On Construction of Continuous Functions with Cusp Singularities

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1 Introduction

In this paper, we study various constructions of continuous functions on $\mathbb{R}$ which have the prescribed cusp singularities at each point. As applications, we get a generalization of the result given in our previous paper [7], which discuss the cusp singularities of the classical Weierstrass functions.

Let $s$ be a positive number, which is not an integer and let $x_0$ be a point in $\mathbb{R}^n$. Then a function $f$ on $\mathbb{R}^n$ belongs to the pointwise Hölder space $C^s(x_0)$, if there exists a polynomial $P$ of degree less than $s$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^s$$

in a neighborhood of $x_0$. The pointwise Hölder exponent of a function $f$ at a point $x_0$ in $\mathbb{R}^n$ is defined as

$$H(f, x_0) = \sup \{s > 0; f \in C^s(x_0)\}.$$ 

If a continuous function $f$ does not belong to $C^s(x_0)$ for every $s > 0$, then $H(f, x_0) = 0$.

However the pointwise Hölder exponent of a function $f$ at a point $x_0$ in $\mathbb{R}^n$ is not stable under the pseudo-differential operators. Similarly it does not fully characterize the oscillatory behavior on a neighborhood of $x_0$. This implies that $f \in C^s(x_0)$ cannot be characterized by size estimates on the wavelet coefficients of $f$.

Here let us recall the definition of the weak scaling exponent characterizing the local oscillatory behavior.

$S_0(\mathbb{R}^n)$ denotes the closed subspace of the Schwartz class $S(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} x^\alpha \psi(x) \, dx = 0$$
for every multi-index $\alpha$ in $\mathbb{Z}_+^n$. Then a tempered distribution $f$ belongs to $\Gamma^s(x_0)$, if for every $\psi$ in $\mathcal{S}_0(\mathbb{R}^n)$, there exists a constant $C(\psi)$ such that

$$\left| \int_{\mathbb{R}^n} f(x) \frac{1}{a^n} \psi \left( \frac{x-x_0}{a} \right) \, dx \right| \leq C(\psi)a^s, \quad 0 < a \leq 1.$$ 

The weak scaling exponent of a function $f$ at a point $x_0$ in $\mathbb{R}^n$ is defined as

$$\beta(f, x_0) = \sup \{ s \in \mathbb{R}; f \text{ locally belongs to } \Gamma^s(x_0) \}.$$ 

Since it is known that the pointwise Hölder space $C^s(x_0)$ is contained in local $\Gamma^s(x_0)$, it is obvious that

$$H(f, x_0) \leq \beta(f, x_0).$$

Now we recall the definition of the two-microlocal spaces $C_{x_0}^{s, s'}$, which characterize this weak scaling exponent.

Let $\varphi$ be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ such that

$$\hat{\varphi}(\xi) = \begin{cases} 1 & \text{on } |\xi| \leq \frac{1}{2} \\ 0 & \text{on } |\xi| \geq 1 \end{cases},$$

where $\hat{\varphi}$ is the Fourier transform of $\varphi$. For every non-negative integer $j$, we define the convolution operator $S_j(f) = f * \varphi_{\frac{1}{2^j}}$ where $\varphi_a(x) = \frac{1}{a^n}\varphi \left( \frac{x}{a} \right)$, and the difference operator $\Delta_j = S_{j+1} - S_j$. Then

$$I = S_0 + \sum_{j=0}^{\infty} \Delta_j.$$ 

Let $\psi = \varphi_{\frac{1}{2}} - \varphi$. Then $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ and

$$\Delta_j(f) = f * \psi_{\frac{1}{2^j}}.$$ 

Let $s$ and $s'$ be two real numbers and $x_0$ a point in $\mathbb{R}^n$. Then a tempered distribution $f$ belongs to the two-microlocal spaces $C_{x_0}^{s, s'}$, if there exists a constant $C$ such that

$$|S_0(f)(x)| \leq C(1 + |x - x_0|)^{-s'}$$

and

$$|\Delta_j(f)(x)| \leq C2^{-js}(1 + 2^j|x - x_0|)^{-s'}$$

for every $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$. 
The following remarkable theorems with respect to the two-microlocal spaces $C_{x_0}^{s,s'}$ and $\Gamma^s(x_0)$ were given in [5].

**Theorem A** [5, Theorem 1.8]. Let $s$ and $s'$ be two real numbers and $x_0$ a point in $\mathbb{R}^n$ and let us assume two positive integers $r$ and $N$ satisfying

$$r + s + \inf(s', n) > 0$$

and

$$N > \sup(s, s+s').$$

Let $\psi$ be a function such that

$$|\partial^\alpha \psi(x)| \leq \frac{C(q)}{(1+|x|)^q}, \quad |\alpha| \leq r, \quad q \geq 1$$

and

$$\int_{\mathbb{R}^n} x^\beta \psi(x) \, dx = 0, \quad |\beta| \leq N - 1.$$

If a function or a distribution $f$ belongs to the two-microlocal spaces $C_{x_0}^{s,s'}$, then we have

$$\left| \int_{\mathbb{R}^n} f(x) \frac{1}{a^n} \overline{\psi\left(\frac{x-b}{a}\right)} \, dx \right| \leq Ca^{s}\left(1 + \frac{|b - x_0|}{a}\right)^{-s'}, \quad 0 < a \leq 1, \quad |b - x_0| \leq 1.$$

**Theorem B** [5, Theorem 1.2]. Let $s$ be a real number and let $f$ be a function or a distribution defined on a neighborhood $V$ of $x_0$.

Then $f$ locally belongs to $\Gamma^s(x_0)$ if and only if $f$ locally belongs to the two-microlocal spaces $C_{x_0}^{s,s'}$ for some $s'$.

Several scientists have been interested in constructing irregular functions. The well-known example is the Weierstrass function [8]. It is an example of a nowhere differentiable continuous function. Hardy gave better estimates of the regularities for the Weierstrass function

$$W_c(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

and its sine series

$$W_s(x) = \sum_{n=0}^{\infty} a^n \sin(b^n \pi x), \quad (2)$$
where \(0 < a < 1, \ b > 1\) and \(ab \geq 1\) [3]. He proved that these functions do not possess finite derivatives at each point \(x\) and showed more precisely that if \(ab > 1\) and \(\xi = \frac{\log(\frac{1}{a})}{\log b}\), then these functions satisfy

\[
\mathcal{W}_c(x + h) - \mathcal{W}_c(x) = O(|h|^\xi) \quad \text{and} \quad \mathcal{W}_s(x + h) - \mathcal{W}_s(x) = O(|h|^\xi)
\]

for each \(x\), but satisfy neither

\[
\mathcal{W}_c(x + h) - \mathcal{W}_c(x) = o(|h|^\xi) \quad \text{nor} \quad \mathcal{W}_s(x + h) - \mathcal{W}_s(x) = o(|h|^\xi)
\]

for any \(x\).

Next let us recall the definition of the Takagi function [6]. Let \(\theta^*\) be the 1-periodic function such that

\[
\theta^*(x) = \left\{
\begin{array}{ll}
x & \text{if } 0 \leq x < \frac{1}{2} \\
1 - x & \text{if } \frac{1}{2} \leq x < 1
\end{array}
\right.
\]

Then the Takagi function is defined by

\[
T(x) = \sum_{n=0}^{\infty} \frac{\theta^*(2^n x)}{2^n}.
\]

It is another example of a nowhere differentiable continuous function.

Using the scaling exponents, Meyer defined two types of singularities of functions as follows [5]: a point \(x_0\) in \(\mathbb{R}^n\) is called a cusp singularity of a function \(f\), when

\[
H(f, x_0) = \beta(f, x_0) < \infty,
\]

while a point \(x_0\) in \(\mathbb{R}^n\) is called an oscillating singularity of a function \(f\), when

\[
H(f, x_0) < \beta(f, x_0).
\]

When a point \(x_0\) is a cusp singularity of a function \(f\), the pointwise Hölder exponent can be found by computing the size estimates on the wavelet coefficients of \(f\) inside the influence cone. Using this fact, we construct continuous functions which have a prescribed cusp singularity at each point \(x_0\) in \(\mathbb{R}\).

Daoudi and his team [2] studied the following problem which was raised by Lévy Véhel:

"Let \(s\) be a function from \([0, 1]\) to \([0, 1]\). Under what conditions on \(s\) does there exist a continuous function \(f\) from \([0, 1]\) to \(\mathbb{R}\) such that \(H(f, x) = s(x)\) for all \(x\) in \([0, 1]\)?"

They solved the problem as follows: "For a function \(s\) from \([0, 1]\) to \([0, 1]\), there exist a continuous function \(f\) on \([0, 1]\) such that \(H(f, x) = s(x)\) for all \(x\) in \([0, 1]\) if and only if \(s\) is a function which can be represented as a limit inferior of a sequence of continuous functions
on $[0, 1]$. Further, they constructed such $f$ by various methods, - as the Weierstrass type function, using Schauder bases and using Iterated Function System.

On the other hand, Andersson [1] proved a similar characterization for a function $s$ from $\mathbb{R}$ to $[0, \infty]$ and constructed $f$ satisfying $H(f, x) = s(x)$ for all $x$ in $\mathbb{R}$ by a method using orthogonal wavelets.

In the rest of the paper we study, for a given function on $\mathbb{R}$, various constructions of a function $f$ satisfying

$$H(f, x) = \beta(f, x) = s(x), \quad x \in \mathbb{R},$$

using orthonormal wavelets in Section 2 and as the Weierstrass type function in Section 3.

## 2 Construction Using Orthonormal Wavelets

In this section, using orthonormal wavelets, we construct a continuous function which has a prescribed cusp singularity at each point in $\mathbb{R}$.

The following Lemma 1 is used in the proof of Theorems 1 and 2.

**Lemma 1.** Let $s$ be a function from $\mathbb{R}$ to $[0, \infty]$, which is the lower limit of a sequence of real continuous functions $\{t_l\}_{l \in \mathbb{N}}$. Then there exists a sequence $\{s_l\}_{l \in \mathbb{Z}_+}$ of infinitely differentiable non-negative functions with compact supports such that

(i) $s_l(x) = \liminf_{l \to \infty} s_l(x), \quad x \in \mathbb{R},$

(ii) For each $x_0$ in $\mathbb{R}$, there exists a positive integer $l_0$ such that

$$s_l(x) \geq \frac{1}{\sqrt{l+1}}, \quad l \geq l_0, \quad |x - x_0| \leq 1.$$

(iii) There exists a sequence $\{C_k\}_{k \in \mathbb{Z}_+} \subset (0, \infty)$ such that

$$\sup_{x \in \mathbb{R}} |s_l^{(k)}(x)| \leq C_k l^{k+1}, \quad l \in \mathbb{Z}_+,$$

where $s_l^{(k)}$ is the $k$-th derivative of $s_l$.

**Proof.** Let $\eta$ be a non-negative infinitely differentiable function supported on $[-1, 1]$ satisfying $\eta(x) = 1$ if $|x| \leq \frac{1}{4}$, $\sup_{x \in \mathbb{R}} \eta(x) = 1$ and $\int_{\mathbb{R}} \eta(x) dx = 1$. If we put

$$\tilde{t}_l(x) = \eta \left(\frac{x}{l}\right) \min \left(\max \left(t_l(x), \frac{1}{\sqrt{l+1}}\right), l\right), \quad l \in \mathbb{N},$$
it is easy to see that \( \{ \tilde{t}_l \}_{l \in \mathbb{N}} \) satisfies

\[
\lim_{l \to \infty} \tilde{t}_l (x) = s(x), \quad x \in \mathbb{R},
\]

\[
\tilde{t}_l (x) \geq \frac{1}{\sqrt{l+1}}, \quad |x| \leq \frac{l}{4},
\]

\[
\tilde{t}_l (x) = 0, \quad |x| \geq l
\]

and

\[
\sup_{x \in \mathbb{R}} \tilde{t}_l (x) \leq l.
\]

Since each \( \tilde{t}_l \) is uniformly continuous, we can choose a strictly increasing sequence of positive integers \( \{ p_l \}_{l \in \mathbb{N}} \) such that

\[
\sup_{|x-y| \leq \frac{1}{p_l}} |\tilde{t}_l (x) - \tilde{t}_l (y)| \leq \frac{1}{l}, \quad l \in \mathbb{N}.
\]

Under these circumstances, we define \( s_l (x) \) for \( l \in \mathbb{Z}_+ \) and \( x \in \mathbb{R} \) by

\[
s_l (x) = \begin{cases} 
0 & \text{if } 0 \leq l < p_1 \\
\int_{\mathbb{R}} p_m \eta(p_m(x-y)) \tilde{t}_m (y) \, dy & \text{if } p_m \leq l < p_{m+1}, \quad m \in \mathbb{N}.
\end{cases}
\]

If we put \( C_k = \int_{\mathbb{R}} |\eta^{(k)}(x)| \, dx \) for \( k \in \mathbb{Z}_+ \), then \( \{ s_l \}_{l \in \mathbb{Z}_+} \) satisfies the required properties (i), (ii) and (iii). To prove (i) we have

\[
|s_l (x) - \tilde{t}_m (x)| = \left| \int_{\mathbb{R}} p_m \eta(p_m(x-y)) \left( \tilde{t}_m (y) - \tilde{t}_m (x) \right) \, dy \right|
\]

\[
\leq \sup_{|x-y| \leq \frac{1}{p_m}} |\tilde{t}_m (y) - \tilde{t}_m (x)| \int_{\mathbb{R}} \eta(y) \, dy
\]

\[
\leq \frac{1}{m}, \quad p_m \leq l < p_{m+1}.
\]

This proves the desired result. To prove (ii) we choose \( m_0 \in \mathbb{N} \) such that \( \frac{m_0}{4} - \frac{1}{m_0} \geq |x_0| + 1 \) and put \( l_0 = p_{m_0} \). For a positive integer \( l \geq l_0 \), choose \( m \in \mathbb{N} \) such that \( p_m \leq l < p_{m+1} \).

Then if \( |x - x_0| \leq 1 \), we have

\[
s_l (x) = \int_{\mathbb{R}} p_m \eta(p_m(x-y)) \tilde{t}_m (y) \, dy
\]

\[
\geq \inf_{|x-y| \leq \frac{1}{p_m}} \tilde{t}_m (y) \int_{\mathbb{R}} \eta(y) \, dy
\]
\[
\begin{align*}
\geq & \inf_{|y| \leq |x_0| + 1 + \frac{1}{m}} \tilde{t}_m(y) \\
\geq & \inf_{|y| \leq \frac{m}{4}} \tilde{t}_m(y) \\
\geq & \frac{1}{\sqrt{m + 1}} \geq \frac{1}{\sqrt{l + 1}}.
\end{align*}
\]

To prove (iii) we choose \( m \in \mathbb{N} \), for a given \( l \in \mathbb{N} \), such that \( p_m \leq l < p_{m+1} \). Then we have
\[
|s^{(k)}_l(x)| = \left| \int_{\mathbb{R}} p_m^{k+1} \eta^{(k)}(p_m(x - y)) \tilde{t}_m(y) \, dy \right|
\leq p_m^k \sup_{|x-y| \leq \frac{1}{p_m}} \tilde{t}_m(y) \int_{\mathbb{R}} |\eta^{(k)}(y)| \, dy
\leq C_k m p_m^k \leq C_k l^{k+1}.
\]

\[\blacksquare\]

**Theorem 1.** Let \( s \) be a function from \( \mathbb{R} \) to \([0, \infty]\), which is the lower limit of a sequence of continuous functions. Then there exists a sequence \( \{s_l\}_{l\in \mathbb{Z}_+} \) of differentiable functions such that
\[
s(x) = \liminf_{l \to \infty} s_l(x), \quad x \in \mathbb{R} \tag{3}
\]
and
\[
\sup_{x \in \mathbb{R}} |s'_l(x)| \leq C_l l^2, \quad l \in \mathbb{Z}_+. \tag{4}
\]

Let \( \psi \) be an orthonormal wavelet in the Schwartz class \( S(\mathbb{R}) \). If we define a continuous function \( f \) by
\[
f(x) = \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} c(l, m) \psi(2^l x - m),
\]
where
\[
c(l, m) = \min(2^{-ls_l(\frac{m}{2^l})}, 2^{-l \log 2^l}),
\]
then we have
\[
H(f, x_0) = \beta(f, x_0) = s(x_0)
\]
at each point \( x_0 \) in \( \mathbb{R} \).
Proof. The existence of \( \{s_j\}_{j \in \mathbb{Z}_+} \) satisfying (3) and (4) follows from Lemma 1. Since

\[
\lim_{j \to \infty} \sup_{|x-y| \leq 2^{-j} \log j} |s_j(x) - s_j(y)| \leq \lim_{j \to \infty} \sup_{x \in \mathbb{R}} |s'_j(x)| \sup_{|x-y| \leq 2^{-j} \log j} |x-y|
\]

\[
\leq C_1 \lim_{j \to \infty} j^2 2^{-j} \log j^2
\]

\[
= 0,
\]

\( H(f, x_0) = s(x_0) \) at each point \( x_0 \in \mathbb{R} \) (cf. [1] p.441, proof of Theorem 1.). We only need to compute the value of \( \beta(f, x_0) \).

Let us assume \( f \) locally belongs to \( \Gamma^s(x_0) \). Then by Theorem B, \( f \) locally belongs to \( \mathcal{C}^{s, s'}_{x_0} \) for some \( s' < 0 \). On the other hand, \( \psi \in \mathcal{S}_0(\mathbb{R}) \) (cf. [4, 2. Corollary 3.7.]). By Theorem A, there exist two constants \( C \in (0, \infty) \) and \( \delta \in (0, \frac{1}{2}) \) such that

\[
\left| \int f(x) \frac{1}{a} \overline{\psi \left( \frac{x-b}{a} \right)} dx \right| \leq C a^s \left( 1 + \frac{|b-x_0|}{a} \right)^{-s'}, \quad 0 < a \leq \delta, \quad |b-x_0| \leq \delta. \quad (5)
\]

Let \( j_0 \) be a positive integer such that \( \frac{1}{2j_0} \leq \delta \). For every \( j \geq j_0 \), there exists \( k_j \in \mathbb{Z} \) such that \( \frac{k_j}{2^j} \leq x_0 < \frac{k_j+1}{2^j} \) and we define \( a_j \) and \( b_j \) by \( a_j = \frac{1}{2^j} \) and \( b_j = \frac{k_j}{2^j} \). Then \( |b_j - x_0| \leq a_j \) and by (5), we have

\[
\left| \int f(x) 2^j \overline{\psi(2^j x - k_j)} dx \right| \leq \frac{C 2^{-s'}}{2^{js}}, \quad j \geq j_0. \quad (6)
\]

We estimate the left hand side of (6) as follows:

\[
\left| \int f(x) 2^j \overline{\psi(2^j x - k_j)} dx \right| = \left| \sum_{l=2}^{\infty} \sum_{m=-\infty}^{\infty} c(l, m) \int \psi(2^j x - m) 2^j \overline{\psi(2^j x - k_j)} dx \right| = c(j, k_j). \quad (7)
\]

By (6) and (7), \( f \in \Gamma^s(x_0) \) implies

\[
c(j, k_j) = \min(2^{-j} 2^{s'}(2^j), 2^{-jd}) \leq \frac{C 2^{-s'}}{2^{js}}, \quad j \geq j_0. \quad (8)
\]

Observe that

\[
\lim_{j \to \infty} \left| s_j \left( \frac{k_j}{2^j} \right) - s_j(x_0) \right| \leq \lim_{j \to \infty} \sup_{x \in \mathbb{R}} |s'_j(x)| \left| \left( x_0 - \frac{k_j}{2^j} \right) \right|
\]

\[
\leq C_1 \lim_{j \to \infty} j^2 2^{-j} \log j^2
\]

\[
= 0.
\]
By (8), we have
\[
s \leq \liminf_{j \to \infty} \max \left( s_j \left( \frac{k_j}{2^j} \right), \frac{1}{\log j} \right)
= \liminf_{j \to \infty} s_j \left( \frac{k_j}{2^j} \right)
= \liminf_{j \to \infty} s_j(x_0) + \lim_{j \to \infty} \left( s_j \left( \frac{k_j}{2^j} \right) - s_j(x_0) \right)
= s(x_0).
\]

Therefore \( \beta(f, x_0) \leq s(x_0) = H(f, x_0) \). Since \( H(f, x_0) \leq \beta(f, x_0) \) is trivial, we have \( H(f, x_0) = \beta(f, x_0) = s(x_0) \).

\[ \square \]

3 Use of Weierstrass Type Functions

In this section, we construct the Weierstrass type continuous function which has a prescribed cusp singularity at each point in \( \mathbb{R} \).

We begin with the following lemma.

**Lemma 2.** Let \( s \in [0, \infty) \), \( l_0 \in \mathbb{Z}_+ \) and \( \{s_l\}_{l \in \mathbb{Z}_+} \subset \mathbb{R} \) be such that

(a) \( \liminf_{l \to \infty} s_l = s \),

(b) \( s_l \geq \frac{1}{\sqrt{l+1}} \), \( l \geq l_0 \).

Suppose \( \lambda > 1 \) and \( \{\theta_l\}_{l \in \mathbb{Z}_+} \subset \mathbb{R} \) are chosen arbitrary.

(i) If \( m \in \mathbb{Z}_+ \) and \( \{\alpha_l\}_{l \in \mathbb{Z}_+} \) is a bounded sequence in \( \mathbb{R} \) and if we define a continuous function \( f \) by

\[
f(x) = \sum_{l=0}^{\infty} \frac{\alpha_l m}{\lambda^{ls_l}} \sin(\lambda^l x + \theta_l), \quad x \in \mathbb{R},
\]

then we have

\( H(f, x_0) \geq s \)

at each point \( x_0 \) in \( \mathbb{R} \).

(ii) If we define a continuous function \( g \) by

\[
g(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l}} \sin(\lambda^l x + \theta_l), \quad x \in \mathbb{R},
\]

then we have

\( H(g, x_0) = \beta(g, x_0) = s \)
at each point $x_0$ in $\mathbb{R}$.

**Proof.** (i) By (b), $f$ is a continuous function on $\mathbb{R}$ and hence we have only to show (i) when $s > 0$.

Let $x_0 \in \mathbb{R}$ be fixed arbitrary.

First, we consider the case $0 < s \leq 1$. Let $\varepsilon \in (0, s)$ be arbitrary. By (a), we can choose $l_0 \in \mathbb{Z}_+$ such that $s > s - \frac{\varepsilon}{2}$ for $l \geq l_0$ and we put $f_1(x) = \sum_{l=l_0}^{\infty} \frac{\alpha_l l^m}{\lambda^{ls_l}} \sin(\lambda^l x + \theta_l)$.

To show $H(f, x_0) \geq s - \varepsilon$, it suffices to show $f_1 \in C^{s-\varepsilon}(x_0)$ since $H(f - f_1, x_0) = \infty$ is obvious. Let $x$ be a real number such that $|x - x_0| < \frac{1}{\lambda^{l_0}}$ and choose $N \in \mathbb{Z}_+$ such that

$$\frac{1}{\lambda^{N+1}} \leq |x - x_0| < \frac{1}{\lambda^{N}}.$$ 

Then we have

$$|f_1(x) - f_1(x_0)| = \sum_{l=l_0}^{\infty} \frac{\alpha_l l^m}{\lambda^{ls_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l))$$

$$\leq \sum_{l=l_0}^{N-1} \frac{\alpha_l l^m}{\lambda^{ls_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l)) + \sum_{l=N}^{\infty} \frac{\alpha_l l^m}{\lambda^{ls_l}} (\sin(\lambda^l x + \theta_l) - \sin(\lambda^l x_0 + \theta_l))$$

$$= A_1 + A_2.$$  \hspace{1cm} (9)

Observe first that there exists a constant $M_1 \in (0, \infty)$ such that

$$|\alpha_l| l^m \leq M_1 \lambda^{\frac{l}{2}}, \quad l \geq l_0.$$  \hspace{1cm} (10)

To estimate $A_1$ and $A_2$ we use (10) to obtain

$$A_1 \leq 2 \sum_{l=l_0}^{N-1} |\alpha_l| l^m \lambda^{l(1-s_l)} |x - x_0|$$

$$\leq M_1 \sum_{l=l_0}^{N-1} \lambda^{l(1-s+l)} |x - x_0|$$

$$= M_1 \frac{\lambda^l(1-s+l)}{\lambda^{l-s} - 1} |x - x_0|$$

$$\leq M_1 \frac{\lambda^{N(1-s+\varepsilon)}}{\lambda^{l-s} - 1} |x - x_0|$$

$$\leq M_1 \lambda^{1-s+\varepsilon} |x - x_0|^{s-\varepsilon}.$$
\[
\leq 2 \sum_{l=N}^{\infty} \frac{|\alpha_l|^m}{\lambda^{ls_l}} \\
\leq 2M_1 \sum_{l=N}^{\infty} \frac{1}{\lambda^{l(s-\epsilon)}} \\
= \frac{2M_1}{\lambda^{N(s-\epsilon)}} \frac{1}{1 - \frac{1}{\lambda^{s-\epsilon}}} \\
\leq \frac{2M_1 \lambda^{2(s-\epsilon)}}{\lambda^{s-\epsilon} - 1} |x - x_0|^{s-\epsilon}.
\]

The estimates for \(A_1\) and \(A_2\) with (9) show that there exists a constant \(M_2 \in (0, \infty)\) such that

\[
|f_1(x) - f_1(x_0)| \leq M_2|x - x_0|^{s-\epsilon}, \quad |x - x_0| < \frac{1}{\lambda^{l_0}}.
\]

Thus \(H(f_1, x_0) \geq s-\epsilon\) and hence \(H(f, x_0) \geq s-\epsilon\). Since \(\epsilon > 0\) is arbitrary, \(H(f, x_0) \geq s\).

Next, we consider the case \(n < s \leq n+1\) for some \(n \in \mathbb{N}\). In this case, \(f\) is \(n\)-times continuously differentiable on \(\mathbb{R}\) and we have

\[
f^{(n)}(x) = \sum_{l=0}^{\infty} \frac{\alpha_l l^m}{\lambda^{l(s_l-n)}} \sin(\lambda^l x + \theta_l + \frac{n\pi}{2}).
\]

Thus \(H(f^{(n)}, x_0) \geq s-n\) by an argument similar to the case where \(0 < s \leq 1\) and hence \(H(f, x_0) \geq s\) holds even for \(1 < s < \infty\).

Finally, we consider the case \(s = \infty\). In this case, \(f\) is obviously infinitely differentiable at \(x_0\) and hence \(H(f, x_0) = \infty\).

(ii) \(H(g, x_0) \geq s\) follows from (i), if we put \(\alpha_l = 1\) for \(l \in \mathbb{Z}_+\) and \(m = 0\) in (i).

For \(\beta(g, x_0)\), let us assume \(g\) locally belongs to \(\Gamma^0(x_0)\). Let \(\psi\) be a function in \(S_0(\mathbb{R})\) such that \(\hat{\psi}(\xi) = 0\) if \(|\xi - 1| \geq \frac{\lambda-1}{\lambda}\) and \(\hat{\psi}(1) = 2\). Then there exist two constants \(M_3 \in (0, \infty)\) and \(\eta \in (0,1]\) such that

\[
|\int g(x) \frac{1}{a} \psi \left( \frac{x-x_0}{a} \right) dx| \leq M_3 a^\rho, \quad 0 < a \leq \eta.
\]

(11)

Let \(j_0\) be a non-negative integer such that \(\frac{1}{\lambda^{j_0}} \leq \eta\). For every \(j \geq j_0\), we put \(a_j = \frac{1}{\lambda^j}\).

By (11), we have

\[
|\int g(x) \lambda^j \psi(\lambda^j (x - x_0)) dx| \leq \frac{M_3}{\lambda^{j\rho}}, \quad j \geq j_0.
\]

(12)

We estimate the left hand side of (12) as follows:

\[
|\int g(x) \lambda^j \psi(\lambda^j (x - x_0)) dx| = \left| \int \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l}} \sin(\lambda^{l-j}x + \lambda^l x_0 + \theta_l) \psi(x) dx \right|
\]
By (12) and (13), \( g \in \Gamma^\rho(x_0) \) implies \( \frac{1}{\lambda^{s_j}} \leq \frac{M}{\lambda^{s_j}} \) for every \( j \geq j_0 \) and hence \( \rho \leq \liminf_{j \to \infty} s_j = s \leq H(g, x_0) \). Therefore \( \beta(g, x_0) \leq s \leq H(g, x_0) \). Since \( H(g, x_0) \leq \beta(g, x_0) \) is trivial, we have \( H(g, x_0) = \beta(g, x_0) = s \).

\[ \square \]

**Theorem 2.** Let \( s \) be a function from \( \mathbb{R} \) to \( [0, \infty] \), which is the lower limit of a sequence of continuous functions and let \( \{s_l\}_{l \in \mathbb{Z}_+} \) be a sequence of continuous functions satisfying part (i), (ii) and (iii) of Lemma 1.

Suppose \( \lambda > 1 \) and \( \{\theta_l\}_{l \in \mathbb{Z}_+} \subset \mathbb{R} \) are chosen arbitrary. If we define a continuous function \( f \) by

\[
f(x) = \sum_{l=0}^\infty \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l),
\]

then we have

\[
H(f, x_0) = \beta(f, x_0) = s(x_0)
\]
at each point \( x_0 \in \mathbb{R} \).

**Proof.** First, we consider the case \( n \leq s(x_0) < n+1 \) for some \( n \in \mathbb{Z}_+ \). Using the Taylor expansion we have

\[
\frac{1}{\lambda^{ls_l(x)}} = \frac{1}{\lambda^{ls_l(x_0)}} + \sum_{j=1}^n \frac{1}{j!} \frac{d^j}{dx^j} \frac{1}{\lambda^{ls_l(x)}} \bigg|_{x=x_0} (x-x_0)^j + \frac{1}{(n+1)!} \frac{1}{\lambda^{ls_l(x)}} \bigg|_{x=\xi_l} (x-x_0)^{n+1},
\]

where \( \xi_l \in (\min(x, x_0), \max(x, x_0)) \). It goes without saying that if \( n = 0 \) the second term in the right hand side of (14) does not appear. By (14), we can write

\[
f(x) = \sum_{l=0}^\infty \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l) = f_1(x) + f_2(x) + f_3(x),
\]
\[ f_1(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls_l(x_0)}} \sin(\lambda^l x + \theta_l), \] (16)

\[ f_2(x) = \sum_{l=0}^{\infty} \sum_{j=1}^{n} \frac{1}{j!} \frac{d^j}{dx^j} \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l)(x-x_0)^j, \] (17)

and

\[ f_3(x) = \frac{1}{(n+1)!} \sum_{l=0}^{\infty} \frac{d^{n+1}}{dx^{n+1}} \frac{1}{\lambda^{ls_l(x)}} \sin(\lambda^l x + \theta_l)(x-x_0)^{n+1}, \] (18)

where \( \xi_l \in (\min(x, x_0), \max(x, x_0)) \).

By part (ii) of Lemma 2, \( H(f_1, x_0) = \beta(f_1, x_0) = s(x_0) \) follows at once. \( f_2 \) does not appear if \( n = 0 \), and if \( n \geq 1 \) we have

\[ f_2(x) = \sum_{l=0}^{\infty} \sum_{j=1}^{n} \sum_{(*)_j} \frac{1}{j!} \frac{(-\log\lambda)^k l^k \alpha_{j,i_1,\ldots,i_k} s_l^{(i_1)}(x_0) \ldots s_l^{(i_k)}(x_0)}{\lambda^{ls_l(x_0)}} \sin(\lambda^l x + \theta_l)(x-x_0)^j, \] (19)

where \( \sum_{(*)_j} \) mean the summation under the condition \( i_1 + \cdots + i_k = j \) with \( i_1 \leq \cdots \leq i_k \) and \( \{\alpha_{j,i_1,\ldots,i_k}\} \) are positive integers satisfying \( \sum_{(*)_j} \alpha_{j,i_1,\ldots,i_k} \leq (k+1)^j \).

By (19) and part (iii) of Lemma 1, we can deduce that \( H(f_2, x_0) \geq s(x_0) + 1 \). For \( f_3 \), we have

\[ f_3(x) = \frac{1}{(n+1)!} \sum_{l=0}^{\infty} \sum_{k=1}^{n+1} \sum_{(*)_{n+1}} \frac{(-\log\lambda)^k l^k \alpha_{n+1,i_1,\ldots,i_k} s_l^{(i_1)}(\xi_l) \ldots s_l^{(i_k)}(\xi_l)}{\lambda^{ls_l(\xi_l)}} \sin(\lambda^l x + \theta_l)(x-x_0)^{n+1}, \] (20)

where \( \sum_{(*)_{n+1}} \) mean the summation under the condition \( i_1 + \cdots + i_k = n+1 \) with \( i_1 \leq \cdots \leq i_k \) and \( \{\alpha_{n+1,i_1,\ldots,i_k}\} \) are positive integers satisfying \( \sum_{(*)_{n+1}} \alpha_{n+1,i_1,\ldots,i_k} \leq (k+1)^{n+1} \).

By (20) and part (iii) of Lemma 1, we can deduce that \( H(f_3, x_0) \geq n+1 \). By the estimates for \( f_1, f_2 \) and \( f_3 \), and (15), we can conclude that \( H(f, x_0) = \beta(f, x_0) = s(x_0) \).

Next, we consider the case \( s(x_0) = \infty \). Let \( n \) be a positive integer and let \( f = f_1 + f_2 + f_3 \), where \( f_1, f_2 \) and \( f_3 \) are defined by (16), (17) and (18), respectively. But in this case, we have \( H(f_1, x_0) = H(f_2, x_0) = \infty \) and \( H(f_3, x_0) \geq n+1 \) by part (iii) of Lemma 1 and part (i) of Lemma 2, since \( \liminf_{larrow \infty} s_l(x_0) = \infty \). By the estimates for \( f_1, f_2 \) and \( f_3 \), and (15), we have \( H(f, x_0) \geq n+1 \). Since \( n \) is arbitrary, we can conclude that \( H(f, x_0) = \beta(f, x_0) = s(x_0) \) even for \( s(x_0) = \infty \).
In the case where $s$ is a continuous function, we have the following result.

**Theorem 3.** Let $s$ be a continuous function from $\mathbb{R}$ to $(0, \infty)$ such that

$$s(x_0) < H(s, x_0)$$

at each point $x_0$ in $\mathbb{R}$. Suppose $\lambda > 1$ and $\{\theta_l\}_{l \in \mathbb{Z}^+} \subset \mathbb{R}$ are chosen arbitrary. If we define a continuous function $f$ by

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls(x)}} \sin(\lambda^l x + \theta_l),$$

then we have

$$H(f, x_0) = \beta(f, x_0) = s(x_0)$$

at each point $x_0$ in $\mathbb{R}$.

**Proof.** Let $x_0 \in \mathbb{R}$ be fixed arbitrary and let $x$ be a real number such that $|x-x_0| < 1$. Then we have

$$f(x) = \sum_{l=0}^{\infty} \frac{1}{\lambda^{ls(x)}} \sin(\lambda^l x + \theta_l) + \sum_{l=0}^{\infty} \left( \frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}} \right) \sin(\lambda^l x + \theta_l)$$

$$= f_1(x) + f_2(x).$$

(21)

By part (ii) of Lemma 2, $H(f_1, x_0) = \beta(f_1, x_0) = s(x_0)$ follows at once. Let $\epsilon$ be a positive number such that $s(x_0) + \epsilon < H(s, x_0)$ and $s(x_0) + \epsilon \notin \mathbb{N}$. Then $s \in C^{s(x_0) + \epsilon}(x_0)$ and there exist a polynomial $P$ of degree at most $[s(x_0) + \epsilon]$, two constants $C \in (0, \infty)$ and $\delta \in (0, 1)$ such that

$$s(x) = s(x_0) + P(x - x_0) + Q(x - x_0)$$

and

$$|Q(x - x_0)| \leq C|x - x_0|^{s(x_0) + \epsilon}, \quad |x - x_0| \leq \delta.$$ 

To estimate $f_2$, using the mean value theorem, we write

$$\frac{1}{\lambda^{ls(x)}} - \frac{1}{\lambda^{ls(x_0)}} = \frac{(- \log \lambda) l(s(x) - s(x_0))}{\lambda^{l\eta}},$$

where $\eta \in [\min(s(x), s(x_0)), \max(s(x), s(x_0))]$. Then we have

$$\left| f_2(x) - \left( -\log \lambda \right) \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\eta}} \sin(\lambda^l x + \theta_l) \right| (x - x_0)$$
\[
= (\log \lambda) \left| \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\tau}} \sin(\lambda^{l}x + \theta_{l}) \right| |Q(x - x_{0})| \\
\leq C(\log \lambda) \sum_{l=0}^{\infty} \frac{l}{\lambda^{l\eta}} |x - x_{0}|^{s(x_{0}) + \epsilon}.
\]

Hence \( H(f_{2}, x_{0}) \geq s(x_{0}) + \epsilon \). By the estimates for \( f_{1} \) and \( f_{2} \), and (21), we can conclude that \( H(f, x_{0}) = \beta(f, x_{0}) = s(x_{0}) \).

**Corollary 1.** Each point in \( \mathbb{R} \) is a cusp singularity of the Weierstrass functions.

**Proof.** Let \( \mathcal{W}_{c} \) and \( \mathcal{W}_{s} \) be the Weierstrass functions (for the definitions of \( \mathcal{W}_{c} \) and \( \mathcal{W}_{s} \), see (1) and (2)). If we put \( \lambda = b \), \( s(x) = \frac{\log(\frac{1}{a})}{\log b} \) and \( \theta_{l} = \frac{\pi}{2} \) for \( l \in \mathbb{Z}_{+} \) or \( \theta_{l} = 0 \) for \( l \in \mathbb{Z}_{+} \), then we have \( H(\mathcal{W}_{c}, x) = \beta(\mathcal{W}_{c}, x) = \frac{\log(\frac{1}{a})}{\log b} = H(\mathcal{W}_{s}, x) = \beta(\mathcal{W}_{s}, x) \) at each point \( x \) in \( \mathbb{R} \) from Theorem 3.

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**References**


