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Integral means of holomorphic mappings in C^n

Kazuyuki Tsurumi
(東京電機大工　鶴見和之)

Tadayuki Sekine
(日本大学薬　関根忠行)

Let $f(z)$ and $g(z)$ be holomorphic functions in the unit disk U with $f(0)=g(0)=0$.

The function $g(z)$ is said to be subordinate to the function $f(z)$ if there exists a function $\psi(z)$ holomorphic in U such that $|\psi(z)| \leq |z|$ for $z \in U$ and $g(z)=f(\psi(z))$. Let $g(z)$ be a holomorphic function in U. For $0<p<\infty$ and $0 \leqq r<1$, let us put

$$M_p(r,\varphi):=\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\varphi(xe^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}.$$

Then we have the theorem

Theorem A (The subordination theorem of Littlewood [2], p.191)

Suppose that f and g are holomorphic in U and $f(0)=g(0)=0$ and that g is subordinate to f, then we have

$$M_p(r,g) \leqq M_p(r,f) \qquad (0<p<\infty,\ 0 \leqq r<1)$$

The purpose of this note is to extend this theorem to the case of C^n.

§1. Preliminaries

Let us denote a point z of the sace C^n by the column vector

$$z := \begin{pmatrix} z_1 \\ \vdots \\ z_2 \end{pmatrix}$$

and z^* denotes the conjugate transposed vector of z. The norm of z is denoted by $\|z\| := \sqrt{z^*z}$. Denoted by $B_n(r, z_0)$ the ball in C^n with radius r and center z_0, i.e,

$$B_n(r, z_0) := \{ z \in C^n \mid |z - z_0| < r \},$$

and let $B := B_n(1, 0)$, $S(r) := \partial B_n(r, 0)$ (the boudary of $B_n(r, 0)$) and $S := S(r)$.

For $z \in C^n$, we will use the following differential forms and operators for the proof of our theorem;

$$dz = \begin{pmatrix} dz_1 \\ \vdots \\ dz_n \end{pmatrix}, \quad d\bar{z} = \begin{pmatrix} d\bar{z}_1 \\ \vdots \\ d\bar{z}_n \end{pmatrix}.$$

$$\omega(z) := dz_1 \wedge \cdots \wedge dz_n \quad (\text{ n-times})$$

$$\eta(z) := \sum_{j=1}^{n} (-1)^{j+1} z_j dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n \qquad (\text{Leray form})$$

$$d\sigma(z) := \frac{1}{(2i)^n}\{\eta(\bar{z}) \wedge \omega(z) + (-1)^{q(n)}\omega(\bar{z}) \wedge \eta(z)\}. \quad (q(n) = \frac{n(n-1)}{2})$$

Let $z = r\zeta$, $r := |z|$ and $|\zeta| = 1$. Then we have

$$\bar{\zeta}_1 d\zeta_1 + \cdots + \bar{\zeta}_n d\zeta_n + \zeta_1 d\bar{\zeta}_1 + \cdots + \zeta_n d\bar{\zeta}_n = 0.$$

Thus $d\sigma(z) = r^{2n-1} d\sigma(\zeta)$, and the form $dS(\zeta) := \frac{1}{v(S)} d\sigma(\zeta)$ is the normalized rotation invariant surface measure on S, where

$$v(S) = \frac{2\pi^n}{(n-1)!} \quad (\text{ the area of } S).$$

$$dv(z) := \frac{1}{(2i)^n}\, \omega(\bar{z}) \wedge \omega(z) \quad (\text{the volume elememt of } C^n).$$

Then we get

$$dv(z) := \frac{1}{(2i)^n}\, \{\eta(\bar{\xi}) \wedge \omega(\zeta) + (-1)^{q(n)}\omega(\bar{\xi}) \wedge \eta(\zeta)\} \wedge r^{2n-1} dr$$

$$= d\,\sigma(\zeta) \wedge r^{2n-1}dr$$

Let us set

$$\frac{\partial}{\partial z} := \left(\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\right), \qquad \frac{\partial}{\partial z^*} := \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} \\ \vdots \\ \frac{\partial}{\partial \bar{z}_n} \end{pmatrix}$$

$$\Delta := 4\frac{\partial^2}{\partial z^* \partial z} = 4\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$$

$$\mathrm{H} := \frac{\partial^2}{\partial z^* \partial z} = \begin{pmatrix} \frac{\partial^2}{\partial \bar{z}_1 \partial z_1} & \cdots & \frac{\partial^2}{\partial \bar{z}_1 \partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial^2}{\partial \bar{z}_n \partial z_1} & \cdots & \frac{\partial^2}{\partial \bar{z}_n \partial z_n} \end{pmatrix}$$

Let $u(z)$ be a real function in a domain D in C^n. The function $u(z)$ is said to be subharmonic in D if the following three conditions hold :

(1) $-\infty \leq u(z) < \infty$

(2) $u(z)$ is upper semicontinuous in D

(3) For any point $z_0 \in$ D, we can take an $r > 0$ such that

$B(r, z_0) \subset$ D and

$$u(z_0) \leq \frac{1}{v(S)} \int_{S(r,z_0)} u(\zeta)\, d\sigma(\zeta)$$

Suppose that $u(z)$ is a function of the class C^2 in D. Then

$$u(z) \text{ is } \left\{ \begin{matrix} \text{subharmonic} \\ \text{plurisubharmonic} \end{matrix} \right\}$$

$$\Downarrow \Uparrow$$

$$\left\{ \begin{matrix} \Delta u = 4\dfrac{\partial^2 u}{\partial z^* \partial z} \geq 0 \\ \mathrm{H}(u) = \dfrac{\partial^2 u}{\partial z^* \partial z} \text{ (complex hessian) is positive definite.} \end{matrix} \right.$$

Thus the plurisubharmonic function is subharmonic.

The mapping function $f(z)$ from a domain in C^n to C^n is denoted by the column vector

$$f(z) = \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix}.$$

The mapping $f(z)$ is said to be holomorphic if each component functions $f_j(z)$ $(j = 1 \cdots n)$ are holomorphic. Let $H_{n,m}$ be the family of holomorphic mappings from B_n to C_m and suppose that $f(z)$ and $g(z)$ are belonging to $H_{n,m}$ and that $f(0) = g(0) = 0$.

The mapping $g(z)$ is said to be subordinate to $f(z)$ if there exists a holomorphic mapping $\Psi(z)$ from B_n to B_n such that $|\Psi(z)| \leq |z|$ ($z \in B_n$) and $g(z) = f(\Psi(z))$.

For a mapping $f(z) \in H_{n,m}$, we set

$$M_p(r,f) := \left\{ \frac{1}{\nu(S_r)} \int_{S_r} |f(z)|^p d\sigma(z) \right\}^{\frac{1}{p}} \qquad (0 < r < 1,\ 0 < p < \infty).$$

§ 2. Theorems for subharmonic functions

For the ball $B_n(z_0, r) \subset C^n$, let us put

$$K(\xi, z) := \frac{1}{\nu(S)} \frac{r^2 - |z - z_0|^2}{r|z - \xi|^{2n}}$$

Then the following theorems hold :

Theorem B ([3], p.32, Theorem 1.16)

Let $\varphi(z)$ be a continuous function on $S(r, z_0)$. Let us put

$$u(z) := \int_{S(r,z_0)} K(\xi, z)\varphi(\xi) d\sigma(\xi) \qquad (z \in B(r, z_0))$$

Then the function $u(z)$ is the solution of the problem of Dirichlet for $B(r, z_0)$ with the boundary value $\varphi(z)$.

Theorem C ([3], p.52, Theorem 2.7)

Let $\varphi(z)$ be a subharmonic function on a domain D in C_n and $u(z)$ is not equal to $-\infty$. Suppose that $B(z_0, r) \subset$ D. Let us put

$$V(z) := \chi_{B(r,z_0)}(z)\int_{S(r,z_0)} K(\xi, z)\varphi(\xi)d\sigma(\xi) + \chi_{D-\overline{B(r,z_0)}}(z)u(z)$$

where χ_A denotes the characteristic function for A. Then $V(x)$ is subharmonic in D and harmonic in $\boldsymbol{B}(r,z_0)$, and we have $u(x) \leqq V(x)$ in $\boldsymbol{B}(r,z_0)$.

Main Theorem. Let $\varphi(z)$ be a subharmonic function in $\boldsymbol{B}$ and let $\psi(z)$ be a holomorphic mapping from $\boldsymbol{B}$ to $\boldsymbol{B}$ such that $\|\Psi(z)\| \leqq \|z\|$. Then we have

$$\int_{S_n(r,0)} \varphi(\psi(z))d\sigma(z) \leqq \int_{S_n(r,0)} \varphi(z)d\sigma(z)$$

As the corollary of our theorem, we obtain the Subordination Theorem of Littlewood for C^n.

Corollary 1. Let f and g be holomorphic mapppings of $H_{n,m}$ such that $f(0) = g(0) = 0$. Suppose that g is subordinate to f, then we have

$$M_p(r, g) \leqq M_p(r, f) \qquad (0 < p < \infty,\ 0 \leqq r < 1)$$

From Corollary 1, we get the following :

Corollary 2. Let the functions $f(z)$ and $g(z)$ be the same as in Corollary 1, then we have

$$\int_{B_n} \|g(z)\|^p dv(z) \leqq K\int_{B_n} \|f(z)\|^p dv(z)$$

where K is a constant.

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