数理解析研究所講究録 1274 巻 2002 年 193-201

An Additive Problem with Piatetski-Shapiro Primes and Almost-Primes

T. P. Peneva

University of Tsukuba, Tsukuba, Japan

Abstract. Suppose that $\frac{662}{755} < \gamma < 1$. We prove a theorem of the Bombieri-Vinogradov type for the Piatetski-Shapiro primes $p = [n^{1/\gamma}]$ and show that every sufficiently large even integer can be written as a sum of a Piatetski-Shapiro prime and an almost-prime.

2000 Mathematics Subject Classification: 11L07, 11L20, 11P32, 11N13, 11N36

Key words: Piatetski-Shapiro primes, Goldbach problem, Bombieri-Vinogradov theorem, exponential sums

1. Introduction and Statement of the Results

In 1937, I. M. Vinogradov [23] solved the *ternary* Goldbach problem by proving that for every sufficiently large odd integer n the equation

$$n = p_1 + p_2 + p_3 \tag{1}$$

has solutions in prime numbers p_1, p_2, p_3.

The *binary* Goldbach problem, which states that every even integer $N \geq 4$ can be represented as the sum of two primes, remains unsettled. An important approach for studying this problem is by the use of sieve methods. Denote, as usual, by P_r any integer with no more than r prime factors, counted according to multiplicity. In 1947, A. Rényi [19] was the first to prove that there exists an r such that every sufficiently large even N is representable in the form

$$N = p + P_r\,, \tag{2}$$

where p is a prime number. The best result in this direction belongs to J.-R. Chen [3] who proved, in 1973, that (2) holds for $r = 2$.

Let γ be a real number such that $\frac{1}{2} < \gamma < 1$. Define

$$\pi_\gamma(x) := |\{p \leq x :\ p = [n^{1/\gamma}] \ \text{ for some } \ n \in \mathbb{N}\}|\,.$$

(here $[t]$ is the integer part of t).

In 1953, I. I. Piatetski-Shapiro [18] showed that

$$\pi_\gamma(x) \sim x^\gamma / \log x \quad (x \to \infty)\,, \tag{3}$$

for $\frac{11}{12} < \gamma < 1$. The prime numbers of the form $p = [n^{1/\gamma}]$ are called *Piatetski-Shapiro primes of type* γ. By using the close connection between the lower bound for γ and the estimates of the exponential sums over primes, a number of authors obtained (3) for longer ranges of γ—G. A. Kolesnik ([11], [12]), D. Leitmann [15], D. R. Heath-Brown [8], H.-Q. Liu and J. Rivat [17] and Rivat [20]. The best known result $\frac{2426}{2817} < \gamma < 1$ is due to J. Rivat and P. Sargos [21].

J. Rivat [20] was the first to consider the problem for obtaining a lower bound for $\pi_\gamma(x)$. By using a sieve method he proved that

$$\pi_\gamma(x) \gg x^\gamma / \log x \quad (x \geq x_0), \tag{4}$$

for $\frac{6}{7} < \gamma < 1$. After that R. C. Baker, G. Harman and J. Rivat [1], C.-H. Jia ([9], [10]) and A. Kumchev [13] improved this result. Finally, J. Rivat and J. Wu [22] showed that (4) holds for $\frac{205}{243} < \gamma < 1$.

In 1992 A. Balog and J. B. Friedlander [2] found an asymptotic formula for the number of solutions of the equation (1) with variables restricted to the Piatetski-Shapiro primes. An interesting corollary of their theorem is that every sufficiently large odd integer can be written as the sum of two primes and a Piatetski-Shapiro prime of type γ, provided that $\frac{8}{9} < \gamma < 1$. Later, A. Kumchev [14] extended this range to $\frac{64}{73} < \gamma < 1$.

Considering the above results, it is interesting to study the solvability of the equation (2) when p is a Piatetski-Shapiro prime. It is naturally expected that a theorem of the Bombieri–Vinogradov type holds for the Piatetski-Shapiro primes. However, the only result in this direction, due to D. Leitmann [16], gives a very low level of distribution which does not allow us to determine the value of the parameter r.

We should also mention the result of D. Fischer and T. Zhan [4], which states a theorem of the Bombieri-Vinogradov type for *almost all* $\gamma \in (\frac{1}{2} + \epsilon, 1)$, where $\epsilon > 0$ is a sufficiently small number.

In the present paper we use D. R. Heath-Brown's approach of [8] to establish the following

Theorem. *Suppose that γ is a real number in the range $\frac{662}{755} < \gamma < 1$, $a \neq 0$ is a fixed integer. Then for any given constant $A > 0$ and any sufficiently small $\epsilon > 0$,*

$$\sum_{\substack{q \leq x^\xi \\ (a,q)=1}} \Bigg| \sum_{\substack{p \leq x \\ p=[n^{1/\gamma}] \\ p \equiv a \,(\mathrm{mod}\, q)}} 1 - \frac{1}{\varphi(q)} \pi_\gamma(x) \Bigg| \ll \frac{x^\gamma}{(\log x)^A}, \tag{5}$$

where

$$\xi = \xi(\gamma) = \begin{cases} \frac{755}{424}\gamma - \frac{331}{212} - \epsilon & \textit{for} \quad \frac{662}{755} < \gamma \leq \frac{608}{675}; \\[2ex] \frac{5}{4}\gamma - \frac{13}{12} - \epsilon & \textit{for} \quad \frac{608}{675} < \gamma < 1. \end{cases} \tag{6}$$

For convenience, we note that $\frac{662}{755} = 0.8768\ldots$, $\frac{608}{675} = 0.9007\ldots$.

An application of [6, Theorem 9.3] gives the following

Corollary. *In the notation of the Theorem, we put* r *to be the least positive integer satisfying the inequality*

$$r + 1 - \frac{\log(4/(1+3^{-r}))}{\log 3} \geq \xi^{-1} + \delta\,,$$

where $\delta > 0$ *is a sufficiently small number. Then every sufficiently large even integer* N *can be represented in the form (2), where* p *is a Piatetski-Shapiro prime number of type* γ *and the least prime factor of* P_r *is* $\geq N^{\xi/4}$.

Notice the two special cases: $r = 7$ for $0.9854 < \gamma < 1$ and $r = 24$ for $\gamma = \frac{608}{675}$.

Throughout this paper x is a sufficiently large number, p is a prime number. We write $\{t\}$ and $\|t\|$ for the fractional part of t and the distance from t to the nearest integer, correspondingly. As usual, $\varphi(n)$ and $\Lambda(n)$ denote Euler's function and von Mangoldt's function, respectively. We write $L = \log x$; $e(t) = \exp(2\pi i t)$; $\psi(t) = \{t\} - \frac{1}{2}$. Instead of $m \equiv n \pmod q$ we write for simplicity $m \equiv n\ (q)$. The notation $n \sim X$ means that n runs through a *sub-interval* of $(X, 2X]$, which endpoints are not necessary the same in the different equations and may depend on the outer summation variables. For positive X and Y, we write $X \asymp Y$ instead of $X \ll Y \ll X$.

2. Outline of the Proof of the Theorem

Step 1: Preliminaries. The first stage of the proof is to transform the problem of estimating the sum in (5) into one involving exponential sums over primes.

For convenience, we put $Q = x^{\xi}$. Clearly, the Theorem will follow, if we can prove that for $X \leq x$,

$$\sum_{\substack{q \leq Q \\ (a,q)=1}} \Bigg| \sum_{\substack{k \sim X \\ k = [n^{1/\gamma}] \\ k \equiv a\,(q)}} \Lambda(k) - \frac{1}{\varphi(q)} \sum_{\substack{k \sim X \\ k = [n^{1/\gamma}]}} \Lambda(k) \Bigg| \ll x^{\gamma} L^{-A}\,. \tag{7}$$

For $1/2 < \gamma < 1$ it is easy to show that

$$[-k^{\gamma}] - [-(k+1)^{\gamma}] = \begin{cases} 1 & if \quad k = [n^{1/\gamma}]\,; \\ 0 & if \quad k \neq [n^{1/\gamma}]\,. \end{cases} \tag{8}$$

Therefore, to prove (7) it is sufficient to demonstrate that

$$\sum_{\substack{q \leq Q \\ (a,q)=1}} \Bigg| \sum_{\substack{n \sim X \\ n \equiv a(q)}} \Lambda(n)\Big((n+1)^{\gamma} - n^{\gamma}\Big) - \frac{1}{\varphi(q)} \sum_{n \sim X} \Lambda(n)\Big((n+1)^{\gamma} - n^{\gamma}\Big) \Bigg| \ll x^{\gamma} L^{-A}\,, \tag{9}$$

$$\sum_{\substack{q \leq Q \\ (a,q)=1}} \Bigg| \sum_{\substack{n \sim X \\ n \equiv a(q)}} \Lambda(n)\Big(\psi(-n^{\gamma}) - \psi(-(n+1)^{\gamma})\Big) \Bigg| \ll x^{\gamma} L^{-A} \tag{10}$$

and

$$\sum_{\substack{q \leq Q \\ (a,q)=1}} \frac{1}{\varphi(q)} \Bigg| \sum_{n \sim X} \Lambda(n)\Big(\psi(-n^{\gamma}) - \psi(-(n+1)^{\gamma})\Big) \Bigg| \ll x^{\gamma} L^{-A}\,. \tag{11}$$

The inequality (9) can be obtained from the Bombieri–Vinogradov theorem by using partial summation and it holds for every $\gamma \in (\frac{1}{2}, 1)$ and $Q = x^{1/2-\epsilon}$, where $\epsilon > 0$ is a sufficiently small number. The inequality (11) follows from the arguments in [8]. Hence, we only have to show (10).

Let $\eta > 0$ be a sufficiently small number. We may assume that $x^{1-\eta} \le X \le x$, otherwise (10) is trivial. Consequently, we have

$$X^{\xi} \le Q \le X^{\xi+\eta/2},$$

for $\xi \le (1-\eta)/2$.

We now use the well-known expansions

$$\psi(t) = -\sum_{0<|h|\le H} \frac{e(th)}{2\pi i h} + \mathcal{O}(g(t,H)), \tag{12}$$

where

$$g(t,H) = \min\left(1, \frac{1}{H\|t\|}\right) = \sum_{h=-\infty}^{\infty} b_h e(th)$$

and

$$b_h \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{|h|^2}\right).$$

We insert (12) into the left-hand side of (10) and evaluate first the contribution of the error term

$$\sum_{\substack{q\le Q\\(a,q)=1}} \sum_{\substack{n\sim X\\ n\equiv a(q)}} \Lambda(n)\Big(g(n^{\gamma},H) + g((n+1)^{\gamma},H)\Big) = R_1 + R_2,$$

say. We treat only R_1, the estimate of R_2 is similar. We have

$$\begin{aligned} R_1 &\ll L \sum_{\substack{q\le Q\\(a,q)=1}} \sum_{\substack{n\sim X\\ n\equiv a(q)}} g(n^{\gamma},H) \\ &\ll L \sum_{\substack{q\le Q\\(a,q)=1}} \sum_{h=-\infty}^{\infty} |b_h| \Big| \sum_{\substack{n\sim X\\ n\equiv a(q)}} e(hn^{\gamma})\Big|. \end{aligned}$$

We now require the next estimate, which is an analogue of [8, Lemma 1] for arithmetic progressions.

Lemma 1. *Let* $1 \le q \le X$, $X < X_1 \le 2X$. *Then*

$$\sum_{\substack{X<n\le X_1\\ n\equiv a(q)}} e(hn^{\gamma}) \ll \min(q^{-1}X,\ |h|^{-1}q^{-1}X^{1-\gamma} + |h|^{1/2}X^{\gamma/2}).$$

We now find

$$\begin{aligned} R_1 &\ll L\sum_{q\le Q}\Big(|b_0|\,q^{-1}X+\sum_{h\ne 0}|b_h|\,(|h|^{-1}q^{-1}X^{1-\gamma}+|h|^{1/2}X^{\gamma/2})\Big)\\ &\ll L^3H^{-1}X+LX^{1-\gamma}\sum_{q\le Q}q^{-1}\sum_{h\ne 0}|h|^{-2}\\ &\qquad +LX^{\gamma/2}Q\Big(\sum_{0<|h|\le H}|h|^{-1/2}+H\sum_{|h|>H}|h|^{-3/2}\Big)\\ &\ll L^3H^{-1}X+L^2X^{1-\gamma}+LH^{1/2}X^{\gamma/2}Q\\ &\ll x^{\gamma}L^{-A}, \end{aligned}$$

on taking

$$H=X^{1-\gamma}L^{2A}$$

and

$$\gamma\ge\frac{1}{2}+\xi+\eta. \tag{13}$$

It remains to show that

$$\sum_{\substack{q\le Q\\(a,q)=1}}\ \sum_{0<|h|\le H}\Big|h^{-1}\sum_{\substack{n\sim X\\ n\equiv a(q)}}\Lambda(n)\Big(e(-hn^{\gamma})-e(-h(n+1)^{\gamma})\Big)\Big|\ll x^{\gamma}L^{-A}.$$

Working similarly to [8, §2], we see that in order to establish the last inequality it is sufficient to prove that

$$\Big|\sum_{k\sim X}\Lambda(k)\,G(k)\Big|\ll XL^{-A}, \tag{14}$$

where

$$G(k)=\sum_{0<h\le H}\Theta_h(k)\,e(hk^{\gamma}) \tag{15}$$

and

$$\Theta_h(k)=\sum_{\substack{q\le Q\\(a,q)=1\\ q|k-a}}c(q,h),\qquad |c(q,h)|=1.$$

Step 2: Combinatorial decomposition. By applying Heath-Brown's identity [7, Lemma 1], we can express $\sum_{k\sim X}\Lambda(k)\,G(k)$ in terms of sums

$$\sum_{\substack{m_1\dots m_{2j}\sim X\\ m\sim M_i}}\cdots\sum\mu(m_1)\dots\mu(m_j)\,\log m_{2j}\,G(m_1\dots m_{2j}),$$

where $1 \le j \le 3$, $M_1 \dots M_{2j} \sim X$, $M_1, \dots, M_j \le X^{1/3}$. By dividing the M_j into two groups we obtain

$$\Big|\sum_{k\sim X} \Lambda(k)\, G(k)\Big| \ll X^{\eta} \max \Big|\sum_{\substack{mn\sim X\\ m\sim M}}\sum a(m)\, b(n)\, G(mn)\Big|, \tag{16}$$

where the maximum is taken over all bilinear forms with coefficients satisfying one of

$$|a(m)| \le 1, \qquad |b(n)| \le 1 \tag{17}$$

or

$$|a(m)| \le 1, \qquad b(n) = 1$$

or

$$|a(m)| \le 1, \qquad b(n) = \log n$$

and in all cases

$$M \le X. \tag{18}$$

We refer to the case (17) as being Type *II* sums and to the other cases as being Type *I* sums. Denote them by $\sum_{II}$ and $\sum_{I}$, respectively.

The following statement belongs to Balog and Friedlander [2, Proposition 1].

Lemma 2. *If we have real numbers $0 < u < 1$, $0 < v < z < 1$ satisfying the inequalities $v < \frac{2}{3}$, $1 - z < z - v$ and $1 - u < \frac{1}{2}z$, then (16) still holds when (18) is replaced by the conditions*

$$M \le X^{u} \qquad \textit{for Type I sums}$$

and

$$X^{v} \le M \le X^{z} \qquad \textit{for Type II sums}.$$

□

Step 3: Estimate of Type I Sums. We have the following

Lemma 3. *Let (κ, l) be an exponent pair for which*

$$4\kappa - 2l + 1 > 0. \tag{19}$$

Suppose that M is such that

$$M \ll \min\left(X^{1-e},\ X^{(2\kappa-2l+2-4\xi-3\eta)/(4\kappa-2l+1)},\ X^{(2l+1+4\xi)/(2l-2\kappa+2)}\right)$$

where

$$e = \frac{6\kappa + 5 - \gamma(4\kappa + 6) + 4\xi + 24\eta}{4\kappa - 2l + 1}. \tag{20}$$

Suppose also that

$$\gamma \geq \frac{7\kappa + 3l + 14}{10\kappa + 2l + 20} + \frac{5\kappa + l + 12}{5\kappa + l + 10}\xi + 5\eta,$$

$$\gamma \geq \frac{5\kappa + 3l + 11}{6\kappa + 2l + 14} + \frac{l - \kappa + 3}{3\kappa + l + 7}\xi + 5\eta$$

and

$$\gamma \geq \frac{5}{6} + \xi + 5\eta.$$

Then

$$\textstyle\sum_I \ll X^{1-2\eta}.$$

□

Step 4: Estimate of Type II Sums. The following statement holds.

Lemma 4. *Suppose that*

$$X^{5-5\gamma+4\xi+15\eta} \leq M \leq X^{\gamma-15\eta}$$

and

$$\gamma \geq \max\left(\frac{1}{2} + 2\xi + 6\eta,\ \frac{5}{6} + \frac{2}{3}\xi + 6\eta\right).$$

Then

$$\textstyle\sum_{II} \ll X^{1-2\eta}.$$

□

Step 5: Conclusion. We now put, in the notation of Lemma 2,

$$u = 1 - e,$$

$$v = 5 - 5\gamma + 4\xi + 15\eta,$$

$$z = \gamma - 15\eta,$$

where the quantity e is defined by (20) and $\eta > 0$ is a sufficiently small number. We take the exponent pair

$$\kappa = \frac{11}{53}, \quad l = \frac{33}{53},$$

which satisfies (19) and define the quantity ξ as in (6) with $\epsilon = 50\eta$.

Then it is not difficult to show that the conditions of Lemma 2, 3 and 4, as well as the inequality (13), hold.

Hence we obtain (14), which suffices to complete the proof of the Theorem. □

3. Proof of the Corollary

We shall show that the conditions of [6, Theorem 9.3] hold. Consider the sequence

$$\mathcal{A} = \{N - p : p \leq N,\ p = [n^{1/\gamma}] \text{ for some } n \in \mathbb{N}\}$$

$$\mathcal{B} = \{p : p \nmid N\}.$$

Define

$$X = \pi_\gamma(N),$$

$$\omega(d) = \begin{cases} d\varphi(d)^{-1} & \text{if } (d,N) = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Now it is easy to prove that the conditions (Ω_1) and $(\Omega_2{}^*(1))$ hold. The condition $(R(1,\alpha))$ follows directly from (5) after we get rid of the extra factor $3^\nu(d)$ using, for example, Cauchy's inequality. As to the condition (Ω_3), we see from the proof of [6, Theorem 9.3] that it is sufficient to establish

$$\sum_{\substack{m \sim M \\ m \equiv N(p^2) \\ m = [n^{1/\gamma}]}} 1 \ll \frac{N^\gamma}{p^2}, \tag{21}$$

for $M \leq N$ and $p \leq N^\xi \leq N^{1/6}$. As in §2, first we apply the identity (8) and introduce the function $\psi(.)$. Then (21) will follow from

$$\sum_{\substack{m \sim M \\ m \equiv N(p^2)}} ((m+1)^\gamma - m^\gamma) \ll \frac{N^\gamma}{p^2} \tag{22}$$

and

$$\sum_{\substack{m \sim M \\ m \equiv N(p^2)}} (\psi(-m^\gamma) - \psi(-(m+1)^\gamma)) \ll \frac{N^\gamma}{p^2}. \tag{23}$$

Obviously, the inequality (22) holds. As to (23), we use formulas (12) with $H = M^{1-\gamma}$ and after that we estimate the contribution of the main term and the error term by applying Lemma 1.

Finally, by [6, Theorem 9.3] we obtain

$$|\{P_r : P_r \in \mathcal{A}\}| \gg \frac{X}{\log X},$$

which suffices to complete the proof of the Corollary. □

Remark. The results in [1], [9], [10], [13], [17], [20] were obtained by an application of the double large sieve, given by E. Fouvry and H. Iwaniec in [5, Theorem 3], which makes use of the summation over h in estimating the Type II sums. We suppose that the same technique can be used to improve the result of our Theorem. However, we are not able to apply the method successfully, since the existence of the quantity $\Theta_h(k)$ in the definition of the function $G(k)$, given by (15), does not allow us to divide the summation over the variables m and n in an effective way.

Acknowledgements. The author would like to thank Dr. Hiroshi Mikawa for his encouragement and valuable suggestions.

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Institute of Mathematics
University of Tsukuba
Tsukuba 305–8571
Japan
e-mail: tpeneva@math.tsukuba.ac.jp