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On systems of linear inequalities (線形不等式系の族について)

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Linear $f_1, f_2, \dots, f_n \in (\mathbb{R} \cap \bar{\mathbb{Q}})[T_1, T_2, \dots, T_n]$

$$f_1 \wedge f_2 \wedge \cdots \wedge f_n \neq 0$$

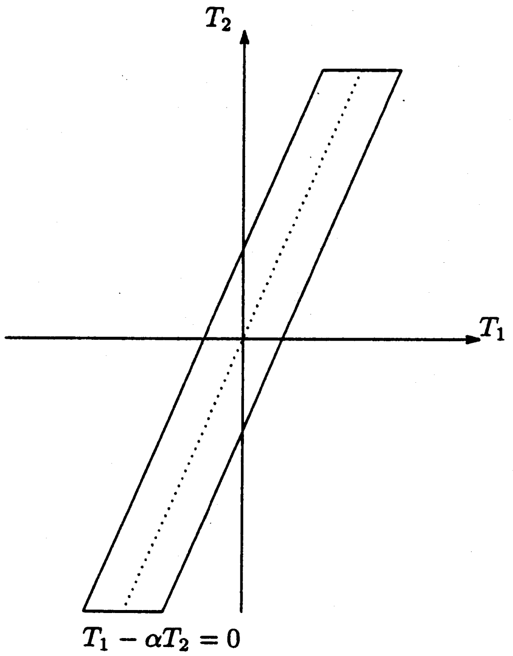
$c(1), c(2), \dots, c(n) \in \mathbb{R}$

Fixed $\delta \in \mathbb{R}$ and variable $Q \in \mathbb{R}_{>1}$

$$|f_i(T_1, \dots, T_n)| < Q^{-c(i)-\delta} \quad (i = 1, \dots, n)$$

<u>Example</u> $(n = 2)$ $\quad \alpha \in \mathbb{R} \cap \bar{\mathbb{Q}}$, $c > \delta > 0$

$$|T_1 - \alpha T_2| < Q^{-c-\delta} \quad |T_2| < Q^{c-\delta}$$



We start with linear forms $f_1, f_2, \dots, f_n$ with real algebraic coefficients in the indeterminates $T_1, T_2, \dots, T_n$. We assume they are linearly independent. Namely, the volume form they define is not 0. We consider real numbers $c(1), c(2), \dots, c(n)$.

For an arbitrarily fixed real number δ and a variable real number Q larger than 1, the following system of linear inequalities is the theme of today's talk: ...

We give the most typical example: ...
We also give the picture: as Q becomes large, the parallelotope stretches.

We are interested in qualitative aspect of the rational integer valued solutions. What can we say about this classical type of inequalities?

FALTINGS' pt of view

$V := \mathbb{Q}T_1 \oplus \mathbb{Q}T_2 \oplus \cdots \oplus \mathbb{Q}T_n$, $L := \mathbb{R} \cap \bar{\mathbb{Q}}$

$\{w(1) < w(2) < \cdots < w(s)\} = \{c(1), \dots, c(n)\}$

$$\begin{aligned} V^{w(j)} &\subset V \otimes_{\mathbb{Q}} L \\ &:= \langle f_i \mid c(i) \geq w(j) \rangle_L \end{aligned}$$

$$V \otimes_{\mathbb{Q}} L = V^{w(1)} \supsetneq V^{w(2)} \supsetneq \cdots \supsetneq V^{w(s)} \supsetneq 0$$

We follow FALTINGS.

V is the vector space of rational linear forms in $T_1, \dots, T_n$. We denote by L the field of real algebraic numbers.

The symbols $w(1), w(2), \dots, w(s)$ are the strictly increasing real numbers such that as a set, it is identical with the set of $c(1), \dots, c(n)$. $V^{w(j)}$ is the subspace over L of the scalar extension of V to L, spanned by all f_i such that $c(i)$ is at least $w(j)$. Thus we obtain a descending filtration on V tensored over $\mathbb{Q}$ by L.

Example (continuation)

$$V \otimes_{\mathbb{Q}} L = V^{-c} \supsetneq V^{c} = L \cdot (T_1 - \alpha T_2) \supsetneq 0$$

In the case of the above example, ...

$$(g_i; d(i))_{i=1}^{n} \Rightarrow (V^{w(j)})_{j=1}^{s} \Leftarrow (f_i; c(i))$$

$\Rightarrow$ Essentially their solutions coincide.

Our observation is that the qualitative nature is determined by the filtration.

In fact, given another system of linear forms g_i and real numbers $d(i)$ which define the same filtration $V^{w(j)}$ as f_i and $c(i)$, then essentially their solutions coincide. To be more precise, one can see easily that the set of solutions to one system is a set of solutions to another system modulo replacement of δ.

The classical theorem of SCHMIDT is concisely stated in this context, for which we need some notation.

Def. (slope)

$$
\begin{aligned}
&M(V;V^{w(j)})\\
&:= \frac{1}{\dim_{\mathbf{Q}} V}\sum_{w\in\mathbf{R}} w\dim_L \mathrm{gr}^w(V^{\cdot})\\
&= \frac{1}{\dim_{\mathbf{Q}} V}\sum_{j=1}^{s-1} w(j)\dim_L(V^{w(j)}/V^{w(j+1)})\\
&\quad +\frac{1}{\dim_{\mathbf{Q}} V} w(s)\dim_L V^{w(s)}\\
&= \frac{c(1)+\cdots+c(n)}{n}\\
&=: M(V)
\end{aligned}
$$

We denote by $M(V;V^{w(j)})$ the slope of the filtration. That is to say, the real number given by the following expression: ...

We write it $M(V)$ if there is no fear of confusion.

$$(\text{pt}) = V^* \quad \leftarrow \text{dual} \rightarrow \quad V = (\text{lin. fn})$$

Next, we introduce the dual vector space V^* to V over $\mathbb{Q}$. The space V^* is the set of points. The space V is the set of linear functions.

$$V^* \supset S^* \neq 0 \text{ ortho. to } W \subsetneq V$$

$$S^* \quad \leftarrow \text{dual} \rightarrow \quad V/W$$

Filtr. on $V\otimes_{\mathbf{Q}} L \rightsquigarrow$ filtr. on $(V/W)\otimes_{\mathbf{Q}} L$

$$M(V/W) := M(V/W;\text{induced filtr.})$$

For a non-zero subspace S^* of V^*, W is the orthogonal subspace of V. The quotient space V/W is dual to S^*, that is, the space V/W is the set of linear functions on S^*. We induce from the filtration on $V\otimes_{\mathbf{Q}} L$ a filtration on $(V/W)\otimes_{\mathbf{Q}} L$, and define the slope of V/W by the slope of V/W with the induced filtration.

Thm (SCHMIDT) If for every $W \subsetneq V$

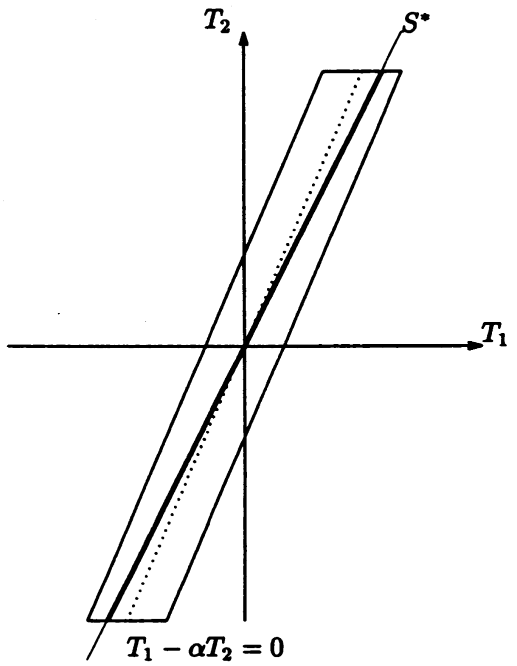
$$M(V/W) \geq M(V) > -\delta$$

then

$$\#\{\text{sols.}\} < \infty$$

Using the notation, the subspace theorem of SCHMIDT is restated as follows.

If for every proper subspace W of V, the slope of V/W is at least the slope of V, and if the slope of V is larger than $-\delta$, then the number of solutions to the system of linear inequalities is finite.



Intuitively, the conditions of the theorem imply that the volume of the convex body in S^* cut out by the given inequalities is small.

Def. (FALTINGS) $(V; V^{w(j)})$ semi-stable
$\Leftrightarrow$ For every $W \subsetneq V$

$$M(V/W) \geq M(V)$$

Example (continuation)

$$V \text{ s.-s.} \Leftrightarrow \alpha \notin \mathbb{Q}$$

Now, FALTINGS found that the first condition of the theorem is nothing but the semi-stability in Geometric Invariant Theory of Mumford. Namely, a filtered vector space $(V; V^{w(j)})$ is semi-stable if and only if for every proper subspace W of V over $\mathbb{Q}$ the slope of V/W is at least the slope of V.

The next assertion is an easy exercise: the V in the example above is semi-stable if and only if the number α is irrational.

As a consequence, the subspace theorem applied to this example gives the famous ROTH's theorem.

Def. (Cat. of lin. ineq.)

$$\mathrm{Obj}(\underline{C}) = \left\{ \begin{array}{l} \text{fin. dim. vect. sp } V/\mathbb{Q} \\ \text{with filtr. } V^{\cdot} \text{ on } V \otimes_{\mathbb{Q}} L \end{array} \right\}$$

$$\mathrm{Hom}_{\underline{C}}(V, S) = \{\mathbb{Q}\text{-lin. } \phi \mid \phi(V^w) \subset S^w\}$$

What is interesting in our formulation? To mention it, we take into account all the systems of linear inequalities, or all the filtered vector spaces.

We propose to define the category $\underline{C}$ of linear inequalities as follows: an object is a finite dimensional vector space V over $\mathbb{Q}$ with a filtration $V^{\cdot}$ on $V \otimes_{\mathbb{Q}} L$, and a morphism in $\underline{C}$ of V to S is a $\mathbb{Q}$-linear map ϕ such that $\phi(V^w)$ is included in S^w for every real number w.

$\phi^* : S^* \to V^*$, sols. $\mapsto$ sols. mod. replacement of δ

$$\mathrm{Hom}_{\underline{C}} = \{\mathbb{Q}\text{-lin., preserving sols.}\}$$

The dual map ϕ^* of S^* to V^* sends the set of solutions of inequalities to a set of solutions of inequalities modulo replacement of δ. So we can regard a morphism in $\underline{C}$ as a $\mathbb{Q}$-linear map preserving solutions.

Thm $\underline{C}_0^{ss}$: full subcat. of $\underline{C}$

$$\mathrm{Obj}(\underline{C}_0^{ss}) = \{\text{s.-s. of slope } 0\}$$

There exists an affine gp scheme $G/\mathbb{Q}$ s.t.

$$\underline{\mathrm{Rep}_{\mathbb{Q}}}(G) \simeq \underline{C}_0^{ss}$$

We get a theorem. Let $\underline{C}_0^{ss}$ be the full subcategory of $\underline{C}$ whose objects are the semi-stable ones of slope 0. Then there exits an affine group scheme G over $\mathbb{Q}$ such that the category of finite dimensional representations over $\mathbb{Q}$ of G is equivalent to the category $\underline{C}_0^{ss}$.

Lem. ([2] [5] [1] [6] [3])

$$V, S \text{ s.-s. } \Rightarrow V \otimes_{\mathbb{Q}} S \text{ s.-s.}$$

Recently a new proof is obtained [4]!

For a proof of Theorem, thanks to the general theory of Tannakian categories, we have only to check several trivial conditions except the following lemma, several proofs of which are already given by some people: when two filtered vector spaces V and S are semi-stable, their tensor product $V \otimes_{\mathbb{Q}} S$ with induced filtration is also semi-stable.

Recently a new proof is obtained by the speaker! The proof is based on MINKOWSKI's theorem in Geometry of Numbers and on the subspace theorem of SCHMIDT.

We would like to close the speech by raising problems and making a remark.

Problem $G=?$

First, simple question: what is the group scheme G? Its meaning?

Problem (FALTINGS [1])

filtr. isocrys. $\leftarrow$ analogy $\rightarrow$ filtr. vect. sp

Second, originally addressed by FALTINGS at the ICM in Munich 1994: how far can we go with the analogy between filtered isocrystals and filtered vector spaces in our sense? I'll attack this problem in future

Rem.
$\underline{C}$ & $\underline{C}_0^{ss}$ indep. of Diophantine Approx. Theory!

The concluding remark: as we see, the categories $\underline{C}$ and $\underline{C}_0^{ss}$ are independent of the parameters δ and Q, hence independent of Diophantine Approximation Theory! We may well interpret them another way!!

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