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Triviality of hierarchical Ising model in four dimensions

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Abstract

A new approach to RG analysis for hierarchical models is proposed by using characteristic functions of single spin distributions. Especially, existence of a critical RG trajectory for a hierarchical Ising model in 4 dimensions is shown and convergence to a Gaussian distribution is confirmed.

1 Introduction

It is widely believed that the Ising model in four dimensions will be Gaussian in a continuum limit (*triviality*). There have been accumulated a number of facts indicating the triviality, but no mathematically rigorous proof has been obtained so far. In fact, in order to study a RG trajectory starting at the Ising model, we have to perform a RG analysis *in the strong coupling region*, since the Ising model is a strong coupling limit of ϕ^4 model. But indispensable techniques seem to be left unknown for such an analysis.

In this situation, we found that, for a *hierarchical approximation* of the Ising model in $d \geq 4$ dimensions, the RG trajectory can be rigorously studied by means of the *characteristic functions* of single spin distributions for effective theories [1]. In this approach, rigorous inequalities due to Newman [2] play an essential role, and numerical calculations are performed by computer to yield rigorous bounds on the RG trajectory in the strong coupling region.

This note describes our basic idea to show the triviality of the hierarchical Ising model in $d \geq 4$ dimensions.

2 Hierarchical model

Hierarchical model is defined as follows. Let Λ be a positive integer, and consider the 2^Λ variables (spin variables) $\phi_\theta = \phi_{\theta_\Lambda,\dots,\theta_1}$ labelled by

$$\theta = (\theta_\Lambda, \dots, \theta_1) \in \{0,1\}^\Lambda, \tag{2.1}$$

Let us define the Hamiltonian H_Λ and the expectation values $\langle\cdot\rangle$, respectively, by

$$H_\Lambda(\phi) = -\frac{1}{2}\sum_{n=1}^{\Lambda}\left(\frac{c}{4}\right)^n \sum_{\theta_\Lambda,\dots,\theta_{n+1}}\left(\sum_{\theta_n,\dots,\theta_1}\phi_{\theta_\Lambda,\dots,\theta_1}\right)^2, \tag{2.2}$$

$$\langle F\rangle_{\Lambda,h} = \frac{1}{Z_{\Lambda,h}}\int d\phi F(\phi)\exp(-\beta H_\Lambda(\phi))\prod_\theta h(\phi_\theta), \tag{2.3}$$

$$Z_{\Lambda,h} = \int d\phi \exp(-\beta H_\Lambda(\phi))\prod_\theta h(\phi_\theta), \tag{2.4}$$

where h is a single spin measure density normalized as

$$\int_{\mathbf{R}} h(x)dx = 1. \tag{2.5}$$

RG transformation

Hierarchical models are so designed that the RG transformation (see (2.11)) has a simple form [3, 4, 5, 6, 7]. Define the block spins ϕ' by

$$\phi'_\tau = \frac{\sqrt{c}}{2}\sum_{\theta_1=0,1}\phi_{\tau\theta_1}\,,\ \tau = (\tau_{\Lambda-1},\dots,\tau_1). \tag{2.6}$$

Then, the equality

$$\sum_{\theta_n,\dots,\theta_1}\phi_{\theta_N,\dots,\theta_1} = \sum_{\theta_n,\dots,\theta_2}\frac{\sqrt{c}}{2}\phi'_{\theta_N,\dots,\theta_2} \tag{2.7}$$

implies

$$H_N(\phi) = H_{N-1}(\phi') - \frac{1}{2}\sum_\tau \phi_\tau'^{\,2}. \tag{2.8}$$

Suppose that a function $F(\phi)$ depends on ϕ through ϕ' only, namely, there is a function $F'(\phi')$ on the block spins such that

$$F(\phi) = F'(\phi'). \tag{2.9}$$

Then it holds that

$$\langle F\rangle_{\Lambda,h} = \langle F'\rangle_{\Lambda-1,\mathcal{R}h}\,, \tag{2.10}$$

where $\mathcal{R}h$ is defined by

$$\mathcal{R}h(x) = \text{const.}\exp\left(\frac{\beta}{2}x^2\right)\int_{\mathbf{R}} h\left(\frac{x}{\sqrt{c}}+y\right)h\left(\frac{x}{\sqrt{c}}-y\right)dy,\ x\in\mathbf{R}. \tag{2.11}$$

Note that

$$h_G(x) = \text{const.}\exp\left(-\frac{1}{4}x^2\right) \tag{2.12}$$

is a fixed point of $\mathcal{R}$, which we shall refer to as the density function of *hierarchical massless Gaussian measure.* By looking into the asymptotics of e.g., susceptibility for the hierarchical massless Gaussian model defined by (2.12), and comparing it with that of the standard nearest neighbor massless Gaussian model on d-dimensional regular lattice, we see that the dimensionality d of the system may be identified (at least for the Gaussian fixed point) as

$$c = 2^{1-2/d}. \tag{2.13}$$

We shall extend the correspondence to hierarchical models with non-Gaussian measures, and use the terminology *d-dimensional hierarchical models* whenever (2.13) holds.

Hierarchical Ising model

Our concern is the hierarchical Ising model, which is defined by the following single spin measure density

$$h_{\mathrm{I},s}(x) = \frac{1}{2}(\delta(x-s) + \delta(x+s)), \tag{2.14}$$

where $s \geq 0$. Hierarchical Ising model has an infinite volume limit $\Lambda \to \infty$, if $0 < c < 2$ ($d > 0$), and has a phase transition, if $1 < c < 2$ ($d > 2$) [3].

The 'continuum limit' of hierarchical Ising model is analyzed through the asymptotic property of the RG trajectory

$$h_N = \mathcal{R}^N h_0\,,\ N = 0, 1, 2, \cdots, \tag{2.15}$$

with the initial point $h_0 = h_{\mathrm{I},s}$.

RG trajectory

Asymptotic properties of the RG trajectories (2.15) are extensively investigated in a weak coupling region i.e., in a 'neighborhood' of h_G [4, 5, 6]. In particular, it is known that, if $d \geq 4$, then there are no non-Gaussian fixed points in a 'neighborhood' of h_G, and that a 'continuum limit' constructed from a critical trajectory with an initial function in a 'neighborhood' of h_G is trivial (Gaussian).

However, the density (2.14) is regarded as a strong coupling limit $\lambda \to \infty$ of the ϕ^4 densities

$$h_{\mu,\lambda}(x) = \text{const.}\exp(-\mu x^2 - \lambda x^4), \quad \mu = -2\lambda s^2, \tag{2.16}$$

and an investigation of the RG trajectory for hierarchical Ising model requires an analysis in the 'strong coupling region' far away from the Gaussian fixed point.

This problem is solved for hierarchical models by using characteristic functions of single spin distributions.

Result

The following theorem claims that the continuum limit of hierarchical Ising model in $d \geq 4$ dimensions is trivial [1].

Theorem 2.1. *If $d \geq 4$ (i.e. $c \geq \sqrt{2}$), there exists a critical trajectory converging to the Gaussian fixed point starting at the hierarchical Ising model. Namely, there exists a positive real number s_c such that if h_N, $N = 0, 1, 2, \cdots$, are defined by (2.15) with $h_0 = h_{\mathrm{I},s_c}$, then the sequence of measures $h_N(x)\,dx$, $N = 0, 1, 2, \cdots$, converges weakly to the massless Gaussian measure $h_G(x)\,dx$.*

Our proof is partially computer-aided and shows for $d = 4$ that the critical value s_c lies in the interval

$$[1.7925671170092624, 1.7925671170092625], \tag{2.17}$$

where we have fixed the so far arbitrary normalization of the spin variables by

$$\beta = \frac{1}{c} - \frac{1}{2} = \frac{1}{2}(2^{2/d} - 1). \tag{2.18}$$

3 Strategy

Characteristic function

Main idea of our proof is to use characteristic functions of single spin distributions:

$$\hat{h}_N(\xi) = \mathcal{F}h_N(\xi) = \int_{\mathbf{R}} e^{i\xi x} h_N(x)\,dx\,. \tag{3.1}$$

The RG transformation for $\hat{h}_N$ is

$$\hat{h}_{N+1} = \mathcal{F}\mathcal{R}\mathcal{F}^{-1}\hat{h}_N\,, \tag{3.2}$$

which has a decomposition

$$\mathcal{F}\mathcal{R}\mathcal{F}^{-1} = \mathcal{T}\mathcal{S}, \tag{3.3}$$

where

$$\mathcal{S}g(\xi) \;=\; g(\frac{\sqrt{c}}{2}\xi)^2, \tag{3.4}$$

$$\mathcal{T}g(\xi) \;=\; \text{const.}\,\exp(-\frac{\beta}{2}\Delta)g(\xi), \tag{3.5}$$

and the constant is so defined that

$$\mathcal{T}g\,(0) = 1\,. \tag{3.6}$$

The transformation (3.2) has the same form as the $N = 2$ case of the Gallavotti hierarchical model [9, 7, 8]. Note that only for $N = 2$ the Gallavotti model is equivalent (by Fourier transform) to the Dyson's hierarchical model.

Newman's inequalities

Let us introduce a 'potential' V_N for the characteristic function $\hat{h}_N$ and its Taylor coefficients $\mu_{n,N}$, respectively, by

$$\hat{h}_N(\xi) = e^{-V_N(\xi)}, \tag{3.7}$$

$$V_N(\xi) = \sum_{n=1}^{\infty} \mu_{n,N}\xi^n. \tag{3.8}$$

Note that

$$\hat{h}_N(0) = 1, \tag{3.9}$$

$$\mu_{2n+1,N} = 0, \quad n = 0, 1, 2, \ldots \tag{3.10}$$

hold. The coefficient $\mu_{n,N}$ is called *a truncated n point correlation.*

The function V_N has a remarkable positivity property, that is, the truncated correlations obey Newman's inequalities:

$$\mu_{2n,N} \geq 0, \ n \geq 1, \tag{3.11}$$

$$\mu_{2n,N} \leq \frac{1}{n}(2\mu_{4,N})^{n/2}, \ n \geq 3. \tag{3.12}$$

These bounds follow from the Lee-Yang property for ferromagnetic systems [2].

Newman's inequalities are extensively used in our proof. We here note the following facts.

(1) The right hand side of (3.8) has non-zero radius of convergence.

(2) It suffices to prove $\mu_{4,N} \to 0$ as $N \to \infty$ in order to show that the trajectory converges to the Gaussian fixed point.

Weak coupling region

The proof of Theorem 2.1 is decomposed into two parts: analyses in the weak coupling region and in the strong coupling region.

Firstly we state the result in the weak coupling region.

It is easily seen that the condition

$$1 \leq \mu_{2,N} \leq 1 + \frac{3}{\sqrt{2}}\mu_{4,N} \tag{3.13}$$

is necessary for the model to be critical. We then put, for $N = 0, 1, 2, \cdots$,

$$\underline{s}_N = \inf\{s > 0 \mid \mu_{2,N} \geq 1\}, \tag{3.14}$$

$$\overline{s}_N = \inf\{s > 0 \mid \mu_{2,N} \geq \min\{1 + \frac{3}{\sqrt{2}}\mu_{4,N}, \ 2 + \sqrt{2}\}\}. \tag{3.15}$$

In the following proposition, a RG flow in a weak coupling region is controlled by means of a *finite* number of truncated correlations, and, in terms of the truncated correlations, a criterion, a set of sufficient conditions, is given for the measure to be in a domain of attraction of the Gaussian fixed point.

Proposition 3.1. *Let $h_0 = h_{I,s}$ and $d = 4$. Assume that there exist integers N_0 and N_1, satisfying $N_0 \le N_1$, such that, for $s \in [\underline{s}_{N_1}, \overline{s}_{N_1}]$, the bounds*

$$\begin{aligned} 0 &\le \mu_{4,N_0} \le 0.0045, &(3.16)\\ 1.6\mu_{4,N_0}^2 &\le \mu_{6,N_0} \le 6.07\mu_{4,N_0}^2, &(3.17)\\ 0 &\le \mu_{8,N_0} \le 48.469\mu_{4,N_0}^3, &(3.18)\end{aligned}$$

and

$$\mu_{2,N} < 2+\sqrt{2}, \quad N_0 \le N < N_1, \tag{3.19}$$

hold. Then there exists an $s_c \in [\underline{s}_{N_1}, \overline{s}_{N_1}]$ such that if $s = s_c$ then

$$\lim_{N\to\infty} \mu_{4,N} = 0, \tag{3.20}$$

$$\lim_{N\to\infty} \mu_{2,N} = 1. \tag{3.21}$$

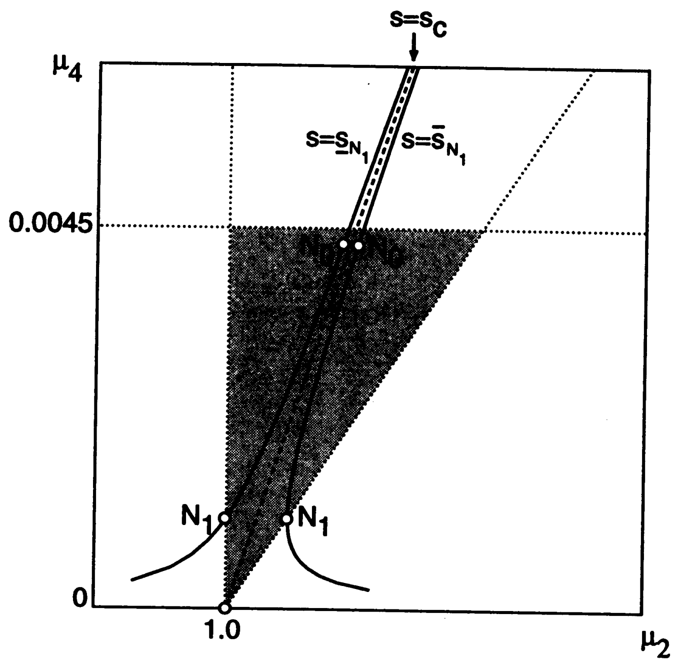

Figure 1. A schematic view of trajectories on (μ_2, μ_4)-plane in Proposition 3.1. Trajectories for $s = \overline{s}_{N_1}$ and for $s = \underline{s}_{N_1}$ (solid lines) and the critical trajectory for $s = s_c$ (broken line) are shown. The Gaussian fixed point corresponds to the point $(1.0, 0)$. The region defined by inequalities for (μ_2, μ_4) analogous to (3.13) and (3.16) (and (3.19)) is shaded.

Our framework in the weak coupling region is designed especially for a critical trajectory starting at the strong coupling region so that the criterion of convergence to the Gaussian fixed point can be checked numerically with mathematical rigor. Note that numerical results are incorporated with the rigorous analysis because the trajectory is controlled by means of finite number of coefficients $\mu_{2n,N}, n = 1,2,3,4$, of V_N that are rigorously estimated by computer.

As a result, we see that the 'effective coupling constant' $\mu_{4,N}$ of a critical model decays as $c_1/(N + c_2)$ after N iterations in $d = 4$ dimensions (exponentially for $d > 4$), which shares the common feature with weakly coupled ϕ^4 model with nearest neighbor interactions.

Proposition 3.1 is a variant of Bleher–Sinai argument [4]. In fact, the criteria introduced in the references [4, 6] seem to be difficult to handle when 'strong coupling constants' are present in the model, as in the Ising models.

In addition, while the original Bleher–Sinai argument takes $N_0 = N_1$, we include the $N_0 < N_1$ case. This generalization makes it possible to complete our proof by evaluating various quantities only at 2 endpoints of the interval in consideration for Ising parameter s, instead of all values in the interval, as is implicit in the assumptions of Proposition 3.1.

Strong coupling region

The following proposition is the result in the strong coupling region that is proved by rigorous computer-aided calculations. In this proposition, it is stated that there is a trajectory whose initial point is an Ising measure and for which the criterion in Proposition 3.1 is satisfied after a small number of iterations.

Proposition 3.2. *The assumptions of Proposition 3.1 are satisfied for $N_0 = 70$ and $N_1 = 100$, where $\underline{s}_{N_1}$ and $\overline{s}_{N_1}$ satisfy*

$$1.7925671170092624 \le \underline{s}_{N_1}, \quad \overline{s}_{N_1} \le 1.7925671170092625\ . \tag{3.22}$$

Proposition 3.2 is proved by basically simple numerical calculations of truncated correlations up to 8 points to ensure the criterion. The results are double checked by Mathematica and `C++` programs, and furthermore they are made mathematically rigorous by means of Newman's inequalities.

Theorem 2.1 follows from Proposition 3.1 and Proposition 3.2.

4 Analysis in the weak coupling region

The operator $\mathcal{S}$ acts on V_N as follows:

$$\left(\mathcal{S}e^{-V_N}\right)(\xi) = e^{-2V_N\left(\frac{\sqrt{c}}{2}\xi\right)}. \tag{4.1}$$

Using (3.8), (3.10), (2.13) we also have

$$2V_N\left(\frac{\sqrt{c}}{2}\xi\right) = \sum_{n=1}^{\infty} \frac{2}{(2\omega)^n}\mu_{2n,N}\xi^{2n}, \tag{4.2}$$

where $\omega = 2^{2/d}$.

Next, write (3.5) as

$$\mathcal{T}g = \text{const.}\, g_{\beta/2}\,, \tag{4.3}$$
$$g_t = \exp(-t\Delta)g, \tag{4.4}$$

where $\Delta g(\xi) = \dfrac{d^2 g}{d\xi^2}(\xi)$, and $\beta = \frac{1}{2}(\sqrt{2}-1)$ for $d=4$. g_t is a solution to

$$\frac{\partial g_t}{\partial t} = -\Delta g_t\,, \tag{4.5}$$
$$g_0 = g. \tag{4.6}$$

Hence, if we put

$$g_t(\xi) = \exp(-V_t(\xi)), \tag{4.7}$$

then V_t satisfies

$$\frac{d}{dt}V_t = (\nabla V_t)^2 - \Delta V_t\,, \tag{4.8}$$

where $\nabla V_t(\xi) = \dfrac{\partial V_t}{\partial \xi}(\xi)$. In other words, V_{N+1} is given as a solution of (4.8) at $t = \beta/2$ (modulo constant term), with the initial condition (4.2) at $t=0$.

Reduction to finite degree of freedoms

If we write

$$V_t(\xi) = \sum_{n=1}^{\infty} \mu_{2n}(t)\xi^{2n}, \tag{4.9}$$

then (4.8) implies

$$\begin{aligned} \frac{d}{dt}\mu_{2n}(t) &= -(2n+2)(2n+1)\mu_{2n+2}(t) \\ &\quad + \sum_{\ell=1}^{n}(2\ell)(2n-2\ell+2)\mu_{2\ell}(t)\,\mu_{2n-2\ell+2}(t). \end{aligned} \tag{4.10}$$

In particular, we have

$$\frac{d}{dt}\mu_2(t) = 4\mu_2(t)^2 - 12\mu_4(t), \tag{4.11}$$
$$\frac{d}{dt}\mu_4(t) = 16\mu_2(t)\mu_4(t) - 30\mu_6(t), \tag{4.12}$$
$$\frac{d}{dt}\mu_6(t) = 24\mu_2(t)\mu_6(t) + 16\mu_4(t)^2 - 56\mu_8(t), \tag{4.13}$$
$$\frac{d}{dt}\mu_8(t) = 32\mu_2(t)\mu_8(t) + 48\mu_4(t)\mu_6(t) - 90\mu_{10}(t). \tag{4.14}$$

Thus, $\mu_{2n,N}$ and $\mu_{2n,N+1}$ are related by

$$\mu_2(0) = \frac{1}{\omega}\mu_{2,N}\,,\ \mu_4(0) = \frac{1}{2\omega^2}\mu_{4,N}\,,\ \mu_6(0) = \frac{1}{4\omega^3}\mu_{6,N}\,,\ \mu_8(0) = \frac{1}{8\omega^4}\mu_{8,N}\,,$$
$$\mu_{2,N+1} = \mu_2(\frac{\beta}{2}),\ \mu_{4,N+1} = \mu_4(\frac{\beta}{2}),\ \mu_{6,N+1} = \mu_6(\frac{\beta}{2}),\ \mu_{8,N+1} = \mu_8(\frac{\beta}{2}).$$

Note that the quantities $\mu_n(t)$ obey Newman's inequalities: by comparing (3.5) and (4.4) we see that the correspondence $V_N \mapsto V(t)$ is obtained by a replacement $\beta \mapsto 2t$ in (2.11). Therefore $\mu_n(t)$ also is a truncated n point correlation of a measure to which arguments in [2] apply, hence analogues of (3.11) and (3.12) hold:

$$\mu_{2n}(t) \geq 0,\ n \geq 1, \tag{4.15}$$
$$\mu_{2n}(t) \leq \frac{1}{n}(2\mu_4(t))^{n/2},\ n \geq 3. \tag{4.16}$$

The positivity of $\mu_{2n}(t)$ implies that if we throw out the last terms of the right hand sides of (4.11)–(4.14), we have upper bounds for $\mu_{2n}(t), n = 1,2,3,4$. Furthermore, replacing the last terms by the corresponding upper bounds, we have lower bounds, and so on.

Integral equations

Proposition 3.1 follows from the bounds described above, and actual calculations are performed in the form of integral equations.

Let us write the solution

$$g_t(\xi) = \exp(-V_t(\xi)) = \exp(-\sum_{n=1}^{\infty}\mu_{2n}(t)\xi^{2n}) \tag{4.17}$$

to (4.5),(4.6) as

$$g_t(\xi) = \sqrt{\frac{\sigma(t)}{\mu}}\exp(-\sigma(t)\xi^2)\psi_z(\eta), \tag{4.18}$$

where μ is a positive constant and

$$\sigma(t) = \frac{\mu}{1-4\mu t} \tag{4.19}$$
$$z = \frac{\sigma(t)t}{\mu} \tag{4.20}$$
$$\eta = \frac{\sigma(t)\xi}{\mu}. \tag{4.21}$$

Then, $\psi_z(\eta)$ obeys

$$\frac{\partial\psi_z(\eta)}{\partial z} = -\frac{\partial^2}{\partial\eta^2}\psi_z(\eta), \tag{4.22}$$
$$\psi_0(\eta) = \exp(\mu\eta^2 - V_0(\eta)). \tag{4.23}$$

In particular, by taking the constant μ as

$$\mu = \mu_2(0) = \frac{1}{\omega}\mu_{2,N}, \tag{4.24}$$

the 'mass term' is separated from the initial potential $V_0(\xi)$.

Let us introduce a 'potential' for $\psi_z(\eta)$ by

$$\psi_z(\eta) \;=\; \exp(-U_z(\eta)) = \exp(-\sum_{n=1}^{\infty} \nu_{2n}(z)\eta^{2n}). \tag{4.25}$$

Then, the coefficients $\nu_{2n}(z)$ obey the same type of equations as (4.11)–(4.14) and they are related with $\mu_{2n}(t)$ by

$$\mu_{2n}(t) \;=\; \left(\frac{\sigma(t)}{\mu_{2,N}}\right)^2 \nu_{2n}(z) + \sigma(t)\delta_{n,1}. \tag{4.26}$$

Note that $\nu_2(z)$ is $\mathcal{O}(\mu_{4,N})$, since the initial value of ν_2 vanishes under (4.24).

Thus, we can obtain necessary bounds by using the following integral equations:

$$\nu_2(z) = \int_0^z (4\nu_2(z)^2 - 12\nu_4(z))dz, \tag{4.27}$$

$$\nu_4(z) = \frac{2}{(2\omega)^2}\mu_{4,N} + \int_0^z (16\nu_2(z)\nu_4(z) - 30\nu_6(z))dz, \tag{4.28}$$

$$\nu_6(z) = \frac{2}{(2\omega)^3}\mu_{6,N} + \int_0^z (24\nu_2(z)\nu_6(z) + 16\nu_4(z)^2 - 56\nu_8(z))dz, \tag{4.29}$$

$$\nu_8(z) = \frac{2}{(2\omega)^4}\mu_{8,N} + \int_0^z (32\nu_2(z)\nu_8(z) + 48\nu_4(z)\nu_6(z) - 90\nu_{10}(z))dz. \tag{4.30}$$

5 Numerical bounds in the strong coupling region

Proposition 3.2 is shown by estimating Taylor coefficients of $\hat{h}_N(\xi)$ instead of $V_N(\xi)$.

Taylor coefficients

Define the Taylor coefficients $a_{n,N}$, $n \geq 0$, of $\hat{h}_N$ by

$$\hat{h}_N(\xi) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n!} a_{n,N}\xi^{2n}. \tag{5.1}$$

In particular, $a_{0,N} = \hat{h}_N(0) = 1$. Obviously $a_{n,N} \geq 0$ holds for $n \geq 0$. Furthermore, they satisfy inequalities of Newman's type (or the Gaussian inequalities) [1, appendix]:

$$a_{n+m,N} \leq a_{n,N}a_{m,N}, \quad n, m \geq 0\,. \tag{5.2}$$

Note that coefficients $\mu_{n,N}$ and $a_{n,N}$ are related, e.g., as

$$\mu_{2,N} = a_{1,N}, \tag{5.3}$$

$$\mu_{4,N} = \frac{a_{1,N}^2 - a_{2,N}}{2}, \tag{5.4}$$

$$\mu_{6,N} = \frac{a_{1,N}^3}{3} - \frac{a_{1,N}\,a_{2,N}}{2} + \frac{a_{3,N}}{6}, \tag{5.5}$$

$$\mu_{8,N} = \frac{a_{1,N}^4}{4} - \frac{a_{1,N}^2\,a_{2,N}}{2} + \frac{a_{2,N}^2}{8} + \frac{a_{1,N} + a_{3,N}}{6} - \frac{a_{4,N}}{24}. \tag{5.6}$$

For Ising measure $h_0 = h_{I,s}$, we have, for $n \geq 0$,

$$\begin{aligned} a_{n,0} &= (-1)^n \frac{n!}{(2n)!} \frac{d^{2n}\hat{h}_0}{d\,\xi^{2n}}(0) \\ &= \frac{n!}{(2n)!} \int x^{2n}\, h_{I,s}(x)dx \\ &= \frac{n!}{(2n)!} s^{2n}. \end{aligned} \tag{5.7}$$

Recursions

Define $b_{n,N}$, $n \geq 0$, by

$$(\mathcal{S}\hat{h}_N)(\xi) = \hat{h}_N(\frac{\sqrt{c}}{2}\xi)^2 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} b_{n,N} \xi^{2n}, \tag{5.8}$$

namely,

$$b_{n,N} = \left(\frac{c}{4}\right)^n \sum_{\ell=0}^{n} \binom{n}{\ell} a_{\ell,N}\, a_{n-\ell,N}, \quad n \geq 0. \tag{5.9}$$

Next, expand the exponential

$$\exp(-\frac{\beta}{2}\Delta) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{\beta}{2}\right)^m \frac{d^{2m}}{d\xi^{2m}} \tag{5.10}$$

and define $\tilde{a}_{n,N}$, $n \geq 0$, by

$$\sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{\beta}{2}\right)^m \frac{d^{2m}}{d\xi^{2m}} \mathcal{S}\hat{h}_N(\xi) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \tilde{a}_{n,N} \xi^{2n}, \tag{5.11}$$

namely,

$$\tilde{a}_{n,N} = \sum_{m=0}^{\infty} \left(\frac{\beta}{2}\right)^m b_{m+n,N} \frac{(2m+2n)!n!}{m!(m+n)!(2n)!}, \quad n \geq 0. \tag{5.12}$$

Then, (3.5) implies

$$\hat{h}_{N+1}(\xi) = \frac{1}{\tilde{a}_{0,N}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \tilde{a}_{n,N} \xi^{2n}, \tag{5.13}$$

where we fixed the constant in the definition of $\mathcal{T}$ by $\hat{h}_{N+1}(0) = 1$. Comparing this with (5.1) we obtain a recursion relation in N for $a_{n,N}$:

$$a_{n,N+1} = \frac{\tilde{a}_{n,N}}{\tilde{a}_{0,N}}, \quad n \geq 0, \; N \geq 0. \tag{5.14}$$

Bounds

We have to bound $a_{n,N}$'s inductively by using (5.9), (5.12), (5.14) with initial data (5.7). This part of our proof is computer-aided.

Note that every coefficient is nonnegative and no cancellation occurs in (5.9) and (5.12). Hence, lower bounds on $b_{n,N}$ are obtained by using lower bounds on $a_{n,N}$ for $n \leq n_*$ and by putting $a_{n,N} = 0$ for $n > n_*$ in the right hand side of (5.9), where n_* is a previously fixed positive integer. Furthermore, lower bounds on $b_{n,N}$ in turn yield lower bounds on $\tilde{a}_{n,N}$ by using (5.12).

On the other hand, upper bounds are obtained as follows. Firstly, we have to append *theoretically* upper bounds on $a_{n,N}$ for $n > n_*$ from those on $a_{n,N}$ for $n \leq n_*$ by means of (5.2). Next, we derive upper bounds on $b_{n,N}$ and $\tilde{a}_{n,N}$ from (5.9) and (5.12), respectively.

As a result, lower and upper bounds on $a_{n,N+1}$ for $n \leq n_*$ are obtained by (5.14).

Finally, we note that (3.16)–(3.19) must hold for *all* $s \in [\underline{s}_{N_1}, \overline{s}_{N_1}]$. Numerical calculations, however, can be done for a *finite* number of s, of course. In fact, $a_{n,N}$ is monotone with respect to s for each n and N. Then, using (5.3)–(5.6), we can bound $\mu_{2n,N}$ from above and below for *all* s in some interval by quatities (obtained by computer) for the endpoints of the interval. Thus, Proposition 3.2 is shown.

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