Unitary Representations for Twisted Product of Matrix Quantum Groups*

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1 Introduction

This paper is a continuation of [JW], where we constructed a family of compact matrix quantum groups in the sense of Woronowicz [SLW2]. The construction followed the scheme provided by Woronowicz in [SLW3], in which the basic role is played by a properly chosen function on permutations. In our case the function is related to counting the number of cycles in permutations. In [JW] we described the C*-algebraic structure of the constructed objects. Here we shall concentrate on the "quantum group" structure (Hopf algebra structure) and unitary representations of the quantum groups.

As defined by Woronowicz in [SLW2], a compact matrix quantum group $(A, u)$ consists of a C*-algebra $A$ and an $N$ by $N$ matrix $u = (u_{jk})_{j,k=1}^{N}$, with the elements $u_{jk} \in A$ generating a dense $*$-subalgebra $A$ of $A$, and with the following additional structure:

1. a C*-homomorphism $\Phi : A \rightarrow A \otimes A$, called the co-multiplication, such that

   $$\Phi(u_{jk}) = \sum_{r=0}^{N} u_{jr} \otimes u_{rk}$$

2. a linear anti-multiplicative mapping $\kappa : A \rightarrow A$, called the co-inverse, such that $\kappa(\kappa(a^*)^*) = a$ for all elements $a \in A$, and

   $$\sum_{r=1}^{N} \kappa(u_{jr})u_{rk} = \delta_{jk}I$$
   $$\sum_{r=1}^{N} u_{jr}\kappa(u_{rk}) = \delta_{jk}I$$

The notion of unitary representation of a quantum group was introduced by Woronowicz in [SLW2]. The definition says that a unitary n-dimensional (co-)representation of a quantum group $(A, u)$ is a unitary element $v = (v_{jk}) \in M_n(A) \simeq M_n(\mathbb{C}) \otimes A$, with $v_{jk} \in A$, which satisfies

$$\Phi(v_{jk}) = \sum_{r=1}^{n} v_{jr} \otimes v_{rk}.$$

Another crucial notion for compact quantum groups is that of a Haar measure. A Haar measure on a compact quantum group $(A, u)$ is a state $h \in A'$ (a linear positive functional normalized by $h(1) = 1$) such that for every element $a \in A$ one has $(id \otimes h)\Phi(a) = (h \otimes id)\Phi(a) = h(a) \cdot 1$. Woronowicz proved in [SLW2] that on every compact quantum group there is the unique Haar measure.

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2 Compact quantum groups associated with cycles in permutations for N=3 and its structure

In this section we describe the structure of our quantum groups as a twisted product of its subgroups.

Let us recall that the quantum group \((A, u)\) we consider is generated by three elements \(a, c, v\), which satisfy the following relations:

\[
\begin{align*}
(1) \quad av &= va \\
(2) \quad cv &= vc \\
(3) \quad ac + tca &= 0 \\
(4) \quad ac^* + tc^*a &= 0 \\
(5) \quad cc^* &= c^*c \\
(6) \quad vv^* &= v^*v = I \\
(7) \quad aa^* + t^2cc^* &= I \\
(8) \quad a^*a + c^*c &= I
\end{align*}
\]

The co-multiplication \(\Phi\) in the quantum group \((A, u)\) is given on generators by

\[
\Phi(a) = a \otimes a + tc^*v^* \otimes c, \quad \Phi(c) = c \otimes a + a^*v^* \otimes c, \quad \Phi(v) = v \otimes v. \quad (2.1)
\]

The co-inverse \(\kappa\) is defined by:

\[
\kappa(a) = a^*v^*, \quad \kappa(c^*v^*) = a, \quad \kappa(c) = tc, \quad \kappa(c^*v^*) = \frac{1}{t}c^*v^*, \quad \kappa(v) = v \quad (2.2)
\]

We are going to show that this group is a twisted product of its two subgroups. Clearly, first we have to explain these notions.

The definition of a quantum subgroup of a quantum group is the following (see [P-W]).

**Definition 2.1** Let \((A, u, \Phi, e, \kappa)\) and \((A_1, u_1, \Phi_1, e_1, \kappa_1)\) be given quantum groups, with the explicite notation of their underlying \(C^*\)-algebras, fundamental representations, co-multiplications, co-units and co-inverses. If there exists a an embedding \(p_j : A_1 \to A\) such that:

\[
\Phi_1 p_1 = p_1 \Phi, \quad e_1 p_1 = p_1 e, \quad \kappa_1 p_1 = p_1 \kappa \quad (2.3)
\]

then we call \((A_1, u_1, \Phi_1, e_1, \kappa_1)\) a quantum subgroup of the quantum group \((A, u, \Phi, e, \kappa)\). The above equalities mean that the restrictions of co-multiplication, co-inverse and co-unit from \((A, u, \Phi, e, \kappa)\) agree with those of \((A_1, u_1, \Phi_1, e_1, \kappa_1)\).

Now, following the work of Podleś and Woronowicz on the quantum Lorentz group [P-W] we shall describe the meaning of twisted product of two quantum groups.

**Definition 2.2** Let \((A, u, \Phi, e, \kappa)\) be a given quantum group and let \((A_1, u_1, \Phi_1, e_1, \kappa_1)\) and \((A_2, u_2, \Phi_2, e_2, \kappa_2)\) be its quantum subgroups with the natural embeddings \(p_j : A_j \to A_1 \otimes A_2, \ j = 1, 2\), given by \(p_1 : A_1 \ni a_1 \mapsto a_1 \otimes 1_{A_2} \in A_1 \otimes A_2\), \(p_2 : A_2 \ni a_2 \mapsto 1_{A_1} \otimes a_2 \in A_1 \otimes A_2\); we assume that \(A = A_1 \otimes A_2\) is the spatial tensor product of the two \(C^*\)-algebras. If there exists a \(*\)-algebra isomorphism \(\sigma : A_1 \otimes A_2 \to A_2 \otimes A_1\), such that:

\[
\Phi = (id_{A_1} \otimes \sigma \otimes id_{A_2})(\Phi_1 \otimes \Phi_2), \quad \kappa = s(\kappa_1 \otimes \kappa_2)\sigma \quad (2.4)
\]

where \(s : A_2 \otimes A_1 \to A_1 \otimes A_2\) is the flip automorphism \(s(a_2 \otimes a_1) = a_1 \otimes a_2\) and \(id_{A_j}\) is the identity map on \(A_j, \ j = 1, 2\), then we say that \((A, u)\) is the twisted product of its subgroups \((A_1, u_1)\) and \((A_2, u_2)\) with the twist \(\sigma\); this will be denoted by
\[ A = A_1 \otimes_\sigma A_2 \] (2.5)

The relations which defined our quantum group can be split in such a way that one can recover two special quantum subgroups inside it.

**Example:** Let \((A_1, u_1, \Phi_1, e_1, \kappa_1)\) be the quantum group defined in the following way:

\[ A_1 = C^*(a, c) \] is the C*-algebra generated by the two elements \(a, c\), which satisfy the relations: \(ac + tca = 0 = ac^* + tc^*a, \quad cc^* = c^*c, \quad aa^* + t^2cc^* = I = a^*a + c^*c = I, \quad u_1 = \begin{pmatrix} a & tc^* \\ c & a^* \end{pmatrix} \)

is the fundamental representation, \(\Phi_1(a) = a \otimes a + tc^* \otimes c, \quad \Phi_1(c) = c \otimes a + a^* \otimes c\)

is the co-multiplication, \(\kappa_1(a) = a^*, \quad \kappa_1(c) = tc\) is the co-inverse and \(e_1(a) = e_1(a^*) = 1, \quad e_1(c) = e_1(c^*) = 0\) is the co-unit.

Then one can easily recognize that \((A_1, u_1)\) is the famous quantum \(SU_q(2)\) group defined by Woronowicz in [SLW1] for \(q = -t\).

**Example:** Let \((A_2, u_2, \Phi_2, \kappa_2, e_2)\) be defined in the following way:

\[ A_2 = C^*(v) \] is the commutative C*-algebra generated by a unitary \(v\), \(u_2 = \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix} \)

is the fundamental representation, \(\Phi_2(v) = v \otimes v\), is the co-multiplication, \(\kappa_2(v) = v^*\) is the co-inverse and \(e_2(v) = 1\) is the co-unit.

Then this definition provides the quantum group \(U(1)\).

A simple computation shows that this two quantum groups are quantum subgroups of our quantum group \((A, u)\) with the natural embeddings. We are going to show that in fact \((A, u)\) is the twisted product of these two subgroups, for a proper choice of the twist \(\sigma\). For this purpose we need the following:

**Definition 2.3** Let \(\sigma : A_1 \otimes A_2 \rightarrow A_2 \times A_1\) be a *-algebra homomorphism defined by putting:

\[
\sigma(a \otimes v^k) = v^k \otimes a, \quad \sigma(c \otimes v^k) = v^{k-1} \otimes c
\]

with \(v^{-1} = v^*\).

Then we have

**Theorem 2.4** The quantum group \(A = A_1 \otimes_\sigma A_2\) is the twisted product of the two quantum subgroups with the twist \(\sigma\).

**Proof:** We should check that the co-multiplications and co-inverses satisfy the definition 2.2. Keeping in mind the identification \(av^k \leftrightarrow a \otimes v^k\) and \(cv^k \leftrightarrow c \otimes v^k\), given by the natural embeddings, we obtain for the co-multiplications:

\[
s(\kappa_2 \otimes \kappa_1)(a \otimes v^k) = s(\kappa_2(v^k) \otimes \kappa_1(a)) = a^* \otimes v^k \quad \text{and} \quad s(\kappa_2 \otimes \kappa_1)(c \otimes v^k) = s(\kappa_2(v^{k-1}) \otimes \kappa_1(c)) = tc \otimes v^{k-1}
\]

which agrees with the corresponding action of \(\kappa\). Since a co-inverse is linear and anti-multiplicative, the above formulas can be extended to the *-subalgebra of \(A\) generated by \(a, c, v\).

For the co-multiplications we have:
\[(id_{A_{1}} \otimes \sigma \otimes id_{A_{2}})(\Phi_{1} \otimes \Phi_{2})(a \otimes v^{k}) = (id_{A_{1}} \otimes \sigma \otimes id_{A_{2}})(a \otimes a \otimes v^{k} \otimes v^{k} + tc^{*} \otimes c \otimes v^{k} \otimes v^{k}) = a \otimes v^{k} \otimes a \otimes v^{k} + tc^{*} \otimes v^{k-1} \otimes c \otimes v^{k}\]

which agrees with
\[\Phi(av^{k}) = (a \otimes a + tc^{*}v^{*} \otimes c)(v^{k} \otimes v^{k}) = av^{k} \otimes av^{k} + tc^{*}v^{k-1} \otimes cv^{k}\]

Since both \((id_{A_{1}} \otimes \sigma \otimes id_{A_{2}})(\Phi_{1} \otimes \Phi_{2})\) and \((I \otimes C')\) are \(C'\)-algebra homomorphisms, and agree on generators, they satisfy the equation 2.7.

\[\square\]

3 Unitary representations of the quantum group

Our description of the unitary representations of the quantum group \((A, u)\) we base on the work by Podleś and Woronowicz [P-W], where a general theorem shows how to construct representations of a quantum group which is twisted product of its quantum subgroups. First we recall this

**Theorem 3.1** Let the quantum group \(A = A_{1} \otimes_{\sigma} A_{2}\) be the twisted product of its quantum subgroups \(A_{1}\) and \(A_{2}\), with the natural embeddings denoted by \(p_{1}\) and \(p_{2}\). Let \(v \in B(K) \otimes A\) be matrix with entries from \(A\) for a finite dimensional complex vector space \(K\). Then the following holds:

1. If \(w\) is a (unitary) representation of \(A\) on \(K\), then \(w^{1} := (id \otimes p_{1})w\) is a (unitary) representation of \(A_{1}\) on \(K\) and \(w^{2} := (id \otimes p_{2})w\) is a (unitary) representation of \(A_{2}\) on \(K\), and the following conditions hold:

\[w = w^{1} \oplus w^{2},\]

\[w^{2} \oplus w^{1} = (id \otimes \sigma)(w^{1} \oplus w^{2})\] (3.1)

where

\[w^{1} \oplus w^{2} = \sum_{j,k} m_{j}^{1}m_{k}^{2} \otimes w_{j}^{1} \otimes w_{k}^{2}\]

for \(w^{i} = \sum_{j} m_{j}^{i} \otimes w_{j}^{i} \in B(K) \otimes A_{i}\)

2. If \(w^{1}\) and \(w^{2}\) are (unitary) representations of \(A_{1}\) and \(A_{2}\) respectively, which are of the same dimension and satisfy the compatibility condition 3.10, then \(w = w^{1} \oplus w^{2}\) is a (unitary) representation of \(A\) on \(K\).

We shall apply this result to our situation to describe the unitary representations of the quantum group \((A, u)\). We will use the theory of irreducible unitary representations of the quantum group \(SU_{q}(2)\), which was described by Woronowicz. The representations \(\{u^{s}\}_{s \in \frac{1}{2}N}\) are indexed by the set \(\frac{1}{2}N = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots\}\), and each \(u^{s}\) act on a \((2s+1)\)
-dimensional Hilbert space. Explicit formulas for matrix elements of these representations are given in [Pu-W], B.19, p. 1616.

Now let us assume that \( w^1 = u^s \), for some \( s \in \frac{1}{2} \mathbb{N} \), and \( w^2 \) is a unitary representation of \( A_2 \) of the dimension \( 2s + 1 \), and that they satisfy the compatibility condition 3.10. This condition implies that, for some positive integer \( r \), \( w^2 = \text{diag}\{v^r, v^{r-1}, \ldots, v^{-2s}\} \) has a diagonal matrix with the decreasing (or, equivalently, increasing) integral powers of the unitary \( v \) on the main diagonal. It follows that then the representation \( w = w^1 \oplus w^2 \) is unitary and irreducible representation of \( A \). This can be seen by using the Haar measure \( h = h_1 \otimes h_2 \) on \( A \), which is the tensor product of the Haar measure on \( SU_q(2) \), \( q = -t \), and the Lebesgue measure on the unit circle, which is the Haar measure on \( A_2 \). Let us recall that the non-trivial action of \( h_1 \) is given by \( h_1((cc^*)^m) = \frac{1-t^2}{1-2t^m+t^{2m+1}} \). Then irreducibility of \( w \) is equivalent to \( h(\chi_w^* \chi_w) = 1 \), where \( \chi_w = \sum_j w_{jj} \) is the character of the representation \( w \). It follows from the form of \( w^2 \) and from the formulas (B.19) of [Pu-W] that the value \( h(\chi_w^* \chi_w) \) is the same as \( h_1(\chi_w^* \chi_w) \), which is 1, by the irreducibility of \( w^1 \).

We shall finish our considerations with the following observation regarding the structure of the irreducible representations of \( (A, u) \). There is a sequence \( \{v^r\}_{r \in \mathbb{Z}} \) - integral powers of \( v \) - of irreducible one-dimensional representations of \( (A, u) \). There representation \( w = \begin{pmatrix} a & tc^*v^* \\ c & a^*v^* \end{pmatrix} = u^\frac{1}{2} \oplus \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix} \) is the fundamental representation of \( (A, u) \), so we can write \( (A, u) = (A, w) \). Its conjugate is the representation \( \overline{w} = \begin{pmatrix} a^* & tc^*v \\ c^* & av \end{pmatrix} = u^\frac{1}{2} \oplus \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \). These two-dimensional representations are not equivalent since they have different characters: \( \chi_w = a + a^*v^* \neq a^* + av = \chi_{\overline{w}} \). The following is the decomposition of their tensor products into irreducible sub-representations:

\[
\begin{align*}
    w \oplus w &= v^* \oplus \left( u^1 \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & v^* & 0 \\ 0 & 0 & v^2 \end{pmatrix} \right) \\
    w \oplus \overline{w} &= 1 \oplus \left( u^1 \oplus \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^* \end{pmatrix} \right) \\
    \overline{w} \oplus \overline{w} &= v \oplus \left( u^1 \oplus \begin{pmatrix} v^2 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)
\end{align*}
\]

References


