数理解析研究所講究録 1278 巻 2002 年 188-193

Unitary Representations for Twisted Product of Matrix Quantum Groups*

JANUSZ WYSOCZAŃSKI
INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY
PL. GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND
E-MAIL: jwys@math.uni.wroc.pl

1 Introduction

This paper is a continuation of [JW], where we constructed a family of compact matrix quantum groups in the sense of Woronowicz [SLW2]. The construction followed the scheme provided by Woronowicz in [SLW3], in which the basic role is played by a properly chosen function on permutations. In our case the function is related to counting the number of cycles in permutations. In [JW] we described the C*-algebraic structure of the constructed objects. Here we shall concentrate on the "quantum group" structure (Hopf algebra structure) and unitary representations of the quantum groups.

As defined by Woronowicz in [SLW2], a compact matrix quantum group (A, u) consists of a C^*-algebra A and an N by N matrix $u = (u_{jk})_{j,k=1}^N$, with the elements $u_{jk} \in A$ generating a dense $*$-subalgebra $\mathcal{A}$ of A, and with the following additional structure:

1. a C^*-homomorphism $\Phi : A \to A \otimes A$, called the co-multiplication, such that

$$\Phi(u_{jk}) = \sum_{r=0}^{N} u_{jr} \otimes u_{rk} \tag{1.1}$$

2. a linear anti-multiplicative mapping $\kappa : \mathcal{A} \to \mathcal{A}$, called the co-inverse, such that $\kappa(\kappa(a^*)^*) = a$ for all elements $a \in \mathcal{A}$, and

$$\sum_{r=1}^{N} \kappa(u_{jr})u_{rk} = \delta_{jk}I \tag{1.2}$$

$$\sum_{r=1}^{N} u_{jr}\kappa(u_{rk}) = \delta_{jk}I \tag{1.3}$$

The notion of unitary representation of a quantum group was introduced by Woronowicz in [SLW2]. The definition says that a unitary n-dimensional (co-)representation of a quantum group (A, u) is a unitary element $v = (v_{jk}) \in M_n(A) \simeq M_n(\mathbb{C}) \otimes A$, with $v_{jk} \in A$, which satisfies $\Phi(v_{jk}) = \sum_{r=1}^{n} v_{jr} \otimes v_{rk}$.

Another crucial notion for compact quantum groups is that of a *Haar measure.* A Haar measure on a compact quantum group (A, u) is a state $h \in A'$ (a linear positive functional normalized by $h(1) = 1$) such that for every element $a \in A$ one has $(id \otimes h)\Phi(a) = (h \otimes id)\Phi(a) = h(a) \cdot 1$. Wor4onowicz proved in [SLW2] that on every compact quantum group there is the unique Haar measure.

*Research partially supported by KBN grant 2P03A05415

2 Compact quantum groups associated with cycles in permutations for N=3 and its structure

In this section we describe the structure of our quantum groups as a twisted product of its subgroups.

Let us recall that the quantum group (A, u) we consider is generated by three elements $a, , c, v$, which satisfy the following relations:

$$\begin{array}{llllll} (1) & av = va & (2) & cv = vc & (3) & ac + tca = 0 \\ (4) & ac^* + tc^*a = 0 & (5) & cc^* = c^*c & (6) & vv^* = v^*v = I \\ (7) & aa^* + t^2cc^* = I & (8) & a^*a + c^*c = I & & \end{array}$$

The co-multiplication Φ in the quantum group (A, u) is given on generators by

$$\Phi(a) = a \otimes a + tc^*v^* \otimes c, \quad \Phi(c) = c \otimes a + a^*v^* \otimes c, \quad \Phi(v) = v \otimes v. \tag{2.1}$$

The co-inverse κ is defined by:

$$\kappa(a) = a^*v^*, \kappa(a^*v^*) = a, \kappa(c) = tc, \kappa(c^*v^*) = \frac{1}{t}c^*v^*, \kappa(v) = v \tag{2.2}$$

We are going to show that this group is a twisted product of its two subgroups. Clearly, first we have to explain these notions.

The definition of a *quantum subgroup* of a quantum group is the following (see [P-W]).

Definition 2. 1 *Let (A, u, Φ, e, κ) and $(A_1, u_1, \Phi_1, e_1, \kappa_1)$ be given quantum groups, with the explicite notation of their underlying C^*-algebras, fundamental representations, co-multiplications, co-units and co-inverses. If there exists a an embedding $p_1 : A_1 \to A$ such that:*

$$\Phi_1 p_1 = p_1 \Phi, \qquad e_1 p_1 = p_1 e, \qquad \kappa_1 p_1 = p_1 \kappa \tag{2.3}$$

then we call $(A_1, u_1, \Phi_1, e_1, \kappa_1)$ a quantum subgroup of the quantum group (A, u, Φ, e, κ). The above equalities mean that the restrictions of co-multiplication, co-inverse and co-unit from (A, u, Φ, e, κ) agree with those of $(A_1, u_1, \Phi_1, e_1, \kappa_1)$.

Now, following the work of Podleś and Woronowicz on the quantum Lorentz group [P-W] we shall describe the meaning of twisted product of two quantum groups.

Definition 2. 2 *Let (A, u, Φ, e, κ) be a given quantum group and let $(A_1, u_1, \Phi_1, e_1, \kappa_1)$ and $(A_2, u_2, \Phi_2, e_2, \kappa_2)$ be its quantum subgroups with the natural embeddings $p_j : A_j \to A_1 \otimes A_2$, $j = 1, 2$, given by $p_1 : A_1 \ni a_1 \mapsto a_1 \otimes 1_{A_2} \in A_1 \otimes A_2$, $p_2 : A_2 \ni a_1 \mapsto 1_{A_1} \otimes a_2 \in A_1 \otimes A_2$; we assume that $A = A_1 \otimes A_2$ is the spatial tensor product of the two C^*-algebras. If there exists a *-algebra isomorphism $\sigma : A_1 \otimes A_2 \to A_2 \otimes A_1$, such that:*

$$\Phi = (id_{A_1} \otimes \sigma \otimes id_{A_2})(\Phi_1 \otimes \Phi_2), \qquad \kappa = s(\kappa_1 \otimes \kappa_2)\sigma \tag{2.4}$$

where $s : A_2 \otimes A_1 \to A_1 \otimes A_2$ is the flip automorphism $s(a_2 \otimes a_1) = a_1 \otimes a_2$ and id_{A_j} is the identity map on A_j, $j = 1, 2$, then we say that (A, u) is the twisted product of its subgroups (A_1, u_1) and (A_2, u_2) with the twist σ ; this will be denoted by

$$A = A_1 \otimes_\sigma A_2 \tag{2.5}$$

The relations which defined our quantum group can be split in such a way that one can recover two special quantum subgroups inside it.

Example: Let $(A_1, u_1, \Phi_1, e_1, \kappa_1)$ be the quantum group defined in the following way:

$A_1 = C^*(a,c)$ is the C*-algebra generated by the two elements a, c, which satisfy the relations: $ac + tca = 0 = ac^* + tc^*a$, $cc^* = c^*c$, $aa^* + t^2cc^* = I = a^*a + c^*c = I$, $u_1 = \begin{pmatrix} a & tc^* \\ c & a^*, \end{pmatrix}$ is the fundamental representation, $\Phi_1(a) = a \otimes a + tc^* \otimes c$, $\Phi_1(c) = c \otimes a + a^* \otimes c$ is the co-multiplication, $\kappa_1(a) = a^*$, $\kappa_1(c) = tc$ is the co-inverse and $e_1(a) = e_1(a^*) = 1$, $e_1(c) = e_1(c^*) = 0$ is the co-unit.

Then one can easily recognize that (A_1, u_1) is the famous quantum $SU_q(2)$ group defined by Woronowicz in [SLW1] for $q = -t$. □

Example: Let $(A_2, u_2, \Phi_2, \kappa_2, e_2)$ be defined in the following way:

$A_2 = C^*(v)$ is the commutative C*-algebra generated by a unitary v, $u_2 = \begin{pmatrix} 1 & 0 \\ 0 & v^*, \end{pmatrix}$ is the fundamental representation, $\Phi_2(v) = v \otimes v$, is the co-multiplication, $\kappa_2(v) = v^*$ is the co-inverse and $e_2(v) = 1$ is the co-unit.

Then this definition provides the quantum group $U(1)$. □

A simple computation shows that this two quantum groups are quantum subgroups of our quantum group (A, u) with the natural embeddings. We are going to show that in fact (A, u) is the twisted product of these two subgroups, for a proper choice of the twist σ. For this purpose we need the following:

Definition 2. 3 *Let $\sigma : A_1 \otimes A_2 \to A_2 \times A_1$ be a *-algebra homomorphism defined by putting:*

$$\sigma(a \otimes v^k) = v^k \otimes a, \quad \sigma(c \otimes v^k) = v^{k-1} \otimes c \tag{2.6}$$

with $v^{-1} = v^*$.

Then we have

Theorem 2. 4 *The quantum group $A = A_1 \otimes_\sigma A_2$ is the twisted product of the two quantum subgroups with the twist σ.*

Proof: We should check that the co-multiplications and co-inverses satisfy the definition 2.2. Keeping in mind the identification $av^k \leftrightarrow a \otimes v^k$ and $cv^k \leftrightarrow c \otimes v^k$, given by the natural embeddings, we obtain for the co-multiplications: $s(\kappa_2 \otimes \kappa_1)\sigma(a \otimes v^k) = s(\kappa_2(v^k) \otimes \kappa_1(a)) = a^* \otimes v^{*k}$ and $s(\kappa_2 \otimes \kappa_1)\sigma(c \otimes v^k) = s(\kappa_2(v^{k-1}) \otimes \kappa_1(c)) = tc \otimes v^{*(k-1)}$ which agrees with the corresponding action of κ. Since a co-inverse is linear and anti-multiplicative, the above formulas can be extended to the *-subalgebra of A generated by a, c, v.

For the co-multiplications we have:

$(id_{A_1} \otimes \sigma \otimes id_{A_2})(\Phi_1 \otimes \Phi_2)(a \otimes v^k) = (id_{A_1} \otimes \sigma \otimes id_{A_2})(a \otimes a \otimes v^k \otimes v^k + tc^* \otimes c \otimes v^k \otimes v^k) =$
$a \otimes v^k \otimes a \otimes v^k + tc^* \otimes v^{k-1} \otimes c \otimes v^k$
which agrees with
$\Phi(av^k) = (a \otimes a + tc^* v^* \otimes c)(v^k \otimes v^k) = av^k \otimes av^k + tc^* v^{k-1} \otimes cv^k$
and
$(id_{A_1} \otimes \sigma \otimes id_{A_2})(\Phi_1 \otimes \Phi_2)(c \otimes v^k) = (id_{A_1} \otimes \sigma \otimes id_{A_2})(c \otimes a \otimes v^k \otimes v^k + a^* \otimes c \otimes v^k \otimes v^k) =$
$c \otimes v^k \otimes a \otimes v^k + a^* \otimes v^{k-1} \otimes c \otimes v^k$
which agrees with
$\Phi(cv^k) = (c \otimes a + a^* v^* \otimes c)(v^k \otimes v^k) = cv^k \otimes av^k + a^* v^{k-1} \otimes cv^k$.

Since both $(id_{A_1} \otimes \sigma \otimes id_{A_2})(\Phi_1 \otimes \Phi_2)$ and Φ are C*-algebra homomorphisms, and agree on generators, they satisfy the equation 2.7. □

3 Unitary representations of the quantum group

Our description of the unitary representations of the quantum group (A, u) we base on the work by Podleś and Woronowicz [P-W], where a general theorem shows how to construct representations of a quantum group which is twisted product of its quantum subgroups. First we recall this

Theorem 3. 1 *Let the quantum group $A = A_1 \otimes_\sigma A_2$ be the twisted product of its quantum subgroups A_1 and A_2, with the natural embeddings denoted by p_1 and p_2. Let $v \in B(K) \otimes A$ be matrix with entries from A for a finite dimensional complex vector space K. Then the following holds:*

1. If w is a (unitary) representation of A on K, then $w^1 := (id \otimes p_1)w$ is a (unitary) representation of A_1 on K and $w^2 := (id \otimes p_2)w$ is a (unitary) representation of A_2 on K, and the following conditions hold:

$$w = w^1 \bigcirc\!\!\!\!\!\perp\, w^2,$$

$$w^2 \bigcirc\!\!\!\!\!\perp\, w^1 = (id \otimes \sigma)(w^1 \bigcirc\!\!\!\!\!\perp\, w^2) \tag{3.1}$$

where

$$w^1 \bigcirc\!\!\!\!\!\perp\, w^2 = \sum_{j,k} m_j^1 m_k^2 \otimes w_j^1 \otimes w_k^2$$

for $w^i = \sum_j m_j^i \otimes w_j^i \in B(K) \otimes A_i$

2. If w^1 and w^2 are (unitary) representations of A_1 and A_2 respectively, which are of the same dimension and satisfy the compatibility condition 3.10, then $w = w^1 \bigcirc\!\!\!\!\!\perp\, w^2$ is a (unitary) representation of A on K.

We shall apply this result to our situation to describe the unitary representations of the quantum group (A, u). We will use the theory of irreducible unitary representations of the quantum group $SU_q(2)$, which was described by Woronowicz. The representations $\{u^s\}_{s \in \frac{1}{2}N}$ are indexed by the set $\frac{1}{2}N = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots\}$, and each u^s act on a $(2s+1)$

-dimensional Hilbert space. Explicit formulas for matrix elements of these representations are given in [Pu-W], B.19, p. 1616.

Now let us assume that $w^1 = u^s$, for some $s \in \frac{1}{2}N$, and w^2 is a unitary representation of A_2 of the dimension $2s+1$, and that they satisfy the compatibility condition 3.10. This condition implies that, for some positive integer r, $w^2 = diag\{v^r, v^{r-1}, \dots, v^{r-2s}\}$ has a diagonal matrix with the decreasing (or, equivalently, increasing) integral powers of the unitary v on the main diagonal. It follows that then the representation $w = w^1 \oplus w^2$ is unitary and irreducible representation of A. This can be seen by using the Haar measure $h = h_1 \otimes h_2$ on A, which is the tensor product of the Haar measure on $SU_q(2)$, $q = -t$, and the Lebesgue measure on the unit circle, which is the Haar measure on A_2. Let us recall that the non-trivial action of h_1 is given by $h_1((cc*)^m) = \frac{1-t^2}{1-t^{2(m+1)}}$. Then irreducibility of w is equivalent to $h(\chi_w^* \chi_w) = 1$, where $\chi_w = \sum_j w_{jj}$ is the *character* of the representation w. It follows from the form of w^2 and from the formulas (B.19) of [Pu-W] that the value $h(\chi_w^* \chi_w)$ is the same as $h_1(\chi_{w^1}^* \chi_{w^1})$, which is 1, by the irreducibility of w^1.

We shall finish our considerations with the following observation regarding the structure of the irreducible representations of (A, u). There is a sequence $\{v^r\}_{r \in Z}$ - integral powers of v - of irreducible one-dimensional representations of (A, u). There representation $w = \begin{pmatrix} a & tc^*v^* \\ c & a^*v^* \end{pmatrix} = u^{\frac{1}{2}} \oplus \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix}$ is the fundamental representation of (A, u), so we can write $(A, u) = (A, w)$. Its conjugate is the representation $\overline{w} = \begin{pmatrix} a^* & tcv \\ c^* & av \end{pmatrix} = u^{\frac{1}{2}} \oplus \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}$. These two-dimensional representations are not equivalent since they have different characters: $\chi_w = a + a^*v^* \neq a^* + av = \chi_{\overline{w}}$. The following is the decomposition of their tensor products into irreducible sub-representations:

$$w \oplus w = v^* \oplus \left(u^1 \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & v^* & 0 \\ 0 & 0 & v^{*2} \end{pmatrix} \right) \tag{3.2}$$

$$w \oplus \overline{w} = 1 \oplus \left(u^1 \oplus \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^* \end{pmatrix} \right) \tag{3.3}$$

$$\overline{w} \oplus \overline{w} = v \oplus \left(u^1 \oplus \begin{pmatrix} v^2 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \tag{3.4}$$

References

[P-W] P. Podleś, S.L. Woronowicz *Quantum deformation of Lorentz group*, Commun. Math. Phys. 130 (1990), 381–431.

[Pu-W] W. Pusz, S. L. Woronowicz, *Representations of quantum Lorentz group on Gelfand spaces*, Rev. Math. Phys. vol. 12, No. 12 (2000), 1551-1625.

[WPu] W. Pusz, *Irreducible unitary representations of quantum Lorentz group*, Commun. Math. Phys. 152 (1993), 591 – 626.

[SLW1] S.L. WORONOWICZ, *Twisted SU(2) group. An example of non-commutative differential calculus*, Publ. RIMS, Kyoto Univ. 23 (1987), 117–181.

[SLW2] S.L. WORONOWICZ, *Compact Matrix Pseudogroups*, Commun. Math. Phys. 111 (1987), 613–665

[SLW3] S.L. WORONOWICZ, *Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups*, Invent. Math. (1988), 35–76.

[JW] J. WYSOCZAŃSKI, *A construction of compact matrix quantum groups and description of the related C^*-algebras*, in "Infinite Dimensional Analysis and Quantum Probability Theory", (ed. Nobuaki Obata), RIMS Kokyuroku 1227 (2001), 209-217.