数理解析研究所講究録 1274 巻 2002 年 174-176

On aperiodic tilings by the projection method

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In 1982 quasi-crystals with icosahedral symmetry were discovered. (published in 1984). It had been axiomatic that the structure of a crystal was periodic, like a wallpaper pattern. Periodicty is another name for translational symmetry. Icosahedral symmetry is incompatible with translational symmetry and therefore quasi-crystals are not periodic. Most famous 2-dimensional mathematical model for a quasi-crystal is a Penrose tiling of the plane. In 1981 de Bruijn introduced projection methods to construct aperiodic tilings such as Penrose tilings.

We recall the definition of tilings by the projection method.

L : a lattice in $\mathbf{R}^d$ with a basis $\{b_i | i = 1, 2, \cdots, d\}$.

E : a p-dimensional subspace of $\mathbf{R}^d$,

$E^{\perp}$: its orthogonal complement.

$\pi : \mathbf{R}^d \to E$, $\pi^{\perp} : \mathbf{R}^d \to E^{\perp}$: the orthogonal projections.

A : a Voronoï cell of L

For any $x \in \mathbf{R}^d$ we put

$$W_x = \pi^{\perp}(x) + \pi^{\perp}(A) = \{\pi^{\perp}(x) + u | u \in \pi^{\perp}(A)\}$$

$$\Lambda(x) = \pi((W_x \times E) \cap L).$$

The Voronoï cell of a point $v \in \Lambda(x)$

$$V(v) = \{u \in \mathbf{R}^n || v - u| \leq |y - u|, \text{forall} y \in \Lambda(x)\}.$$

$\mathcal{V}(x)$: the Voronoï tiling induced by $\Lambda(x)$, which consists of the Voronoï cells of $\Lambda(x)$.

For a vertex v in $\mathcal{V}(x)$

$$S(v) = \bigcup \{P \in \mathcal{V}(x) | v \in P\}.$$

The tiling $T(x)$ given by the projection method is defined as the collection of tiles Conv $(S(v) \cap \Lambda(x))$, where Conv (B) denotes the convex hull of a set B. Note that $\Lambda(x)$ is the set of the vertices of $T(x)$.

In order to state theorems we recall several definitions. The dual lattice L^* is defined by the set of vectors $y \in \mathbf{R}^d$ such that $\langle y, x\rangle \in \mathbf{Z}$ for all $x \in L$, where $\langle\ ,\ \rangle$ denotes standard inner product. A lattice L is called integral if all its vectors satisfy that $\langle x, y\rangle \in \mathbf{Z}$ for all $x, y \in L$. The standard lattice is both integral and self dual.

For $L = \mathbf{Z}^d$, C. Hillman characterized the number of periods of the tilings. He also constructed periods for given tilings.

One of Hillman's results is extended to the case that L is integral.

Theorem. *Let $T(x)$ be the tiling by the projection method and assume that L is integral. Then,* rank Ker $(\pi^{\perp}|L)$ *is equal to the dimension of the linear space of the periods of $T(x)$.*

For the general lattices Theorem is not true. We have the following example;

L : a lattice in $\mathbf{R}^2$ with a basis $\{(1, \sqrt{2}), (1, -1)\}$,

E : the x-axis of $\mathbf{R}^2$.

In this case it is easy to see that all tilings in $\mathbf{R}^1$ obtained by the projection method are periodic and rank Ker $(\pi^{\perp}|L) = 0$.

The following property is analogous to classical uniform distribution of sequences.

Theorem (de Bruijn and Senechal, 1995)

Assume that $\pi^{\perp}(L)$ is dense in $E^{\perp}$.

K_1, K_2:$(d-p)$-dimensional cubes in $E^{\perp}$

$J \subset E$:a p-dimensional cube centered at the origin.

For any positive real number λ, we set

$$P_\lambda^1 = K_1 \times \lambda J, P_\lambda^2 = K_2 \times \lambda J.$$

Then,

$$\lim_{\lambda \to \infty} \frac{\textbf{card } P_\lambda^1 \cap L}{\textbf{card } P_\lambda^2 \cap L} = \frac{Vol(K_1)}{Vol(K_2)}$$

A tiling space $\mathcal{T}(E)$ is defined by a space of tilings consisting of all translates by $E = \mathbf{R}^p$ of the tilings $T(x)$ for all $x \in E^\perp$. Tiling spaces are topological dynamical systems, with a continuous $\mathbf{R}^p$ translation action and a topology defined by a tiling metric on tilings of $\mathbf{R}^p$

Let $\mathrm{Orb}(T(x))$ denote the orbit of $T(x)$ in $\mathcal{T}(E)$ by the $\mathbf{R}^p$ translation action and $\mathrm{span}(A)$ denote the $\mathbf{R}$-linear span of a set A.

Uniform distribution of the projection method is closely related to the ergodicity of the tiling space.

Theorem *Let $\mathcal{T}(E)$ be the tiling by the projection method in terms of a p-dimensional subspace E of $\mathbf{R}^d$ and $p' : E^\perp \to \mathrm{span}(L^* \cap E^\perp)$ be the orthogonal projection. Define $p : L \to \mathrm{span}(L^* \cap E^\perp)$ by $p = p' \circ (\pi^\perp|L)$. We take a basis $x_1, \ldots, x_k$ of the direct summand K such that $L = p^{-1}(\{0\}) \oplus K$. Then $\mathcal{T}(E)$ decomposes into a k parameter family of orbit closures $\overline{\mathrm{Orb}(T(t_1x_1 + \cdots + t_kx_k))}$ for $t_1, \ldots, t_k \in \mathbf{R}$.*

In particular, we obtain that k is equal to $\mathrm{rank}\ (L^* \cap E^\perp)$.

Note that $\pi^\perp(L)$ is dense in $E^\perp$ if and only if $E^\perp \cap L^* = \{0\}$. A. Hof(1988) proved that $E^\perp \cap L^* = \{0\}$ if and only if $\mathcal{T}(E) = \overline{\mathrm{Orb}(T(0))}$. Assume that L is integral. Then we see that $\mathrm{rank}\ (L^* \cap E^\perp) = \mathrm{rank}\ (L \cap E^\perp) = \mathrm{rank\ Ker}\ (\pi|L)$ because $L \subset L^*$ and L^*/L is finite. The number of independent periods of the tiling space $\mathcal{T}(E^\perp)$ is equal to $\mathrm{rank\ Ker}\ (\pi|L)$. We immediately obtain the following theorem in the case that L is integral:

Theorem *Let $\mathcal{T}(E)$ (resp. $\mathcal{T}(E^\perp)$) be the tiling space by the projection method in terms of a p-dimensional subspace E (resp. $(d-p)$-dimensional subspace $E^\perp$) of $\mathbf{R}^d$ and assume that L is an integral lattice. Then $\mathcal{T}(E)$ decomposes into a k parameter family of orbit closures, where k is equal to the number of independent periods of the tiling space $\mathcal{T}(E^\perp)$.*