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The corona type decomposition of Hardy-Orlicz spaces

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Abstract

The H^p -corona type problem in several complex variables has been solved affirmatively by Amar [1], Andersson [2], Andersson-Carlesson [3, 4], Krantz-Li [11] and so on. Especially, Andersson-Carlsson [4] proved the H^p -norm estimates of the corona solutions which are constructed by a concrete integral representation formula. In this paper, we give some Orlicz space versions for interpolation theorems of Marcinkiewicz type and prove the H_ϕ-norm estimates of the corona solutions for $\phi \in \Delta_2 \cap \nabla_2$. Moreover we also show that the Δ_2-condition is reasonable in a sense.

1 Introduction

In this paper, we consider a candidate of holomorphic space, in which we discuss the corona type problem. The corona problem was conjectured by S.Kakutani as early as 1941 and was solved affirmatively by L.Carleson in 1962. Here, the corona problem is meant to be a problem about the structure of the maximal ideal space $\mathcal{M}$ of $H^\infty(D)$. That is, open unit disc D is dense in $\mathcal{M}$ with respect to the Gelfand topology ? This question is equivalent to the existence problem as follows. For any $f_1, \cdots, f_m \in H^\infty(D)$ such that $\inf_{z\in D} \sum_{k=1}^m |f_k(z)| \geq \delta > 0$, is there exist $g_1, \cdots, g_m \in H^\infty(D)$ such that $\sum_{k=1}^m f_k(z)g_k(z) = 1$? $f_1, \cdots, f_m$ and $g_1, \cdots, g_m$ are refered to as the corona data and the corona solutions respectively. Let X be a holomorphic space. We consider the question whether the mapping defined by

$$X \times \cdots \times X \ni (g_1, \cdots, g_m) \mapsto \sum_{k=1}^m f_k g_k \in X$$

is surjective. We say that X has the X-corona solution (for the corona data $f_1, \cdots, f_m$) if this mapping is serjective. Then, let $T_k : X \to X$, $(k = 1, \cdots, m)$ be an operator such that

$$h(z) = \sum_{k=1}^m f_k(z) \cdot T_k h(z), \quad (h \in X,\, z \in \Omega)$$

if X has the X-corona solution for the corona data $f_1, \cdots, f_m$. In particular we refer to $T_k h$, $(k = 1, \cdots, m)$ as the X-corona solution if T_k is bounded on X in such sense as $\|T_k h\|_X \leq C\|h\|_X$.

Then the corona theorem asserts that $H^\infty(D)$ has the $H^\infty(D)$-corona solutions for any corona data. On the other hand, the corona problem in several complex variables has not been solved yet. In some studies of the corona problem in several complex variables so far, the H^p -corona type problem has been solved affirmatively. That is, it is shown that H^p has the H^p-corona solution. (For details, see Amar [1], Andersson [2], Andersson-Carlsson [3, 4], Krantz-Li [11] and so on.)

Now, we are motivated by the question whether H^∞ can be approximated by some holomorphic spaces X having the X-corona solution. And we consider the Hardy-Orlicz space $H_\phi(\Omega)$, which is a generalization of Hardy spaces H^p , as a candidate of such space. In what follows, we let $\Omega \subset \boldsymbol{C}^n$ be a bounded strictly pseudoconvex domain with a smooth boundary of class C^3.

At first, we review some convex functions. We refer to a convex function $\phi : \boldsymbol{R} \to \boldsymbol{R}_+ \cup \{\infty\}$ as a Young function if (1) $\phi(x) = \phi(-x)$, (2) $\phi(0) = 0$ and (3) $\lim_{x\to\infty} \phi(x) = \infty$. Moreover, a continuous Young function ϕ is called an N-function if (1) $\phi(x) = 0$ iff $x = 0$ and (2) $\lim_{x\to 0} \frac{\phi(x)}{x} = 0$, $\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty$. Then, we introduce two classifications for convex functions which play an important role below. A Young function $\phi : \boldsymbol{R} \to \boldsymbol{R}_+$ satisfies the Δ_2-condition ($\phi \in \Delta_2$) if there exists a positive constant K such that

$$\phi(2x) \leq K\phi(x), \quad (x \geq 0).$$

And a Young function $\phi : \boldsymbol{R} \to \boldsymbol{R}_+$ satisfies the ∇_2 -condition ($\phi \in \nabla_2$) if there exists a positive constant $a > 1$ such that

$$\phi(x) \leq \frac{1}{2a}\phi(ax), \quad (x \geq 0).$$

Let ϕ be an N-function satisfying the Δ_2 and ∇_2-condition. Then, the Hardy-Orlicz space $H_\phi(\Omega)$ is defined as follows.

$$H_\phi(\Omega) = \left\{ f \in \mathcal{O}(\Omega) : \limsup_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} \phi(|f|)d\sigma_\varepsilon < \infty \right\}.$$

Since $f \in H_\phi(\Omega)$ belongs to the Nevanlinna class, f has the nontangential limit $f(\zeta)$ at almost every $\zeta \in \partial\Omega$. From now on, we identify $H_\phi(\Omega)$ with a function space on the boundary $\partial\Omega$.

2 Main results

We use the real variable methods such as an Orlicz space version of the interpolation theorem of Marcinkiewicz type, Hardy-Littlewood maximal operator, nontangential maximal operator and Orlicz space theory to characterize the Hardy-Orlicz space. Our main results are as follows.

Theorem 1 *Suppose that $\phi \in \Delta_2 \cap \nabla_2$. Then every function in Hardy-Orlicz space $H_\phi(\Omega)$ can be approximated by some functions holomorphic up to the boundary with respect to Luxemberg norm:*

$$H_\phi(\Omega) \quad \cong \quad [A(\partial\Omega)]_{L_\phi(\partial\Omega)},$$

where we recall that $A(\partial\Omega)$ is the restriction of $C(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ to the boundary $\partial\Omega$ and we mean $[A(\partial\Omega)]_{L_\phi(\partial\Omega)}$ as the closure of $A(\partial\Omega)$ with respect to the Luxemberg norm.

Theorem 2 *Suppose that $\phi \in \Delta_2 \cap \nabla_2$. Then the image of Orlicz space $L_\phi(\partial\Omega)$ by the Szegö projection S coincides with Hardy-Orlicz space $H_\phi(\Omega)$, that is,*

$$SL_\phi(\partial\Omega) \quad = \quad H_\phi(\Omega).$$

By combining the theorem above and an Orlicz space version of the interpolation theorem of Marcinkiewicz type, we obtain an interpolation theorem for Hardy-Orlicz spaces.

Theorem 3 *Let $\phi, \phi_2 \in \Delta_2 \cap \nabla_2$ be satisfying that $\sup_{\lambda>0} \frac{\varphi(\lambda)\phi_2(\lambda)}{\phi(\lambda)\varphi_2(\lambda)} < 1$, where φ and φ_2 are the left derivatives of ϕ and ϕ_2 respectively. We suppose that a sublinear operator B defined on $H^1(\Omega)$ and $H_{\phi_2}(\Omega)$ is of weak type $(1,1)$ and of weak type (ϕ_2, ϕ_2) respectively. Then B is defined on $H_\phi(\Omega)$ and the following holds:*

$$\int_{\partial\Omega} \phi(|Bf|)d\sigma \leq C \inf\left\{\int_{\partial\Omega} \phi(|g|)d\sigma \,:\, g \in L_\phi(\partial\Omega)\, s.t.\, f = Sg\right\},$$

where S is the Szegö projection.

Before the corona type decomposition of Hardy-Orlicz spaces $H_\phi(\Omega)$, we review the corona type decomposition of Hardy spaces $H^p(\Omega)$ as follows. Andersson-Carlsson [4] shows that an explicit integral formula due to Berndtsson [5] provides the H^p-corona solutions.

Theorem 4 *(Andersson-Carlsson [4])*
Let $1 \leq p < \infty$. If $f_1, \cdots, f_m \in H^\infty(\Omega)$ satisfies that $\sum_{i=1}^m |f_i(z)| \geq \delta > 0$ for all $z \in \Omega$, then there exist integral operators $T_i : H^p(\Omega) \to H^p(\Omega)$, $(i = 1, \cdots, m)$ such that $\sum_{i=1}^m f_i(z)T_ih(z) = h(z)$, $(z \in \Omega)$ and $\|T_ih\|_p \leq C\|h\|_p$ for a positive constant C.

By combining the theorems above, we can show that this integral formula due to Berndtsson [5] admits H_ϕ-estimates if $\phi \in \Delta_2 \cap \nabla_2$.

Corollary 1 *Let $\phi \in \Delta_2 \cap \nabla_2$. If $f_1, \cdots, f_m \in H^\infty(\Omega)$ are corona data, that is, they satisfy that $\sum_{i=1}^m |f_i(z)| \geq \delta > 0$ for all $z \in \Omega$, then there exist integral operators $T_i : H_\phi(\Omega) \to H_\phi(\Omega)$, $(i = 1, \cdots, m)$ such that $\sum_{i=1}^m f_i(z)T_ih(z) = h(z)$, $(z \in \Omega)$. Furthermore it follows that there exists a positive constant C such that*

$$\int_{\partial\Omega} \phi(|T_ih|)d\sigma \leq C \inf\left\{\int_{\partial\Omega} \phi(|g|)d\sigma : g \in L_\phi(\partial\Omega) \text{ such that } h = Sg\right\},$$

where S is the Szegö projection.

From the theorems above, we may say that the Hardy-Orlicz space $H_\phi(\Omega)$ with a moderate growth condition (i.e. $\phi \in \Delta_2 \cap \nabla_2$) has the $H_\phi(\Omega)$ -corona solution. On the other hand, a question whether the condition that $\phi \in \Delta_2$ is too strong occurs. Then we investigate the relation between the boundedness of the Szegö projection and the operators constructing the corona solutions and the gorwthness of the N-function ϕ in order to find a reasonable condition with respect to the growthness of ϕ.

Theorem 5 *Let ϕ be an N-function. We suppose that S is the Szegö projection on Ω. If S is of weak type (ϕ, ϕ) :*

$$\phi(\lambda)\sigma(\{|Sf| > \lambda\}) \leq C_1 \int_{\partial\Omega} \phi(C_2|f|)d\sigma, \quad (\lambda > 0,\, f \in L_\phi(\partial\Omega)),$$

then ϕ satisfies the Δ_2-condition.

Theorem 6 *Let $f_1, \cdots, f_m \in H^\infty(\Omega)$ be the corona data satisfying that $\sum_{i=1}^m \|f_i\|_\infty < 1$. We suppose that $T_i : H^\infty(\Omega) \to H^1(\Omega)$, $(i = 1, \cdots, m)$ is a linear operator such that $h(z) = \sum_{i=1}^m f_i(z)T_ih(z)$, $(z \in \Omega)$. If every operator T_i satisfies that*

$$\phi(\lambda)\sigma(\{|T_ih| > \lambda\}) \leq C \int_{\partial\Omega} \phi(|h|)d\sigma, \quad (\lambda > 0,\, h \in H_\phi(\Omega)),$$

then ϕ satisfies the Δ_2-condition.

3 Preliminaries

Most main theorems are obtained as applications of an Orlicz space version of the interpolation theorem of Marcinkiewicz type. At first, we give a definition of weak type inequality in $L_\phi(X)$ to improve the interpolation theorem in Gallardo [7], where X is a space of homogeneous type. We denote the quasi-distance over X by d and the Borel regular measure on X with doubling condition by μ. Let us recall that an operator T is said to be quasi-additive if $|T(f+g)| \le C(|Tf| + |Tg|)$ for a constant $C > 0$. If $C = 1$ here, then T is called sublinear.

Definition 1 *A sublinear operator T defined on an Orlicz space $L_\phi(X)$ is of weak type (ϕ, ϕ) if there exists positive constants C_1 and C_2 such that*

$$\phi(\lambda)\mu(\{x \in X : |Tf| > \lambda\}) \le C_1 \int_X \phi(C_2|f|)d\mu, \quad (f \in L_\phi(X),\ \lambda > 0).$$

Lemma 1 *Let ϕ, ϕ_1 and ϕ_2 be three N-functions satisfying the following growth conditions:*

$$\begin{aligned} \sup_{\lambda>0} \frac{\varphi(\lambda)\phi_1(\lambda)}{\phi(\lambda)\varphi_1(\lambda)} &< 1, \\ \inf_{\lambda>0} \frac{\varphi(\lambda)\phi_2(\lambda)}{\phi(\lambda)\varphi_2(\lambda)} &> 1, \end{aligned}$$

where φ, φ_1 and φ_2 are the left derivatives of ϕ, ϕ_1 and ϕ_2 respectively. Then, there exist positive constants C_1 and C_2 such that

$$\begin{aligned} \int_u^\infty \frac{\varphi(t)}{\phi_1(t)}dt &\le C_1\frac{\phi(u)}{\phi_1(u)}, \quad (u > 0), \\ \int_0^u \frac{\varphi(t)}{\phi_2(t)}dt &\le C_2\frac{\phi(u)}{\phi_2(u)}, \quad (u > 0). \end{aligned}$$

Proof: We may take a positive number r such that

$$\sup_{\lambda>0} \frac{\varphi(\lambda)\phi_1(\lambda)}{\phi(\lambda)\varphi_1(\lambda)} < r < 1.$$

Then it follows that

$$\frac{\varphi(\lambda)}{\phi_1(\lambda)} < r\phi(\lambda)\frac{\varphi_1(\lambda)}{\phi_1(\lambda)^2} = -r\phi(\lambda)\frac{d}{d\lambda}\left(\frac{1}{\phi_1(\lambda)}\right), \quad (\lambda > 0).$$

On the other hand, for any $\lambda_0 > 0$, the following holds:

$$\log\frac{\phi(\lambda)}{\phi(\lambda_0)} = \int_{\lambda_0}^{\lambda} \frac{\varphi(t)}{\phi(t)}dt \le r\int_{\lambda_0}^{\lambda} \frac{\varphi_1(t)}{\phi_1(t)}dt = \log\left(\frac{\phi_1(\lambda)}{\phi_1(\lambda_0)}\right)^r, \quad (\lambda \ge \lambda_0).$$

Hence we obtain that

$$\int_u^\infty \frac{\varphi(\lambda)}{\phi_1(\lambda)}d\lambda \le -r\left[\frac{\phi(\lambda)}{\phi_1(\lambda)}\right]_u^\infty + r\int_u^\infty \frac{\varphi(\lambda)}{\phi_1(\lambda)}d\lambda = r\frac{\phi(u)}{\phi_1(u)} + r\int_u^\infty \frac{\varphi(\lambda)}{\phi_1(\lambda)}d\lambda, \quad (u>0),$$

since $\frac{\phi(\lambda)}{\phi_1(\lambda)} \le \frac{\phi(\lambda_0)}{\phi_1(\lambda_0)^r}\phi_1(\lambda)^{r-1} = C\phi_1(\lambda)^{r-1} \to 0$, $(\lambda \to \infty)$. Thus we conclude that

$$\int_u^\infty \frac{\varphi(\lambda)}{\phi(\lambda)}d\lambda \le \frac{r}{1-r}\frac{\phi(u)}{\phi_1(u)}, \quad (u>0).$$

We can show the another inequality in the same way as above. □

Using Lemma 1, we can improve the interpolation theorem in Gallardo [7] to prove the next theorem.

Theorem 7 *Let ϕ, ϕ_1 and ϕ_2 be as in the lemma above and $\phi_1, \phi_2 \in \Delta_2$. We suppose that a sublinear operator T is of weak type (ϕ_1, ϕ_1) and of weak type (ϕ_2, ϕ_2). Then T is bounded on the Orlicz space $L_\phi(X)$:*

$$\int_X \phi(|Tf|)d\mu \le C_1 \int_X \phi(C_2|f|)d\mu, \quad (f \in L_\phi(X)).$$

Moreover we can obtain the same conclusion if T is of type (∞, ∞) and of weak type (ϕ_2, ϕ_2).

Proof. From the weak type inequality and the sublinearity in the hypothesis, we can assume that

$$\begin{aligned} |T(f+g)| &\le |Tf| + |Tg|, \\ \phi_i(\lambda)\nu(|Tf|>\lambda) &\le C_i \int \phi_i(|f|)d\mu, \quad (i=1,2). \end{aligned}$$

For any $f \in L_\phi(X)$ and any $\lambda > 0$, we take f_λ and f^λ as follows:

$$\begin{aligned} f_\lambda &= f\chi_{\{|f|>\frac{\lambda}{2}\}}. \\ f^\lambda &= f - f_\lambda. \end{aligned}$$

Then, since $\nu(|Tf|>\lambda) \le \nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right) + \nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right)$, the following holds.

$$\begin{aligned} \int \phi(|Tf|)d\nu &= \int_0^\infty \varphi(\lambda)\nu(|Tf|>\lambda)d\lambda \\ &\le \int_0^\infty \varphi(\lambda)\nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right)d\lambda + \int_0^\infty \varphi(\lambda)\nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right)d\lambda. \end{aligned}$$

It may be noted that $f_\lambda \in L_{\phi_2}$ and $f^\lambda \in L_{\phi_1}$. In fact, $\phi_2(x) \le C_R\phi(x)$, $(\frac{\lambda}{2} = R \le x)$ and $\phi_1(x) \le C'_R\phi(x)$, $(x \le R = \frac{\lambda}{2})$, it follows that $\phi_2(|f_\lambda|) \le C_R\phi(|f|)$ and $\phi_1(|f^\lambda|) \le C'_R\phi(|f|)$. From the weak type inequality, the first term in the right hand side above is less than

$$\int_0^\infty \varphi(\lambda)d\lambda \int C_2\frac{\phi_2(|f_\lambda|)}{\phi_2(\frac{\lambda}{2})}d\mu \le C_2\int \phi_2(|f|)d\mu \int_0^{2|f|} \frac{\varphi(\lambda)}{\phi_2(\frac{\lambda}{2})}d\lambda.$$

We note that there exists $K > 0$ such that $K\phi_2(\frac{\lambda}{2}) \geq \phi_2(\lambda)$ since $\phi_2 \in \Delta_2$. Then, by using Lem
ve obtain that

$$\begin{aligned}\int_0^{2|f|} \frac{\varphi(\lambda)}{\phi_2(\frac{\lambda}{2})} d\lambda &\leq K \int_0^{2|f|} \frac{\varphi(\lambda)}{\phi_2(\lambda)} d\lambda \\ &\leq K' \frac{\phi(2|f|)}{\phi_2(2|f|)} \\ &\leq K' \frac{\phi(2|f|)}{\phi_2(|f|)}.\end{aligned}$$

Hence the following holds.

$$\begin{aligned}\int_0^\infty \varphi(\lambda)\nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right) d\lambda &\leq C_2 K' \int \phi_2(|f|) \frac{\phi(2|f|)}{\phi_2(|f|)} d\mu \\ &\leq C_2 K' \int \phi(2|f|) d\mu.\end{aligned}$$

In a similar way as above, we can obtain that

$$\int_0^\infty \varphi(\lambda)\nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right) d\lambda \leq C_1 K' \int \phi(2|f|) d\mu.$$

In the case that T is of type (∞, ∞), we may assume that

$$\begin{aligned}\|Tf\|_\infty &\leq C_1 \|f\|_\infty. \\ \phi_2(\lambda)\nu(|Tf| > \lambda) &\leq C_2 \int \phi_2(|f|) d\mu.\end{aligned}$$

For any $f \in L_\phi(X)$ and any $\lambda > 0$, we take f_λ and f^λ as follows:

$$\begin{aligned}f_\lambda &= f\chi_{\{|f| > \frac{\lambda}{2C_1}\}}. \\ f^\lambda &= f - f_\lambda.\end{aligned}$$

We note that $\nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right) = 0$ since $\|Tf^\lambda\|_\infty \leq C_1\|f^\lambda\|_\infty \leq C_1 \frac{\lambda}{2C_1} = \frac{\lambda}{2}$. Thus we obtain that

$$\nu(|Tf| > \lambda) \leq \nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right) + \nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right) = \nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right).$$

Therefore it follows that

$$\begin{aligned}\int \phi(|f|) d\nu &= \int_0^\infty \varphi(\lambda)\nu(|Tf| > \lambda) d\lambda \\ &\leq \int_0^\infty \varphi(\lambda)\nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right) d\lambda \\ &\leq C_2 \int_0^\infty \varphi(\lambda) d\lambda \frac{\int \phi_2(|f_\lambda|) d\mu}{\phi_2(\frac{\lambda}{2})} \\ &\leq C_2 \int \phi_2(|f|) d\mu \int_0^{2C_1|f|} \frac{\varphi(\lambda)}{\phi_2(\frac{\lambda}{2})} d\lambda.\end{aligned}$$

Since $\phi_2 \in \Delta_2$, there exists $K > 0$ such that $K\phi_2(\frac{\lambda}{2}) \geq \phi_2(\lambda)$. Then, using Lemma 3, the following holds.

$$\begin{aligned} C_2 \int \phi_2(|f|)d\mu \int_0^{2C_1|f|} \frac{\varphi(\lambda)}{\phi_2(\frac{\lambda}{2})}d\lambda &\leq C_2K \int \phi_2(|f|)d\mu \int_0^{2C_1|f|} \frac{\varphi(\lambda)}{\phi_2(\lambda)}d\lambda \\ &\leq C_2K \int \phi_2(|f|)\frac{\phi(2C_1|f|)}{\phi_2(2C_1|f|)}d\mu. \end{aligned}$$

Now we should note that $\phi_2(|f|) \leq \phi_2(2C_1|f|)$ if $2C_1 \geq 1$ and that $\phi_2(|f|) \leq L\phi_2(2C_1|f|)$ for an $L > 0$ if $2C_1 < 1$ since $\phi_2 \in \Delta_2$. Hence we obtain that

$$C_2K \int \phi_2(|f|)\frac{\phi(2C_1|f|)}{\phi_2(2C_1|f|)}d\mu \leq C_2KL \int \phi(2C_1|f|)d\mu.$$

This completes the proof. □

Furthermore, a small modification of the proof in Coifman-Weiss [6] leads us to the following.

Theorem 8 *Let $\phi \in \Delta_2 \cap \nabla_2$ and ϕ_2 be an N-function . We suppose that* $\sup_{\lambda>0} \frac{\varphi(\lambda)\phi_2(\lambda)}{\phi(\lambda)\varphi_2(\lambda)} < 1$ *and that a sublinear operator $B : H^1_{Re}(X) + L_{\phi_2}(X) \to M(X)$ is of weak type $(H^1_{Re}, 1)$ and of weak type (ϕ_2, ϕ_2), where $M(X)$ is the set of all measurable functions on X. If X is bounded, then the following holds:*

$$\int_X \phi(|Bf|)d\mu \leq C \int_X \phi(|f|)d\mu, \quad (f \in L_\phi(X)).$$

If X is unbounded , then the following holds:

$$\|Bf\|_{(\phi)} \leq C\|f\|_{(\phi)}, \quad (f \in L_\phi(X)).$$

4 Proofs

Proof of Theorem 1. We give a sketch of the proof here. Details are left to Imai [8]. Firstly we let $f \in [A(\partial\Omega)]_{L_\phi(\partial\Omega)}$. Then we can take a sequence $f_n \in A(\partial\Omega)$ such that $\|f - f_n\|_{(\phi)} \to 0$, $(n \to \infty)$. Using the Poisson kernel $P(z,\zeta)$, we define a function F by

$$F(z) = \int_{\partial\Omega} P(z,\zeta)f(\zeta)d\sigma(\zeta), \quad (z \in \Omega).$$

In the same way as is shown in Imai [8], we know that F is holomorphic in Ω. Moreover it follows that

$$|F_\varepsilon(\zeta)| \leq CM_{HL}f(\zeta), \quad (a.e. \zeta \in \partial\Omega)$$

in Stein [15]. Since the Hardy-Littlewood maximal operator M_{HL} is of weak type $(1,1)$ and of type (∞,∞), it follows that $\phi(M_{HL}f)$ is integrable from Theorem 7. And, since $F_\varepsilon(\zeta)$ converges to $f(\zeta)$ pointwisely at almost every $\zeta \in \partial\Omega$ by means of the well-known property of the Poisson integral, the

Lebesgue dominated convergence theorem shows that $\int_{\partial\Omega}\phi(|F_\varepsilon|)d\sigma \to \int_{\partial\Omega}\phi(|f|)d\sigma$, $(\varepsilon \to 0)$. Therefore we have that $\|F_\varepsilon - f\|_{(\phi)} \to 0$, $(\varepsilon \to 0)$. (For details, see Rao-Ren [14].) This shows that $[A(\partial\Omega)]_{L^*_\phi(\partial\Omega)} \subset H_\phi(\Omega)$.

Conversely, we let $f \in H_\phi(\Omega)$. And we choose a finite open covering $\mathcal{U} = \{U_1, \cdots, U_q\}$ of $\partial\Omega$ and a point $p_j \in U_j$ for every $j = 1, \cdots, q$. If $1 = \gamma_1 + \cdots + \gamma_q$ is a partition of unity subordinate to the open covering $\mathcal{U} = \{U_1, \cdots, U_q\}$, we define f_j by

$$f_j(z) = \int_{\partial\Omega} \frac{K(\zeta, z)}{\Phi(\zeta, z)^n} f(\zeta)\gamma_j(\zeta)d\sigma(\zeta), \quad (z \in \Omega),$$

where $\frac{K(\zeta,z)}{\Phi(\zeta,z)^n}$ is the Henkin-Ramirez reproducing kernel. Then it is trivial that f_j is holomorphic in a neighborhood of $\Omega \cup (\partial\Omega \setminus U_j)$. Moreover we may write that

$$\begin{aligned} f_j(z) &= \int_{\partial\Omega} f(\zeta)\{\gamma_j(\zeta) - \gamma_j(z)\}\frac{K(\zeta, z)}{\Phi(\zeta, z)^n}d\sigma(\zeta) + f(z)\gamma_j(z) \\ &= T_j f(z) + f(z)\gamma_j(z). \end{aligned}$$

Since it is proved that the operator T_j is of type $(1,1)$ and of type (∞,∞) when $T_j f$ is resticted to $\partial\Omega_\varepsilon$ for sufficiently small $\varepsilon > 0$ by Stout [18], Theorem 7 shows that

$$\limsup_{\varepsilon\to 0} \int_{\partial\Omega} \phi(|(T_j f)_\varepsilon|)d\sigma \le C \int_{\partial\Omega} \phi(|f|)d\sigma.$$

Hence it follows that $f_j \in H_\phi(\Omega)$.

Now, for any sufficient small $\varepsilon > 0$, we suppose that

$$f_j^{(\varepsilon)}(\zeta) = f_j(\zeta - \varepsilon\nu_j),$$

where ν_j is the outer unit vector transversal to $\partial\Omega$ at the point p_j. Then $f_j^{(\varepsilon)} \in \mathcal{O}(\overline{\Omega})$ and we know that

$$|f_j^{(\varepsilon)}(\zeta)| \le C + CM_{HL}f_j(\zeta)$$

in the same way as is shown in Imai [8]. Since $f_j \in L_\phi(\partial\Omega)$, Theorem 7 shows that $C + CM_{HL}f_j \in L_\phi(\partial\Omega)$. Hence it follows that $\int_{\partial\Omega}\phi(|f_j^{(\varepsilon)}|)d\sigma \to \int_{\partial\Omega}\phi(|f_j|)d\sigma$, $(\varepsilon \to 0)$ from the Lebesgue dominated convergence theorem. From this convergence we have $\|f_j^{(\varepsilon)} - f_j\| \to 0$, $(\varepsilon \to 0)$. (For details, see Rao-Ren [14].) This shows that $f \in [A(\partial\Omega)]_{L^*_\phi(\partial\Omega)}$ since $f = f_1 + \cdots + f_q$. □

Proof of Theorem 2. Since $\phi \in \Delta_2 \cap \nabla_2$, there exist ϕ_1 and $\phi_2 \in \Delta_2 \cap \nabla_2$ such that $\sup_{\lambda>0} \frac{\varphi(\lambda)\phi_1(\lambda)}{\phi(\lambda)\varphi_1(\lambda)} < 1$ and $\inf_{\lambda>0} \frac{\varphi(\lambda)\phi_2(\lambda)}{\phi(\lambda)\varphi_2(\lambda)} > 1$. (For details, see Gallardo [7] and Rao-Ren [14].) Hence we can apply Theorem 7 to the Szegö projection S in order to complete the proof. □

Proof of Theorem 3. We consider the composition operators $A = B \circ S$ of a sublinear operators B and the Szegö projection S. Then, since A is bounded on real Hardy space $H^1_{Re}(\partial\Omega)$ and on an Orlicz space $L_{\phi_2}(\partial\Omega)$, we can apply Theorem 8 to the operator A in order to show that

$$\int_{\partial\Omega} \phi(|Ag|)d\sigma \le C \int_{\partial\Omega} \phi(|g|)d\sigma, \quad (g \in L_\phi(\partial\Omega)).$$

Since $H_\phi(\Omega) = SL_\phi(\partial\Omega)$ as shown in Theorem 2, we can take any $g \in L_\phi(\partial\Omega)$ such that $f = Sg$ for $f \in H_\phi(\Omega)$ to obtain that

$$\int_{\partial\Omega} \phi(|Bf|)d\sigma = \int_{\partial\Omega} \phi(|Ag|)d\sigma \le C \int_{\partial\Omega} \phi(|g|)d\sigma.$$

Since g is arbitrary function in $L_\phi(\partial\Omega)$ such that $f = Sg$, we can conclude that

$$\int_{\partial\Omega} \phi(|Bf|)d\sigma \le C \inf\left\{\int_{\partial\Omega} \phi(|g|)d\sigma \,:\, g \in L_\phi(\partial\Omega)\, s.t.\, f = Sg\right\}.$$

□

Proof of Corollary 1. Since $\phi \in \Delta_2 \cap \nabla_2$, there exist ϕ_1 and $\phi_2 \in \Delta_2 \cap \nabla_2$ such that $\sup_{\lambda>0} \frac{\varphi(\lambda)\phi_1(\lambda)}{\phi(\lambda)\varphi_1(\lambda)} < 1$ and $\inf_{\lambda>0} \frac{\varphi(\lambda)\phi_2(\lambda)}{\phi(\lambda)\varphi_2(\lambda)} > 1$. (For details, see Gallardo [7] and Rao-Ren [14].) Hence we can apply Theorem 7 to operators T_i in Theorem 4 in order to complete the proof. □

Before giving the proofs of Theorem 5 and 6, we show a lemma as follows.

Lemma 2 *Let ϕ be an N-function. We suppose that a sublinear operator T on $L_\phi(\partial\Omega)$ is of weak type (ϕ, ϕ), that is,*

$$\phi(\lambda)\sigma(|Tf| > \lambda) \le C_1 \int_{\partial\Omega} \phi(C_2|f|)d\sigma, \quad (f \in L_\phi(\partial\Omega),\ \lambda > 0).$$

If $\sup_{\|f\|_\infty \le 1} \|Tf\|_\infty > C_2$, then ϕ satisfies the Δ_2-condition.

Proof. From the hypothesis, there exist $r > 1$ and $\|f\|_\infty \le 1$ such that

$$K = \sigma(\{|Tf| > rC_2\}) > 0.$$

Then, for any $\lambda > 0$, we define a function $g \in L_\phi(\partial\Omega)$ by

$$g(\zeta) = \frac{\lambda}{rC_2} f(\zeta).$$

By applying the inequality of weak type to g, we obtain that

$$\phi(\lambda)\sigma\{|Tg| > \lambda\} \le C_1 \int_{\partial\Omega} \phi(C_2|g|)d\sigma.$$

Since $\{|Tg| > \lambda\} = \{|Tf| > rC_2\}$, we have that $\sigma(\{|Tg| > \lambda\}) = \sigma(\{|Tf| > rC_2\}) = K > 0$. Therefore, we have that

$$\begin{aligned} \phi(\lambda) &\le \sigma(\{|Tf| > rC_2\})^{-1} C_1 \int_{\partial\Omega} \phi\left(C_2 \frac{\lambda}{rC_2}\|f\|_\infty\right) d\sigma \\ &\le C_1 K^{-1} \|\sigma\| \cdot \phi\left(\frac{\lambda}{r}\right). \end{aligned}$$

This inequality shows that ϕ satisfies the Δ_2 -condition. □

Now we are ready to prove Theorem 5 and 6.
Proof of Theorem 5. Since $SL^\infty(\partial\Omega) = BMOA \supset H^\infty$, it follows that

$$\sup\{\|Sf\|_\infty : f \in L^\infty \text{ such that } \|f\|_\infty \leq 1\} = \infty.$$

Therefore we can apply Lemma 2 to the Szegö projection S. □

Proof of Theorem 6. We suppose that $\sup\{\|T_i f\|_\infty : f \in H^\infty \text{ such that } \|f\|_\infty \leq 1\} \leq 1$ for every $i = 1, \cdots, m$. Now we choose a bounded holomorphic function $h \in H^\infty(\Omega)$ such that $\sum_{i=1}^m \|f_i\|_\infty < \|h\|_\infty \leq 1$. Then we have that

$$\begin{aligned} \|h\|_\infty &\leq \sum_{i=1}^m \|f_i\|_\infty \|T_i h\|_\infty \\ &\leq \sum_{i=1}^m \|f_i\|_\infty \\ &< \|h\|_\infty. \end{aligned}$$

This is a contradiction. Therefore there exist a certain $k \in \{1, \cdots, m\}$ such that

$$\sup\{\|T_k f\|_\infty : f \in H^\infty \text{ such that } \|f\|_\infty \leq 1\} > 1.$$

Then we can apply Lemma 2 to the operator T_k. □

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