\varphi(\gamma_i{}^{-1}\gamma_j) = d$ if $i = j$. So $\Pi_{i=1}^{d-1}\varphi(\gamma_i{}^{-1}\gamma_j)^{n_i} - 1 = d^{n_j} - 1 \neq 0$. Since $R(\gamma_j)Q_1(\gamma_j) \neq 0$, we obtain $A(\gamma_j) = 0$ for any j. Therefore we may put

$$A(z) = \Pi_{i=1}^{d-1}(1-\gamma_i{}^{-1}z)^{n_i}A_1(z),$$

where $A_1(z) \in C[z]$ such that $A_1(\gamma_i) \neq 0$ for any i. The proof of the lemma is completed.

Now we return to the proof in Case III. It follows from (9) and (13) that

$$1 \leq \max\{d-1, \frac{h}{d+1}\} \leq q \leq \frac{h}{d-1} - 1. \tag{14}$$

In paticular, we have

$$2 \leq d(d-1) \leq h. \tag{15}$$

In the case of $h = 2$, we have $d = 2$ by (15) and so $q = 1$ by (14). We have $R(z) = 1$ by (10) and $Q(z) = 1 - z$ by Lemma 2.1, which implies $A(z) = 1 - z$ by (12), and so $H(z) = 1 - z^2$ by (7). This gives the functional equation

$$f(z) = \sum_{k \geq 0} \frac{z^{2^k}}{1 - z^{2^{k+1}}} = \frac{z}{1-z},$$

which is the rational function given in the case (iv).

If $h = 3$, we have $d = 2$ by (15), and hence $q = 1$ or 2 by (14). Assume that $q = 1$. Then we have $Q(z) = 1 - z$, $R(z) = 1$, and $A(z) = 1 - z$ by (12), which contradicts (9). If $q = 2$, we have $Q(z) = (1-z)(1-bz)$, where $b \neq 1$, $A(z) = 1 - z$, and $R(z) = 1 - cz$, which implies $(1-bz)^2 = (1-cz)(1+z)(1-bz^2) - z(1-cz^2)(1-bz)$. Letting $z = 1$ we have $b = c$ since $b \neq 1$, which is impossible since P, R are coprime. The proof of Theorem 1.1 is completed.

2.2 Proof of Theorem 1.2

The proof is the same as these of Case I and II in Theorem 1.1, since the condition $\deg H \leq 3$ is not used there.

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