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T-system and thermodynamic Bethe ansatz equations for solvable lattice models associated with superalgebras

東京大学大学院数理科学研究科　　坪井禅吾 (Zengo Tsuboi)
Graduate School of Mathematical Sciences, University of Tokyo

Abstract

An analytic Bethe ansatz is carried out related to the Lie superalgebra $osp(1|2s)$. We present an eigenvalue formula of a transfer matrix in dressed vacuum form (DVF) labeled by a Young (super) diagram. Remarkable duality among DVFs is found. A complete set of transfer matrix functional relations (T-system) is proposed as a reduction of a Hirota-Miwa equation. We also derive a thermodynamic Bethe ansatz (TBA) equation from this T-system and the quantum transfer matrix method. This TBA equation is identical to the one from the string hypothesis.

1 Introduction

Solvable lattice models related to Lie superalgebras [1] have received much attentions [2, 3, 4, 5, 6, 7, 8, 9]. To construct eigenvalue formulae of transfer matrices for such models is an important problem in mathematical physics. To achieve this program, the Bethe ansatz has been often used.

Nowadays, there is much literature (see for example, [4, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and references therein.) on Bethe ansatz analysis for solvable lattice models related to Lie superalgebras. However, most of it deals only with models related to simple representations like fundamental ones. Only a few people (see for example, [15, 17]) tried to deal with more complicated models such as fusion models [25] by the Bethe ansatz; while there was no systematic study on this subject.

To address such situations, we have recently executed [26, 27, 28, 29, 30] an analytic Bethe ansatz [31, 32, 33, 34] systematically related to the Lie superalgebras $sl(r+1|s+1), B(r|s), C(s), D(r|s)$ cases. Namely, we have

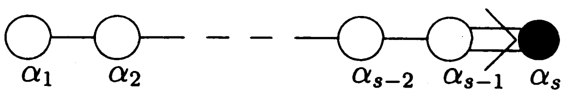

Figure 1: Dynkin diagram for the Lie superalgebra $B(0|s) = osp(1|2s)$ ($s \geq 1$): white circles denote even roots; a black circle denotes an odd root.

proposed a set of dressed vacuum forms (DVFs) and a class of functional relations (T-system) for it. Moreover we have also studied thermodynamic Bethe ansatz (TBA) equations [35] related to $osp(1|2)$ [36, 37, 38] and $osp(1|2s)$ [39] from the point of view of the string hypothesis [40, 41] and the quantum transfer matrix (QTM) method [42, 43, 44, 45, 46, 21].

In this paper, we briefly review the T-system and the TBA equation related to the Lie superalgebra $osp(1|2s) = B(0|s)$ based on [30, 39]. After a brief review on the Lie superalgebra $osp(1|2s)$, we introduce a QTM for $osp(1|2s)$ model[16] in section 3. In section 4, we carry out an analytic Bethe ansatz based on the Bethe ansatz equation (BAE) (13) and obtain the eigenvalue formula for the QTM. We define the dressed vacuum form (DVF) $T_{\lambda\subset\mu}(v)$ labeled by a skew-Young (super) diagram $\lambda \subset \mu$ as a summation over semi-standard tableaux. This DVF has a determinant expression (quantum supersymmetric Jacobi-Trudi formula). In particular, for a rectangular Young (super) diagram, this DVF satisfies a kind of Hirota-Miwa equation[47, 48]. By considering a reduction to this equation, we derive the $osp(1|2s)$ version of the T-system. Based on this T-system, we derive the TBA equation from the QTM method in section 5. Namely, we consider a dependant variable transformation, and derive the Y-system from the T-system. Then we transform the Y-system with certain analytical conditions into the TBA equation. Moreover we find that this TBA equation coincides with the one from the string hypothesis. This indicates the validity of the string hypothesis for the $osp(1|2s)$ model.

2 The Lie superalgebra $osp(1|2s)$

In this section, we briefly mention the Lie superalgebra $B(0|s) = osp(1|2s)$ for $s \in \mathbb{Z}_{\geq 1}$ (see for example [1, 49, 50]).

In contrast to other Lie superalgebras, the simple root system of $osp(1|2s)$ is unique and given as follows (see Figure 1):

$$\begin{aligned} &\alpha_i = \delta_i - \delta_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, s-1, \\ &\alpha_s = \delta_s \end{aligned} \tag{1}$$

where $\delta_1, \ldots, \delta_s$ are the bases of the dual space of the Cartan subalgebra with the bilinear form $(\ |\ )$ such that

$$(\delta_i|\delta_j) = -\delta_{ij} \tag{2}$$

$\{\alpha_i\}_{i\neq s}$ are even roots and α_s is an odd root with $(\alpha_s|\alpha_s) \neq 0$. Let $\lambda \subset \mu$ be a skew-Young (super) diagram labeled by the sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu_i \geq \lambda_i : i = 1, 2, \ldots$; $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$; $\mu_1 \geq \mu_2 \geq \cdots \geq 0$ and $\mu' = (\mu'_1, \mu'_2, \ldots)$ be the conjugate of μ. In particular, for $\lambda = \phi$, $\mu_1 \leq s$ case, the Kac-Dynkin label $[b_1, b_2, \ldots, b_s]$ is related to the Young (super) diagram with shape $\mu = (\mu_1, \mu_2, \ldots)$ as follows:

$$\begin{aligned} b_i &= \mu'_i - \mu'_{i+1} \qquad \text{for} \qquad i \in \{1, 2, \ldots, s-1\}, \\ b_s &= 2\mu'_s. \end{aligned} \tag{3}$$

An irreducible representation with the Kac-Dynkin label $[b_1, b_2, \ldots, b_s]$ is finite dimensional if and only if

$$\begin{aligned} b_j &\in \mathbb{Z}_{\geq 0} \qquad \text{for} \qquad j \in \{1, 2, \ldots, s-1\}, \\ b_s &\in 2\mathbb{Z}_{\geq 0}. \end{aligned} \tag{4}$$

3 $osp(1|2s)$ model and QTM method

In this section, we introduce an integrable spin chain[16, 18] associated with the fundamental representation of $osp(1|2s)$, and define a QTM. The $\check{R}$-matrix[5, 8, 9, 18] of the model is given as

$$\check{R}(v) = I + vP - \frac{2v}{2v-g}E, \tag{5}$$

where $g = 2s+1$; $P_{ab}^{cd} = (-1)^{p(a)p(b)}\delta_{ad}\delta_{bc}$; $E_{ab}^{cd} = \alpha_{ab}(\alpha^{-1})_{cd}$; $a, b, c, d \in J = \{1, 2, \ldots, s, 0, \overline{s}, \ldots, \overline{2}, \overline{1}\}$ $(1 \prec 2 \prec \cdots \prec s \prec 0 \prec \overline{s} \prec \cdots \prec \overline{2} \prec \overline{1})$; α is $(2s+1) \times (2s+1)$ anti-diagonal matrix whose non-zero elements are $\alpha_{a,\overline{a}} = 1$ for $a \in \{1, 2, \ldots, s, 0\}$ and $\alpha_{a,\overline{a}} = -1$ for $a \in \{\overline{s}, \overline{s-1}, \ldots, \overline{1}\}$; $\overline{\overline{a}} = a$; $p(a) = 0$ for $a = 0$; $p(a) = 1$ for $a \in \{1, 2, \ldots, s\} \sqcup \{\overline{s}, \ldots, \overline{2}, \overline{1}\}$. The Hamiltonian of the present model for the periodic boundary condition is given by

$$H = \mathcal{J}\sum_{k=1}^{L}\left(P_{k,k+1} + \frac{2}{g}E_{k,k+1}\right), \tag{6}$$

where L is the number of the lattice sites; $P_{k,k+1}$ and $E_{k,k+1}$ act nontrivially on the k th site and $k+1$ th site. There are several formulations of QTM for graded vertex models. We consider the case where the transfer matrix is defined as the ordinary trace of a monodromy matrix. The QTM is defined as

$$T_1^{(1)}(u,v) = \mathrm{Tr}_j \prod_{k=1}^{\frac{N}{2}} R_{a_{2k},j}(u+iv)\widetilde{R}_{a_{2k-1},j}(u-iv), \tag{7}$$

where $R_{ab}^{cd}(v) = \check{R}_{ba}^{cd}(v)$; $\widetilde{R}_{jk}(v) ={}^{t_k}R_{kj}(v)$ (t_k is the transposition in the k-th space); N is the Trotter number and assumed to even. By using the largest eigenvalue $T_1^{(1)}(u_N,0)$ of the QTM (7), the free energy density is expressed as

$$\mathcal{F} = -\frac{1}{\beta}\lim_{N\to\infty}\log T_1^{(1)}(u_N,0), \tag{8}$$

where $u_N = -\frac{\mathcal{J}\beta}{N}$ ($\beta = 1/(k_BT)$; k_B: the Boltzmann constant; T: the temperature). From now on, we abbreviate the parameter u in $T_1^{(1)}(u,v)$.

4 Analytic Bethe ansatz and T-system for QTM

One can obtain the eigenvalue formulae of the QTM (7) by replacing the vacuum part of the DVF for the row-to-row transfer matrix [16, 18] with that of the QTM. Explicitly we have

$$T_1^{(1)}(v) = \sum_{a\in J} \boxed{a}_v, \tag{9}$$

where the functions $\{\boxed{a}_v\}_{a\in J}$ are defined as

$$\begin{aligned}
\boxed{a}_v &= \psi_a(v)\frac{Q_{a-1}(v+\frac{i}{2}(a+1))Q_a(v+\frac{i}{2}(a-2))}{Q_{a-1}(v+\frac{i}{2}(a-1))Q_a(v+\frac{i}{2}a)} \\
&\qquad \text{for} \quad a\in\{1,2,\dots,s\}, \\
\boxed{0}_v &= \psi_0(v)\frac{Q_s(v+\frac{i}{2}(s-1))Q_s(v+\frac{i}{2}(s+2))}{Q_s(v+\frac{i}{2}(s+1))Q_s(v+\frac{i}{2}s)}, \\
\boxed{\overline{a}}_v &= \psi_{\overline{a}}(v)\frac{Q_{a-1}(v-\frac{i}{2}(a-2s))Q_a(v-\frac{i}{2}(a-2s-3))}{Q_{a-1}(v-\frac{i}{2}(a-2s-2))Q_a(v-\frac{i}{2}(a-2s-1))} \\
&\qquad \text{for} \quad a\in\{1,2,\dots,s\},
\end{aligned} \tag{10}$$

where $Q_0(v) := 1$; $\psi_a(v)$ is the vacuum part

$$\psi_a(v) = \begin{cases} \zeta_1\frac{\phi_+(v)\phi_-(v+i)\phi_+(v-\frac{2s-1}{2}i)}{\phi_+(v-\frac{2s+1}{2}i)} & \text{for} \quad a=1, \\ \zeta_a\phi_+(v)\phi_-(v) & \text{for} \quad 2\preceq a\preceq \overline{2}, \\ \zeta_{\overline{1}}\frac{\phi_-(v)\phi_+(v-i)\phi_-(v+\frac{2s-1}{2}i)}{\phi_-(v+\frac{2s+1}{2}i)} & \text{for} \quad a=\overline{1}, \end{cases} \tag{11}$$

where $\phi_{\pm}(v) = (v \pm iu)^{\frac{N}{2}}$; ζ_a is a phase factor:

$$\zeta_a = \begin{cases} (-1)^{N-M_1} & \text{if} \quad a = 1 \\ (-1)^{M_{a-1}-M_a} & \text{if} \quad a \in \{2, 3, \dots, s\} \\ 1 & \text{if} \quad a = 0 \\ (-1)^{M_{\overline{a}-1}-M_{\overline{a}}} & \text{if} \quad a \in \{\overline{s}, \dots, \overline{3}, \overline{2}\} \\ (-1)^{N-M_1} & \text{if} \quad a = \overline{1}, \end{cases} \tag{12}$$

where $\overline{\overline{a}} = a$. The complex variables $\{v_k^{(a)}\}$ are roots of the following Bethe ansatz equation

$$\prod_{j=1}^{N} \left(\frac{v_k^{(a)} - w_j^{(a)} + \frac{i}{2}\delta_{a1}}{v_k^{(a)} - w_j^{(a)} - \frac{i}{2}\delta_{a1}} \right) = -(-1)^{M_{a-1}-M_{\sigma(a+1)}} \prod_{d=1}^{s+1} \frac{Q_{\sigma(d)}(v_k^{(a)} + \frac{i}{2}B_{ad})}{Q_{\sigma(d)}(v_k^{(a)} - \frac{i}{2}B_{ad})}, \tag{13}$$

where $k \in \{1, 2, \dots, M_a\}$; $a \in \{1, 2, \dots, s\}$; $\sigma(d) = d$ for $1 \le d \le s$; $\sigma(s+1) = s$; $B_{ad} = 2\delta_{ad} - \delta_{a,d+1} - \delta_{a,d-1}$; $Q_a(v) = \prod_{k=1}^{M_a}(v - v_k^{(a)})$; $M_a \in \mathbb{Z}_{\ge 0}$; $M_0 = N$. The parameter σ expresses an effect of 'a peculiar two-body self-interaction for the root $\{v_k^{(s)}\}$' [18], which originates from the odd simple root α_s with $(\alpha_s|\alpha_s) \neq 0$. One may interpret the QTM as a transfer matrix of an inhomogeneous vertex model. In our case, the inhomogeneity parameters $w_j^{(a)} \in \mathbb{C}$ take the values: $w_j^{(a)} = iu\delta_{a1}$ for $j \in 2\mathbb{Z}_{\ge 1}$; $w_j^{(a)} = (-iu + \frac{ig}{2})\delta_{a1}$ for $j \in 2\mathbb{Z}_{\ge 0} + 1$. The dress part of the DVF (9) is free of poles under the BAE (13). This is a requirement from the analytic Bethe ansatz [31].

Now we will present a DVF $T_{\lambda \subset \mu}(v)$ for a 'fusion QTM'. We can derive the explicit expression of $T_{\lambda \subset \mu}(v)$ by modifying the vacuum part of the DVF in Ref. [30] so that the vacuum part is compatible with the left hand side of the BAE (13). We assign coordinates $(i, j) \in \mathbb{Z}^2$ on the skew-Young (super) diagram $\lambda \subset \mu$ such that the row index i increases as we go downwards and the column index j increases as we go from the left to the right and that $(1, 1)$ is on the top left corner of μ. We define an admissible tableau b on the skew-Young (super) diagram $\lambda \subset \mu$ as a set of elements $b(i, j) \in J$ labeled by the coordinates (i, j) mentioned above, with the following rule (admissibility conditions).

$$b(i, j) \prec b(i, j+1), \qquad b(i, j) \preceq b(i+1, j). \tag{14}$$

Let $B(\lambda \subset \mu)$ be the set of admissible tableaux on $\lambda \subset \mu$. For any skew-Young (super) diagram $\lambda \subset \mu$, define $T_{\lambda \subset \mu}(v)$ as follows

$$T_{\lambda \subset \mu}(v) = \sum_{b \in B(\lambda \subset \mu)} \prod_{(j,k) \in (\lambda \subset \mu)} \boxed{b(j, k)}_{v - \frac{i}{2}(-\mu_1 + \mu'_1 - 2j + 2k)}, \tag{15}$$

$$\boxed{1} \xrightarrow{(1,1)} \boxed{2} \xrightarrow{(2,2)} \boxed{0} \xrightarrow{(2,3)} \boxed{\overline{2}} \xrightarrow{(1,4)} \boxed{\overline{1}}$$

Figure 2: The Bethe-strap structure of $T_1^{(1)}(v)$ for $osp(1|4)$: The pair (a, b) denotes the common pole $v_k^{(a)} - \frac{i}{2}b$ of the pair of the tableaux connected by the arrow. This common pole vanishes under the BAE (13). The leftmost tableau corresponds to the 'highest weight ', which is called the *top term*. This term carries the $osp(1|4)$ weight δ_1.

where the product is taken over the coordinates (j, k) on $\lambda \subset \mu$. Let $T_m^{(a)}(v) := T_{(a^m)}(v)$. The following determinant formula (quantum supersymmetric Jacobi-Trudi formula) should be valid (cf. [34]).

$$\begin{aligned} T_{\lambda\subset\mu}(v) &= \det_{1\leq j,k\leq\mu_1'}(T_1^{(\mu_k-\lambda_j+j-k)}(v - \\ &\quad \frac{i}{2}(-\mu_1+\mu_1'+\mu_k'+\lambda_j'-j-k+1))). \end{aligned} \tag{16}$$

We may think of (15) as an $osp(1|2s)$ version of the Bazhanov and Reshetikhin's eigenvalue formula [32]. In particular, for $\lambda = \phi$, $\mu_1 \leq s$ case, the 'top term' of $T_\mu(v)$ will be the term corresponding to the tableau $b(i, j) = j$ $(1 \leq i \leq \mu_j'$, $1 \leq j \leq s)$. This term carries the $osp(1|2s)$ weight with the Kac-Dynkin label (3) (in the sense in Ref. [33]). DVFs have so called *Bethe-strap* structures [33] and we confirmed, for several examples, that $T_{\lambda\subset\mu}(v)$ coincides with the Bethe-strap of the minimal connected component which includes the top term as the examples in Figure 2, Figure 3 and Figure 4. $T_{\lambda\subset\mu}(v)$ may be viewed as a prototype of a 'q-supercharacter' (cf. [51]).

Now we introduce the functional relations among DVFs. The following relation follows from the determinant formula (16).

$$T_m^{(a)}(v+\frac{i}{2})T_m^{(a)}(v-\frac{i}{2}) = T_{m+1}^{(a)}(v)T_{m-1}^{(a)}(v) + T_m^{(a-1)}(v)T_m^{(a+1)}(v), \tag{17}$$

where $a, m \in \mathbb{Z}_{\geq 1}$. This functional relation is a kind of Hirota-Miwa equation [47, 48] and can be proved by the Jacobi identity. The following theorem follows from the admissible condition (14).

Theorem 1 $T_{\lambda\subset\mu}(v) = 0$ *if* $\lambda \subset \mu$ *contains* $m \times a$ *rectangular subdiagram (m: the number of row, a: the number of column) with* $a \in \mathbb{Z}_{\geq 2s+2}$ *and* $m \in \mathbb{Z}_{\geq 1}$. *In particular, we have*

$$T_m^{(a)}(v) = 0 \qquad \text{if} \quad a \in \mathbb{Z}_{\geq 2s+2} \quad \text{and} \quad m \in \mathbb{Z}_{\geq 1}. \tag{18}$$

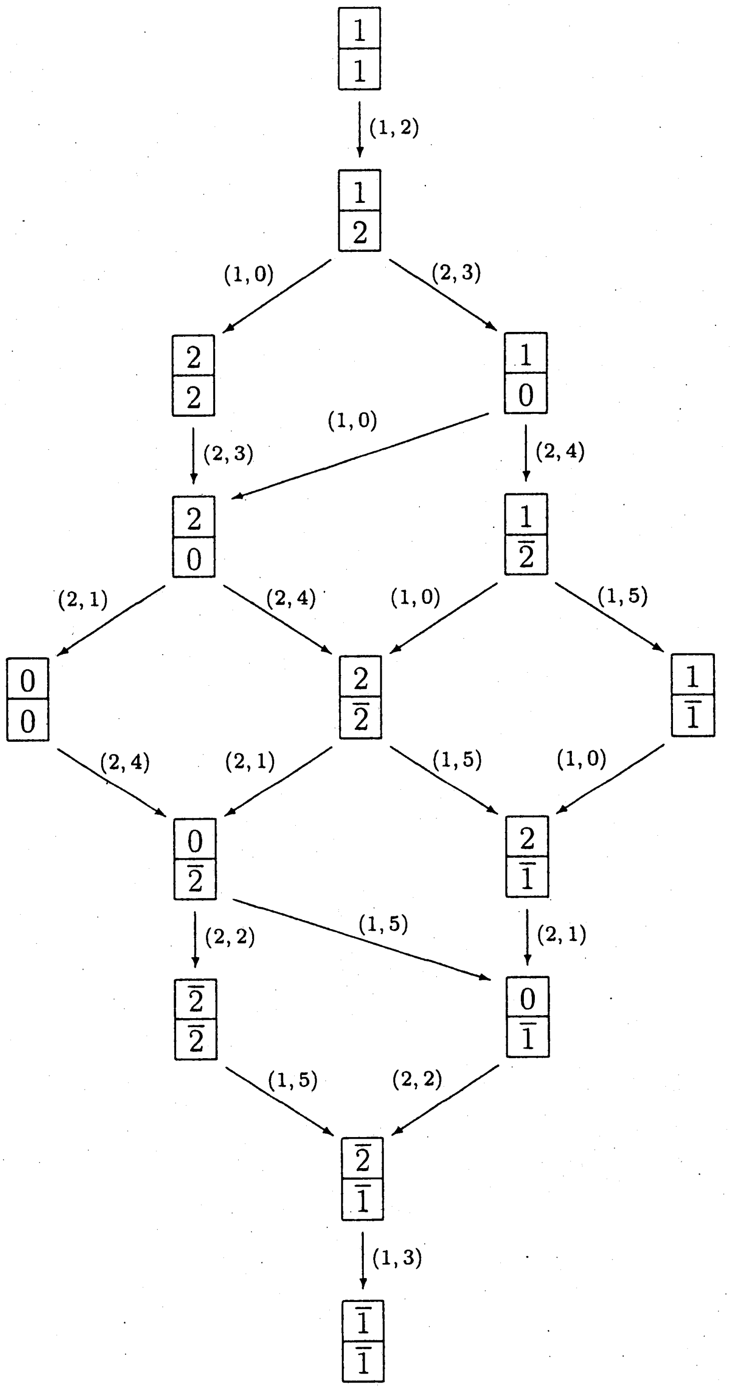

Figure 3: The Bethe-strap structure of $T_2^{(1)}(v)$ for $osp(1|4)$: The topmost tableau corresponds to the 'highest weight ', which is called the *top term*. This term carries the $osp(1|4)$ weight $2\delta_1$

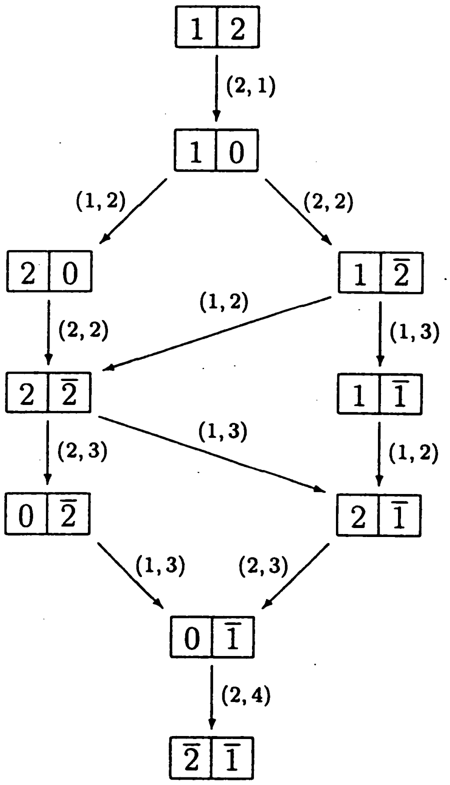

Figure 4: The Bethe-strap structure of $T_1^{(2)}(v)$ for $osp(1|4)$: The topmost tableau corresponds to the 'highest weight ', which is called the *top term*. This term carries the $osp(1|4)$ weight $\delta_1 + \delta_2$

There is a remarkable duality for $T_m^{(a)}(v)$.

Theorem 2 *For any $a \in \{1,\dots,s\}$ and $m \in \mathbb{Z}_{\geq 0}$, we have*

$$T_m^{(a)}(v) = \mathcal{M}_m^{(a)}(v) T_m^{(2s-a+1)}(v), \tag{19}$$

where $\mathcal{M}_m^{(a)}(v)$ is given as

$$\begin{aligned}\mathcal{M}_m^{(a)}(v) &= \prod_{j=1}^{m} \left\{ \frac{\psi_1(v-\frac{i}{2}(m-a-2j+2))}{\psi_1(v-\frac{i}{2}(m-2s+a-2j+1))} \right. \\ &\quad \left. \times \frac{\prod_{k=2}^{a} \psi_2(v-\frac{i}{2}(m-a-2j+2k))}{\prod_{k=2}^{2s-a+1} \psi_2(v-\frac{i}{2}(m-2s+a-2j+2k-1))} \right\}. \end{aligned} \tag{20}$$

For $a \in \{1,2,\dots,s\}$ and $m \in \mathbb{Z}_{\geq 1}$, we define a normalization function

$$\mathcal{N}_m^{(a)}(v) = \frac{\prod_{j=1}^{m}\prod_{k=1}^{a} \phi_-(v-\frac{m-a-2j+2k}{2}i)\phi_+(v-\frac{m-a-2j+2k}{2}i)}{\phi_-(v-\frac{m-a}{2}i)\phi_+(v+\frac{m-a}{2}i)}. \tag{21}$$

We reset $T_m^{(a)}(v)/\mathcal{N}_m^{(a)}(v)$ to $T_m^{(a)}(v)$, where $T_m^{(a)}(v)$ is defined by (15). By using the Theorem 1,2, we can obtain the T-system as a reduction of the Hirota-Miwa equation (17).

$$\begin{aligned} T_m^{(a)}(v+\frac{i}{2})T_m^{(a)}(v-\frac{i}{2}) &= T_{m+1}^{(a)}(v)T_{m-1}^{(a)}(v) + T_m^{(a-1)}(v)T_m^{(a+1)}(v) \\ &\qquad \text{for} \quad a \in 1,2,\dots,s-1, \\ T_m^{(s)}(v+\frac{i}{2})T_m^{(s)}(v-\frac{i}{2}) &= T_{m+1}^{(s)}(v)T_{m-1}^{(s)}(v) + g_m^{(s)}(v)T_m^{(s-1)}(v)T_m^{(s)}(v), \end{aligned} \tag{22}$$

where

$$\begin{aligned} T_0^{(a)}(v) &= \phi_-(v+\frac{a}{2}i)\phi_+(v-\frac{a}{2}i) \quad \text{for} \quad a \in \mathbb{Z}_{\geq 1}, \\ T_m^{(0)}(v) &= \phi_-(v-\frac{m}{2}i)\phi_+(v+\frac{m}{2}i) \quad \text{for} \quad m \in \mathbb{Z}_{\geq 1}, \\ g_m^{(s)}(v) &= \frac{\phi_-(v+\frac{m+s+1}{2}i)\phi_+(v-\frac{m+s+1}{2}i)}{\phi_-(v+\frac{m+s}{2}i)\phi_+(v-\frac{m+s}{2}i)} \quad \text{for} \quad m \in \mathbb{Z}_{\geq 1}. \end{aligned} \tag{23}$$

For $s=1$, $g_m^{(1)}(v)T_m^{(0)}(v)$ coincides with the function $T_m^{(0)}(v)$ in Ref. [37]. Since the dress part of the DVF $T_m^{(a)}(v)$ is same as the row-to-row case, this functional equation (22) has essentially the same form as the $osp(1|2s)$ T-system in Ref.[30].

5 TBA equation

For $m \in \mathbb{Z}_{\geq 1}$, we define the Y-functions:

$$
\begin{aligned}
Y_m^{(a)}(v) &= \frac{T_{m+1}^{(a)}(v)T_{m-1}^{(a)}(v)}{T_m^{(a-1)}(v)T_m^{(a+1)}(v)} \quad \text{for} \quad a \in \{1,2,\ldots,s-1\}, \\
Y_m^{(s)}(v) &= \frac{T_{m+1}^{(s)}(v)T_{m-1}^{(s)}(v)}{g_m^{(s)}(v)T_m^{(s-1)}(v)T_m^{(s)}(v)}.
\end{aligned} \tag{24}
$$

By using the T-system (22), one can show that the Y-functions satisfy the following Y-system:

$$
Y_m^{(a)}(v+\frac{i}{2})Y_m^{(a)}(v-\frac{i}{2}) = \frac{(1+Y_{m+1}^{(a)}(v))(1+Y_{m-1}^{(a)}(v))}{\prod_{d=1}^{s}(1+(Y_m^{(d)}(v))^{-1})^{I_{ad}}}, \tag{25}
$$

where $Y_0^{(a)}(v)=0$, $a \in \{1,2,\ldots,s\}$ and $m \in \mathbb{Z}_{\geq 1}$; $I_{ad} = \delta_{a,d-1}+\delta_{a,d+1}+\delta_{ad}\delta_{as}$. A numerical analysis for finite N, u, s indicates that a two-string solution (for every color) in the sector $N = M_1 = M_2 = \cdots = M_s$ of the BAE (13) provides the largest eigenvalue of the QTM (7) at $v = 0$. Moreover, we expect the following conjecture is valid for this two-string solution.

Conjecture 1 *For small u ($|u| \ll 1$) and $a \in \{1,2,\ldots,s\}$, every zero of $T_m^{(a)}(v)$ is located outside of the physical strip* $\mathrm{Im}v \in [-\frac{1}{2}, \frac{1}{2}]$.

Based on this conjecture, we shall establish the ANZC property in some domain for the Y-functions (24) to transform the Y-system (25) to nonlinear integral equations. Here ANZC means Analytic NonZero and Constant asymptotics in the limit $|v| \to \infty$. One can show that the Y-function has the following asymptotic value

$$
\lim_{|v|\to\infty} Y_m^{(a)}(v) = \frac{m(g+m)}{a(g-a)}, \tag{26}
$$

which is identified to the solution of the constant Y-system

$$
(Y_m^{(a)})^2 = \frac{(1+Y_{m-1}^{(a)})(1+Y_{m+1}^{(a)})}{\prod_{d=1}^{s}(1+(Y_m^{(d)})^{-1})^{I_{ad}}}, \tag{27}
$$

where $Y_0^{(a)} := 0$, $a \in \{1,2,\ldots,s\}$ and $m \in \mathbb{Z}_{\geq 1}$. From the Conjecture 1 and (26), we find that the functions $1+Y_m^{(a)}(v)$, $1+(Y_m^{(a)}(v))^{-1}$ in the domain $\mathrm{Im}v \in [-\delta, \delta]$ $(0 < \delta \ll 1)$ and $Y_m^{(a)}(v)$ for $(a,m) \neq (1,1)$ in the domain

$\mathrm{Im}v \in [-\frac{1}{2}, \frac{1}{2}]$ (physical strip) have the ANZC property. On the other hand, $Y_1^{(1)}(v)$ has zeros of order $N/2$ at $\pm i(\frac{1}{2}-u)$ if $u>0$ ($\mathcal{J}<0$), poles of order $N/2$ at $\pm i(\frac{1}{2}+u)$ if $u<0$ ($\mathcal{J}>0$) in the physical strip. Then we must modify $Y_1^{(1)}(v)$ as

$$\widetilde{Y}_m^{(a)}(v) = Y_m^{(a)}(v)\left\{\tanh\frac{\pi}{2}(v+i(\frac{1}{2}\pm u))\tanh\frac{\pi}{2}(v-i(\frac{1}{2}\pm u))\right\}^{\pm\frac{N\delta_{a1}\delta_{m1}}{2}}, \tag{28}$$

where the sign $\pm$ is identical to that of $-u$. Taking note on the relation

$$\tanh\frac{\pi}{4}(v+i)\tanh\frac{\pi}{4}(v-i) = 1, \tag{29}$$

one can modify the lhs of the Y-system (25) as

$$\widetilde{Y}_m^{(a)}(v-\frac{i}{2})\widetilde{Y}_m^{(a)}(v+\frac{i}{2}) = \frac{(1+Y_{m+1}^{(a)}(v))(1+Y_{m-1}^{(a)}(v))}{\prod_{d=1}^{s}(1+(Y_m^{(d)}(v))^{-1})^{I_{ad}}}, \tag{30}$$

$$\text{for} \quad m\in\mathbb{Z}_{\geq 1} \quad \text{and} \quad a\in\{1,2,\ldots,s\}.$$

Now that the ANZC property has been established for the Y-system, we can transform (30) into a system of nonlinear integral equations by a standard procedure.

$$\begin{aligned}\log Y_m^{(a)}(v) &= \mp\frac{N\delta_{a1}\delta_{m1}}{2}\log\left\{\tanh\frac{\pi}{2}(v+i(\frac{1}{2}\pm u))\tanh\frac{\pi}{2}(v-i(\frac{1}{2}\pm u))\right\} \\ &\quad +K*\log\left\{\frac{(1+Y_{m-1}^{(a)})(1+Y_{m+1}^{(a)})}{\prod_{d=1}^{s}(1+(Y_m^{(d)})^{-1})^{I_{ad}}}\right\}(v),\end{aligned} \tag{31}$$

where $Y_0^{(a)}(v)=0$, $a\in\{1,2,\ldots,s\}$ and $m\in\mathbb{Z}_{\geq 1}$; $*$ is a convolution

$$(f*h)(v) = \int_{-\infty}^{\infty} dw f(v-w)h(w), \tag{32}$$

and the kernel is

$$K(v) = \frac{1}{2\cosh\pi v}. \tag{33}$$

Substituting $u=-\frac{\beta\mathcal{J}}{N}$ and taking the Trotter limit $N\to\infty$, we obtain the TBA equation

$$\log Y_m^{(a)}(v) = \frac{\pi\mathcal{J}\beta\delta_{ap}\delta_{mb}}{\cosh\pi v} + K*\log\left\{\frac{(1+Y_{m-1}^{(a)})(1+Y_{m+1}^{(a)})}{\prod_{d=1}^{s}(1+(Y_m^{(d)})^{-1})^{I_{ad}}}\right\}(v), \tag{34}$$

where $a \in \{1, 2, \ldots, s\}$, $m \in \mathbb{Z}_{\geq 1}$, $Y_0^{(a)}(v) := 0$. This TBA equation (34) is identical to the one from the string hypothesis. Taking note on the relations

$$
\begin{aligned}
C_{ad}(v) &= \sum_{l=1}^{\min(a,d)} G_{|a-d|+2l-1}(v), \\
G_a(v) &= \frac{4}{2s+1} \frac{\cos\frac{(2s+1-2a)\pi}{4s+2}\cosh\frac{2\pi v}{2s+1}}{\cos\frac{(2s+1-2a)\pi}{2s+1} + \cosh\frac{4\pi v}{2s+1}}, \\
\widehat{C}_{ad}(k) &= \int_{-\infty}^{\infty} dv C_{ad}(v) e^{-ikv}, \\
\sum_{c=1}^{s} \widehat{C}_{ac}(k)\widehat{D}_{cd}(k) &= \delta_{ad}, \qquad (35) \\
\widehat{D}_{cd}(k) &= 2\delta_{cd}\cosh\frac{k}{2} - I_{cd},
\end{aligned}
$$

one can also rewrite this TBA equation as

$$
\begin{aligned}
\log Y_m^{(a)}(v) &= 2\pi\beta\mathcal{J}\delta_{m1}G_a(v) \\
&\quad + \sum_{b=1}^{s} C_{ab} * \log\left\{\frac{(1+Y_{m-1}^{(b)})(1+Y_{m+1}^{(b)})}{\prod_{d=1}^{s}(1+Y_m^{(d)})^{I_{bd}}}\right\}(v), \qquad (36)
\end{aligned}
$$

where $Y_0^{(a)}(v) = 0$, $a \in \{1, 2, \ldots, s\}$ and $m \in \mathbb{Z}_{\geq 1}$. In contrast to (34), (36) does not contain $1 + (Y_m^{(a)}(v))^{-1}$ which is not relevant to evaluate the central charge for the case $\mathcal{J} < 0$. One can also derive the following relation from (22) for $m = 1$, (24) and (35).

$$
\begin{aligned}
\log T_1^{(1)}(v) &= \log\phi_-(v+i)\phi_+(v-i) + \sum_{a=1}^{s} G_a * \log(1+Y_1^{(a)}) \\
&\quad + N\int_0^{\infty} dk \frac{2e^{-\frac{k}{2}}\sinh(ku)\cos(kv)\cosh(\frac{2s-1}{4}k)}{k\cosh(\frac{2s+1}{4}k)}. \qquad (37)
\end{aligned}
$$

Taking the Trotter limit $N \to \infty$ with $u = -\frac{\mathcal{J}\beta}{N}$, we obtain the free energy density $\mathcal{F} = -\frac{1}{\beta}\log T_1^{(1)}(0)$ without infinite sum.

$$
\begin{aligned}
\mathcal{F} &= \mathcal{J}\left\{\frac{2}{2s+1}\left(2\log 2 - \psi(\frac{1}{2s+1}) + \psi(\frac{3+2s}{2+4s})\right) - 1\right\} \\
&\quad - k_B T \sum_{a=1}^{s}\int_{-\infty}^{\infty} dv G_a(v)\log(1+Y_1^{(a)}(v)), \qquad (38)
\end{aligned}
$$

where $\psi(z)$ is the digamma function

$$\psi(z) = \frac{d}{dz}\log\Gamma(z). \tag{39}$$

The first term in the rhs of (38) for $\mathcal{J} = -1$ coincides with the grand state energy of the $osp(1|2s)$ model in [16]. Using the result of this section, we can show that the central charge of the corresponding system is s.

6 Discussion

In this paper, we have derived the TBA equation from the $osp(1|2s)$ version of the T-system. The $osp(r|2s)$ integrable spin chain is related to interesting physical problems, such as the loop model which is related to statistical properties of polymers[22], and the fractional quantum Hall effect [52], etc. So it is desirable to study the $osp(r|2s)$ integrable spin chain beyond the $osp(1|2s)$ case. For $r > 0$ case, we have only the T-system for tensor-like representations [30]. To construct a complete set of the T-system which is relevant for the QTM method, we have to treat spinorial representations.

In closing this paper, we shall mention the $sl(r+1|s+1)$ version of the T-system [26, 27, 28] which is omitted in this paper. The $osp(1|2s)$ T-system is obtained as a reduction of a kind of Hirota-Miwa equation. This is also the case with $sl(r+1|s+1)$. For $m, a \in \mathbb{Z}_{\geq 1}$, $sl(r+1|s+1)$ T-system leads as follows.

$$\begin{aligned}
&T_m^{(a)}(v-1)T_m^{(a)}(v+1) = T_{m+1}^{(a)}(v)T_{m-1}^{(a)}(v) + T_m^{(a-1)}(v)T_m^{(a+1)}(v) \\
&\text{for} \quad 1 \leq a \leq r \quad \text{or} \quad 1 \leq m \leq s \quad \text{or} \quad (a,m) = (r+1, s+1),
\end{aligned}$$

$$T_m^{(r+1)}(v-1)T_m^{(r+1)}(v+1) = T_{m+1}^{(r+1)}(v)T_{m-1}^{(r+1)}(v) \quad \text{for} \quad m \geq s+2,$$

$$T_{s+1}^{(a)}(v-1)T_{s+1}^{(a)}(v+1) = T_{s+1}^{(a+1)}(v)T_{s+1}^{(a-1)}(v) \quad \text{for} \quad a \geq r+2.$$

where,

$$\begin{aligned}
&T_{s+1}^{(a)}(v) = \epsilon_a T_{a+s-r}^{(r+1)}(v) \quad \text{for} \quad a \geq r+1, \\
&T_m^{(0)}(v) = T_0^{(a)}(v) = 1.
\end{aligned}$$

Here we omit the vacuum part which can be easily recovered so as to be compatible with the lhs (vacuum part) of the BAE. The phase factor ϵ_a depends on the definition of the transfer matrix. For example, if the transfer matrix is defined as a supertrace of a monodromy matrix, we have $\epsilon_a = (-1)^{(s+1)(a+r+1)}$. Note that above functional equation reduces to the T-system for sl_{r+1} [32] (see also [53, 54]) if we set $s = -1$.

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