On aperiodic tilings by the projection method

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In 1982 quasi-crystals with icosahedral symmetry were discovered. (published in 1984). It had been axiomatic that the structure of a crystal was periodic, like a wallpaper pattern. Periodicty is another name for translational symmetry. Icosahedral symmetry is incompatible with translational symmetry and therefore quasi-crystals are not periodic. Most famous 2-dimensional mathematical model for a quasi-crystal is a Penrose tiling of the plane. In 1981 de Bruijn introduced projection methods to construct aperiodic tilings such as Penrose tilings.

We recall the definition of tilings by the projection method.

$L$ : a lattice in $\mathbb{R}^d$ with a basis $\{b_i|i=1,2,\cdots,d\}$.
$E$ : a $p$-dimensional subspace of $\mathbb{R}^d$,
$E^\perp$ : its orthogonal complement.
$\pi : \mathbb{R}^d \rightarrow E$, $\pi^\perp : \mathbb{R}^d \rightarrow E^\perp$ : the orthogonal projections.
$A$ : a Voronoi cell of $L$

For any $x \in \mathbb{R}^d$ we put

$$W_x = \pi^\perp(x) + \pi^\perp(A) = \{\pi^\perp(x) + u|u \in \pi^\perp(A)\}$$

$$\Lambda(x) = \pi((W_x \times E) \cap L).$$

The Voronoi cell of a point $v \in \Lambda(x)$

$$V(v) = \{u \in \mathbb{R}^n||v-u| \leq |y-u|, \text{forally} \in \Lambda(x)\}.$$ 

$\mathcal{V}(x)$ : the Voronoï tiling induced by $\Lambda(x)$, which consists of the Voronoï cells of $\Lambda(x)$.

For a vertex $v$ in $\mathcal{V}(x)$

$$S(v) = \bigcup\{P \in \mathcal{V}(x)|v \in P\}.$$

The tiling $T(x)$ given by the projection method is defined as the collection of tiles $\text{Conv}\ (S(v) \cap \Lambda(x))$, where $\text{Conv}\ (B)$ denotes the convex hull of a set $B$. Note that $\Lambda(x)$ is the set of the vertices of $T(x)$. 
In order to state theorems we recall several definitions. The dual lattice $L^*$ is defined by the set of vectors $y \in \mathbb{R}^d$ such that $\langle y, x \rangle \in \mathbb{Z}$ for all $x \in L$, where $\langle \, , \, \rangle$ denotes standard inner product. A lattice $L$ is called integral if all its vectors satisfy that $\langle x, y \rangle \in \mathbb{Z}$ for all $x, y \in L$. The standard lattice is both integral and self dual.

For $L = \mathbb{Z}^d$, C. Hillman characterized the number of periods of the tilings. He also constructed periods for given tilings.

One of Hillman's results is extended to the case that $L$ is integral.

Theorem. Let $T(x)$ be the tiling by the projection method and assume that $L$ is integral. Then, rank $\text{Ker}(\pi^\perp|L)$ is equal to the dimension of the linear space of the periods of $T(x)$.

For the general lattices Theorem is not true. We have the following example;
$L$ : a lattice in $\mathbb{R}^2$ with a basis $\{(1, \sqrt{2}), (1, -1)\}$,
$E$ : the $x$-axis of $\mathbb{R}^2$.

In this case it is easy to see that all tilings in $\mathbb{R}^1$ obtained by the projection method are periodic and rank $\text{Ker}(\pi^\perp|L) = 0$.

The following property is analogous to classical uniform distribution of sequences.

Theorem (de Bruijn and Senechal, 1995)
Assume that $\pi^\perp(L)$ is dense in $E^\perp$.
$K_1, K_2$ : $(d - p)$-dimensional cubes in $E^\perp$
$J \subset E$ : a $p$-dimensional cube centered at the origin.
For any positive real number $\lambda$, we set
$P_\lambda^1 = K_1 \times \lambda J, P_\lambda^2 = K_2 \times \lambda J$.

Then,
$$\lim_{\lambda \to \infty} \frac{\text{card} P_\lambda^1 \cap L}{\text{card} P_\lambda^2 \cap L} = \frac{Vol(K_1)}{Vol(K_2)}$$
A tiling space $T(E)$ is defined by a space of tilings consisting of all translates by $E = \mathbb{R}^p$ of the tilings $T(x)$ for all $x \in E^\perp$. Tiling spaces are topological dynamical systems, with a continuous $\mathbb{R}^p$ translation action and a topology defined by a tiling metric on tilings of $\mathbb{R}^p$.

Let $\text{Orb}(T(x))$ denote the orbit of $T(x)$ in $T(E)$ by the $\mathbb{R}^p$ translation action and $\text{span}(A)$ denote the $\mathbb{R}$-linear span of a set $A$.

Uniform distribution of the projection method is closely related to the ergodicity of the tiling space.

Theorem Let $T(E)$ be the tiling by the projection method in terms of a $p$-dimensional subspace $E$ of $\mathbb{R}^d$ and $p' : E^\perp \to \text{span}(L^* \cap E^\perp)$ be the orthogonal projection. Define $p : L \to \text{span}(L^* \cap E^\perp)$ by $p = p' \circ (\pi^\perp|L)$. We take a basis $x_1, \ldots, x_k$ of the direct summand $K$ such that $L = p^{-1}(\{0\}) \oplus K$. Then $T(E)$ decomposes into a $k$ parameter family of orbit closures $\text{Orb}(T(t_1x_1 + \cdots + t_kx_k))$ for $t_1, \ldots, t_k \in \mathbb{R}$.

In particular, we obtain that $k$ is equal to $\text{rank}(L^* \cap E^\perp)$.

Note that $\pi^\perp(L)$ is dense in $E^\perp$ if and only if $E^\perp \cap L^* = \{0\}$. A. Hof (1988) proved that $E^\perp \cap L^* = \{0\}$ if and only if $T(E) = \overline{\text{Orb}(T(0))}$. Assume that $L$ is integral. Then we see that $\text{rank}(L^* \cap E^\perp) = \text{rank}(L \cap E^\perp) = \text{rank Ker}(\pi|L)$ because $L \subset L^*$ and $L^*/L$ is finite. The number of independent periods of the tiling space $T(E^\perp)$ is equal to $\text{rank Ker}(\pi|L)$.

We immediately obtain the following theorem in the case that $L$ is integral:

Theorem Let $T(E)$ (resp. $T(E^\perp)$) be the tiling space by the projection method in terms of a $p$-dimensional subspace $E$ (resp. $(d-p)$-dimensional subspace $E^\perp$) of $\mathbb{R}^d$ and assume that $L$ is an integral lattice. Then $T(E)$ decomposes into a $k$ parameter family of orbit closures, where $k$ is equal to the number of independent periods of the tiling space $T(E^\perp)$. 

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