0. Preliminaries

$PSL(2, \mathbb{Z})$ is the group of $2 \times 2$ matrices over $\mathbb{Z}$ with determinant 1 modulo $\pm E$. This group has the following generators

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

satisfying the relations

$S^2 = (TS)^3 = E.$

Any element of $PSL(2, \mathbb{Z})$ can be presented as follows by using $S$ and $T$,

$PSL(2, \mathbb{Z}) \ni T^{b_1}ST^{b_2}S \cdots T^{b_l}S.$

From now on, we use the following sequence of integers to indicate the element.

$[b_1, b_2, \cdots, b_l]$

Then we get the following relations by using this symbol.

$[b_1, b_2, \cdots, b_i, 0, b_{i+2}, \cdots, b_l] = [b_1, b_2, \cdots, b_i + b_{i+2}, \cdots, b_l]$

$[b_1, b_2, \cdots, b_i, 1, 1, 1, b_{i+4}, \cdots, b_l] = [b_1, b_2, \cdots, b_i, b_{i+4}, \cdots, b_l]$

It is known that two symbols present the same element in $PSL(2, \mathbb{Z})$ if and only if they can be transformed to each other by finite sequence of the above relations.

1. The definition of the finite type invariant of $PSL(2, \mathbb{Z})$
Let $\overline{\Gamma}$ denote the free abelian group generated by all the elements in $PSL(2, \mathbb{Z})$ and $\overline{\Gamma}_n$ denote the group spanned by the following set

$$\left\{ \sum_{c_{ij} = \pm 1} (-1)^{\text{the number of } (-1)\text{'s in } \{c_{ij}\}} \times [b_1, b_2, \cdots, b_l]_{c_1, c_2, \cdots, c_n} \right\},$$

where

$$[b_1, b_2, \cdots, b_1, b_{i_1}, \cdots, b_{i_2}, \cdots, b_{i_n}, \cdots, b_l]_{c_1, c_2, \cdots, c_n} = [b_1, b_2, \cdots, b_{i_1} - c_{i_1} + 1, \cdots, b_{i_2} - c_{i_2} + 1, \cdots, b_{i_n} - c_{i_n} + 1, \cdots, b_l].$$

Note that if $c_{ij}$ is 1, then $b_{ij}$ does not change and that if $c_{ij}$ is $-1$, then $b_{ij}$ is changed to $b_{ij} + 2$.

Now we define the finite type invariant of $PSL(2, \mathbb{Z})$ as following.

**Definition.** An additive map from $\overline{\Gamma}/\overline{\Gamma}_{n+1}$ to $\mathbb{Q}$ is called an invariant of type $n$.

Let $\sim_n$ (we call this $n$-equivalence) denote the equivalence relation defined by $\overline{\Gamma}_{n+1}$ in $\overline{\Gamma}$.

**2. ON TYPE 0, 1 AND 2 INVARIANTS**

**Theorem 1.**

$$\overline{\Gamma}/\overline{\Gamma}_1 = \mathbb{Z}[[1], [0], [1], [0, 1], [1, 0], [1, 1]].$$

Moreover, 0-equivalence class of

$$\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)$$

is determined by its congruence class modulo 2.

From now on, we restrict ourselves to the matrices 0-equivalent to the identity $E$ and consider finite type invariants. Let $\overline{\Gamma}(2)$ be the span over $\mathbb{Z}$ of matrices 0-equivalent to the identity. We know that any element of $\overline{\Gamma}(2)$ can be presented as a sequence of even integers with even length, subject to the following relation

$$[2a_1, 2a_2, \cdots, 2a_i, 0, 2a_{i+2}, \cdots, 2a_{2m}] = [2a_1, 2a_2, \cdots, 2(a_i + a_{i+2}), \cdots, 2a_{2m}].$$

By similar calculation, we have the following
Theorem 2.

$$\overline{\Gamma}(2)/\overline{\Gamma}(2)_2 = \mathbb{Z}\{[\ ], [0, 2], [2, 0]\}.$$  

In fact, any element of $\overline{\Gamma}(2)$ is 1-equivalent to

$$(1 - A)[\ ] + A_0[0, 2] + A_1[2, 0],$$

where

$$A = \sum_{i=1}^{2m}a_i, \quad A_0 = \sum_{i=1}^{m}a_{2i}, \quad A_1 = \sum_{i=1}^{m}a_{2i-1}.$$  

Moreover, $1 - A$, $A_0$, $A_1$ are well-defined.

If $[2a_1, 2a_2, \ldots, 2a_{2m}] = (\alpha \beta \gamma \delta)$, then

$$A_0 = \sum_{i=1}^{\gamma/2}(-1)^{(2i-1)\frac{\alpha}{\gamma}}, \quad A_1 = \sum_{i=1}^{\gamma/2}(-1)^{(2i-1)\frac{\delta}{\gamma}}.$$  

Where $[\ ]$ denotes the greatest integer function.

To prove the formulas, we use Tuler's result of the linking number of a 2-bridge link ([2]).

Corollary 2.1. Any type 1 invariant is of the form

$$c_1(1 - A) + c_2A_0 + c_3A_1,$$

where $c_i$'s are constants.

Theorem 3.

$$\overline{\Gamma}(2)/\overline{\Gamma}(2)_3 = \mathbb{Z}\{[\ ], [0, 2], [2, 0], [2, 2], [0, 2, 2, 0], [0, 4], [4, 0]\}.$$  

In fact, any element of $\overline{\Gamma}(2)$ is 2-equivalent to

$$\frac{(A-1)(A-2)}{2} [\ ] - A_0(A-2)[0, 2] + A_1(A-2)[2, 0]$$

$$+ \sum_{i=1}^{m} \sum_{j=i}^{m} a_{2i-1}a_{2j} [2, 2] + \sum_{i=1}^{m-1} \sum_{j=i}^{m-1} a_{2i}a_{2j+1} [0, 2, 2, 0]$$
\[ + \frac{A_0(A_0-1)}{2} [0, 4] + \frac{A_1(A_1-1)}{2} [4, 0]. \]

If \( [2a_1, 2a_2, \ldots, 2a_{2m}] = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), then
\[
\sum_{i=1}^{m-1} \sum_{j=i}^{m-1} a_{2i}a_{2j+1} = \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)\frac{\beta}{2}]+[2j\frac{\delta}{2}]},
\]

To prove the formulas, we use the result of the Casson knot invariant of a 2-bridge knot ([1]).

**Corollary 3.1.** Any type 2 invariant is of the form
\[
d_1 \frac{(A-1)(A-2)}{2} + d_2 A_0(A-2) + d_3 A_1(A-2)
\]
\[
+ d_4 \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)\frac{\beta}{2}]+[2j\frac{\delta}{2}]},
\]
\[
+ d_5 \sum_{i=1}^{(\alpha-1)/2} \sum_{j=i}^{(\alpha-1)/2} (-1)^{[(2i-1)\frac{\gamma}{2}]+[2j\frac{\delta}{2}]},
\]
\[
+ d_6 \frac{A_0(A_0-1)}{2} + d_7 \frac{A_1(A_1-1)}{2},
\]

where \( d_i \)'s are constants.

Detail will appear elsewhere.

**REFERENCES**
