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# Discrete indefinite improper affine spheres

NOZOMU MATSUURA*

Graduate School of Information Sciences, Tohoku University

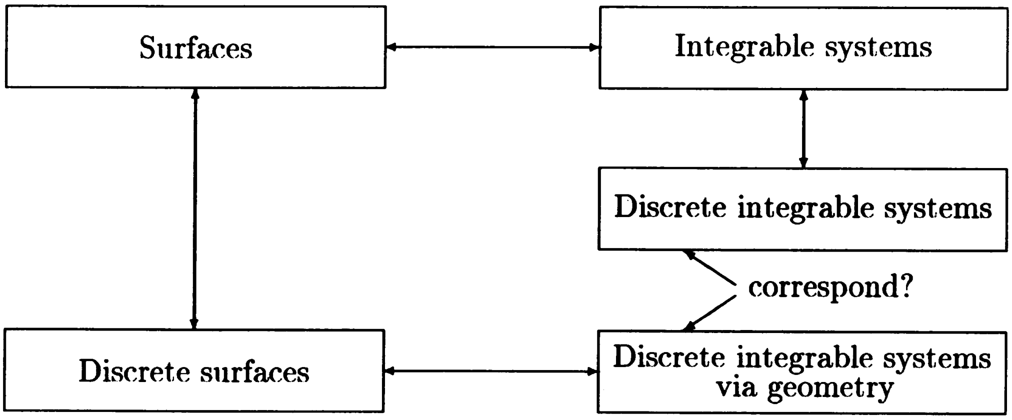
**Abstract**

The purpose of this note is to discretize improper affine spheres and to investigate them in detail. We clarify a link between discrete improper affine spheres and Hirota's discrete Liouville equation.

## 1 Introduction

In recent years, there has been explosive progress in the theory of discrete integrable systems. In this connection, discrete surfaces have been studied one after another with strong ties to physics and great potential for computer analysis. Those relationship between geometry and integrable systems can be diagramed as on the next page: the right arrows mean integrability contitions, the left ones geometric correspondent, the up ones continuum limit, and the down ones discretization. In general, one differential equation may have many discrete models, so how can we find a good one among them? A possible strategy is to discretize it via geometry. Such a link between discrete integrable systems and particular classes of discrete surfaces has been established. For example, Hirota's discrete sine-Gordon equation arises as the discrete integrability condition for discrete pseudo-spherical surfaces [1][7].

*松浦 望, 東北大学大学院情報科学研究科 (D1)
E-mail: nozomu@ims.is.tohoku.ac.jp




Affine spheres are those surfaces for which the affine shape operator is a scalar multiple of the indentity, but unlike the Euclidean case they are by no means simple or easy to determine [6]. A discrete integrable analogue of *proper* affine spheres was given by A. Bobenko and W. Schief [2][3], who presented a natural geometric discretization of them and investigated the corresponding discrete Gauss-Codazzi equations in detail. But, improper affine spheres make also an abundant and important class including ruled surfaces. Furthermore, every solution to the Liouville equation

$$(\log \omega)_{uv} + \omega^{-2} = 0 \tag{1}$$

describes an improper affine sphere in $\mathbf{R}^3$. The solution $\omega$ of (1) becomes the volume element of the affine metric. In this note, we discretize *improper* affine spheres. A discrete integrable analogue of the equation (1) was constructed by R. Hirota [4] without using any relation to geometry. We show that Hirota's discrete Liouville equation

$$2\sinh \frac{W_{12} - W_1 - W_2 + W}{2} + \exp \frac{-W_{12} - W_1 - W_2 - W}{2} = 0 \tag{2}$$

describes our discrete improper affine spheres, and this observation permits the diagram given above to commute.

## 2 Preliminary

In this section, let us recall basic notation of affine differential geometry. Let $M$ be a two dimensional smooth manifold and $D$ the usual flat affine connection on $\mathbf{R}^3$. For an immersion $f: M \to (\mathbf{R}^3, D)$, we choose an arbitrary transversal vector field $\xi$ on $M$, that is,

$$T_{f(x)}\mathbf{R}^3 = f_*(T_xM) \oplus \mathbf{R}\xi_x$$

at each point $x \in M$. The formulas of Gauss and Weingarten

$$\begin{aligned} D_X(f_*Y) &= f_*(\nabla_X Y) + h(X,Y)\xi, \\ D_X\xi &= -f_*(SX) + \tau(X)\xi \end{aligned}$$

induce on $M$ an affine connection $\nabla$, a symmetric $(0,2)$-tensor field $h$, a $(1,1)$-tensor field $S$ and a 1-form $\tau$. The determinant function of $\mathbf{R}^3$ induces a volume form $\theta$ on $M$ via

$$\theta(X,Y) = \det(f_*X, f_*Y, \xi).$$

The rank of the affine fundamental form $h$ is independent of the choice of transversal vector field $\xi$. We assume that the rank is 2, so that $h$ can be treated as a nondegenerate metric on $M$. This is a basic assumption on which Blaschke developed affine differential geometry of hypersurfaces. For each point $x \in M$, there is a transversal field $\xi$ defined in a neighborhood of $x$ satisfying the conditions

$$\omega = \theta, \quad \nabla\theta = 0. \tag{3}$$

Here $\omega$ denotes the volume element of the nondegenerate metric $h$. The former is called volume condition and the latter equiaffine condition. Since the determinant function is parallel relative to $D$, the equation $\nabla\theta = \tau\theta$ holds. Therefore, the equiaffine condition is equivalent to $\tau = 0$.

A transversal field satisfying (3) is called a *Blaschke normal field*, which is uniquely determined up to sign locally. The immersion $f: (M, \nabla) \to (\mathbf{R}^3, D)$ with Blaschke normal field is called *Blaschke immersion* and $h$ is called *affine metric*.

**Lemma 2.1** *The Laplacian of a Blaschke immersion, $\Delta f$ relative to the affine metric is equal to $2\xi$.*

**Definition 2.2** A Blaschke immersion $f$ is called an *improper* affine sphere if $S$ is identically 0. If $S = \lambda I$, where $\lambda$ is a nonzero constant, then $f$ is called a *proper* affine

An affine sphere has the following characteristic property (cf. [6, p. 43]), which helps us to discretize affine spheres.

**Lemma 2.3** *Let $f\colon M \to \boldsymbol{R}^3$ be a Blaschke immersion. Then $(f, M)$ is an improper affine sphere if and only if the Blaschke normals are parallel in $\boldsymbol{R}^3$, and $(f, M)$ is a proper affine sphere if and only if the Blaschke normals meet at one point in $\boldsymbol{R}^3$.*

Since a discretization of surfaces essentially depends on a choice of a coordinate system, we need consider separately the cases of which metric is indefinite or definite.

# 3 Discrete indefinite improper affine sphere

Assume now that a Blaschke immersion $f$ is an improper affine sphere and the affine metric $h$ is indefinite. We shall also say $f$ an indefinite improper affine sphere. We choose an asymptotic coordinate system $(D, (u, v))$ with respect to $h$ so that $h = 2\omega du dv$. By the volume condition, we have that $\omega(u, v) = \det(f_u, f_v, \xi)$. Applying if necessary a transformation $(u, v) \mapsto (v, -u)$, we can always achieve $\omega > 0$.

**Proposition 3.1** *Let $f\colon D \subset M \to \boldsymbol{R}^3$ be an indefinite improper affine sphere. Then the Gauss equations are as follows:*

$$f_{uu} = \frac{\omega_u}{\omega} f_u + \frac{a}{\omega} f_v, \tag{4}$$

$$f_{uv} = \omega \xi, \tag{5}$$

$$f_{vv} = \frac{b}{\omega} f_u + \frac{\omega_v}{\omega} f_v, \tag{6}$$

*where $\xi$ is a nonzero constant vector in $\boldsymbol{R}^3$, and three functions $a, b$ and $\omega$ satisfy the Gauss-Codazzi equations*

$$(\log \omega)_{uv} + ab\omega^{-2} = 0, \quad a_v = 0, \quad b_u = 0. \tag{7}$$

**Proof** We choose asymptotic coordinate system $(D, (u, v))$ and obtain

$$f_{uv} = \left| h(\partial/\partial u, \partial/\partial v) \right| \xi = \omega \xi$$

by Lemma 2.1. □

Since $a$ is a function only in $u$, we obtain $adu^3 = d\tilde{u}^3$, where $\tilde{u} = \int a^{1/3}du$. Namely, we can take $a = 1 = b$ without a loss of generality in the case that $ab \neq 0$. Then, the compatibility condition (7) is reduced to the Liouville equation

$$w_{uv} + e^{-2w} = 0, \tag{8}$$

where $\omega = e^w$. Consider the relations

$$(\widetilde{w} - w)_u = -\beta e^{-\widetilde{w}-w}, \quad (\widetilde{w} + w)_v = \frac{1}{\beta} e^{\widetilde{w}-w}, \tag{9}$$

where $\beta \in \mathbf{R}$ is a nonzero constant which is known as a Bäcklund parameter. The integrability condition of (9) produces $\widetilde{w}_{uv} = 0$. Thus, the implicit relations (9) give a link between the nonlinear equation and the linear equation. This connection may be exploited to solve the Liouville equation in full generality. Inserting the general solution $\widetilde{w}(u, v) = p(u) + q(v)$ into the Bäcklund relations (9), and subsequent integration produces a general solution of the Liouville equation in the form

$$f(u, v) = \xi \int_{u_0}^{u} \int_{v_0}^{v} \left( \beta \int_{s_0}^{s} e^{-2p(\sigma)} d\sigma + \frac{1}{\beta} \int_{t_0}^{t} e^{2q(\sigma)} d\sigma + \alpha \right) e^{p(s)-q(t)} \, dt \, ds + \eta(u) + \zeta(v),$$

where $\alpha \in \mathbf{R}$ is a constant and $\eta(u), \zeta(v)$ are vectors in $\mathbf{R}^3$.

In the case that $ab = 0$, we easily obtain

$$f(u, v) = \xi \int_{u_0}^{u} e^{p(\sigma)} d\sigma \int_{v_0}^{v} e^{q(\sigma)} d\sigma + \eta(u) + \zeta(v).$$

The following proposition is well known (cf. [6, pp. 92, 116]).

**Proposition 3.2** *If $f$ is a ruled improper affine sphere, then it is locally of the form $z = xy + \varphi(x)$, where $\varphi$ is an arbitarary function of $x$. Conversely, the graph of $z = xy + \varphi(x)$ is a ruled improper affine sphere.*

Now we discretize improper affine spheres in a purely geometric manner. For a map $F\colon \mathbf{Z}^2 \to \mathbf{R}^3$, we denote increments of the discrete variables by subscripts, namely,

$$F = F(n, m), \quad F_1 = F(n+1, m), \quad F_2 = F(n, m+1), \quad F_{12} = F(n+1, m+1).$$

Moreover, decrements are indicated by subscripts with overbars, that is

$$F_{\overline{1}} = F(n-1, m), \quad F_{\overline{2}} = F(n, m-1).$$

Taking the Gauss equations (4)–(6) into account, we give the following definition. Discretizing surfaces is nothing less than discretizing the coordinate system.

**Definition 3.3** A map $F\colon \mathbf{Z}^2 \to \mathbf{R}^3$ is called a *discrete indefinite improper affine sphere* if it has the following properties: every five points $F = F(n,m)$ and its neighbours $F_1, F_2, F_{\bar{1}}, F_{\bar{2}}$ lie on one plane. The vectors $F_{12} + F - F_1 - F_2$ are all parallel in $\mathbf{R}^3$.

**Proposition 3.4** *Let $F\colon \mathbf{Z}^2 \to \mathbf{R}^3$ be a discrete indefinite improper affine sphere. Then the discrete Gauss equations are as follows:*

$$
\begin{aligned}
&(10) & (F_1 - F) - (F - F_{\bar{1}}) &= \frac{\Omega - \Omega_{\bar{1}}}{\Omega}(F_1 - F) + \frac{A}{\Omega}(F_2 - F), \\
&(11) & F_{12} + F - F_1 - F_2 &= \Omega\, \Xi, \\
&(12) & (F_2 - F) - (F - F_{\bar{2}}) &= \frac{B}{\Omega}(F_1 - F) + \frac{\Omega - \Omega_{\bar{2}}}{\Omega}(F_2 - F),
\end{aligned}
$$

*where $\Xi$ is a nonzero constant vector in $\mathbf{R}^3$ and three functions $A, B$ and $\Omega$ satisfy the discrete Gauss-Codazzi equations*

$$(13) \qquad \Omega_{12}\Omega - \Omega_1\Omega_2 + A_1B_2 = 0, \quad A_2 - A = 0, \quad B_1 - B = 0.$$

*Moreover, these equations (10)–(13) become continuous ones (4)–(7) in the continuum limit*

$$(14) \qquad F = f, \quad \Omega = \omega\varepsilon_1\varepsilon_2, \quad A = a{\varepsilon_1}^3, \quad B = b{\varepsilon_2}^3,$$

*where smooth variables are correlated to discrete ones as $(u,v) = (\varepsilon_1 n, \varepsilon_2 m)$ for small positive numbers $\varepsilon_1$ and $\varepsilon_2$.*

**Proof** Since $F$ is a discrete indefinite improper affine sphere, there exist functions on $\mathbf{Z}^2$ such that

$$
\begin{aligned}
F_{11} - F_1 &= P(F_1 - F) + Q(F_{12} - F_1), \\
F_{12} + F - F_1 - F_2 &= \Omega\, \Xi, \\
F_{22} - F_2 &= R(F_2 - F) + S(F_{12} - F_2),
\end{aligned}
$$

where $\Xi$ is a nonzero constant vector in $\mathbf{R}^3$. The compatibility condition of $F$ is equivalent to the system

$$
\begin{aligned}
0 &= P_2 + Q_2 S - P, \\
0 &= Q_2 R - Q, \\
0 &= P_2 \Omega + Q_2 S \Omega + Q_2 \Omega_2 - Q\Omega - \Omega_1, \\
0 &= S_1 P - S, \\
0 &= R_1 + S_1 Q - R, \\
0 &= R_1 \Omega + S_1 Q \Omega + S_1 \Omega_1 - S\Omega - \Omega_2.
\end{aligned}
$$

The aimed equations (13) are obtained by setting $A_1 = Q\Omega$ and $B_2 = S\Omega$.

Next, we regard a discrete map $F$ as an approximation of a smooth map $f$, that is

$$F(n,m) = f(\varepsilon_1 n, \varepsilon_2 m)$$

for small $\varepsilon_1, \varepsilon_2$, then, the Taylor expansions

$$F_1 - F = \varepsilon_1 f_u + \frac{{\varepsilon_1}^2}{2} f_{uu} + O({\varepsilon_1}^3), \quad F_2 - F = \varepsilon_2 f_v + \frac{{\varepsilon_2}^2}{2} f_{vv} + O({\varepsilon_2}^3)$$

apply. Thus, the discrete Gauss and Gauss-Codazzi equations (10)–(13) produce continuous ones (4)–(7) in the continuum limit $\varepsilon_1, \varepsilon_2 \to 0$. □

In the case that $AB \neq 0$, the first equation of the systems (13) is locally written down as

$$2\sinh \frac{W_{12} - W_1 - W_2 + W}{2} + A_1 B_2 \exp \frac{-W_{12} - W_1 - W_2 - W}{2} = 0, \tag{15}$$

where $\Omega = \pm \exp W$.

**Remark 3.5** The equation (15) is exactly Hirota's discrete Liouville equation (2) when $AB = 1$. He constructed a discrete integrable analogue to the Liouville equation (8) without using any relation to geometry, and it has been revealed that his method to discretize nonlinear partial differential equations produces good difference ones in the view of discrete integrability. Thus, our discrete indefinite improper affine spheres are described in terms of discrete integrable systems.

In the case that $AB = 0$, we obtain the following theorem, which is a discrete analogue of Proposition 3.2.

**Theorem 3.6** *Let $F\colon \boldsymbol{Z}^2 \to \boldsymbol{R}^3$ be a ruled discrete indefinite improper affine sphere, that is, the points $F(n, m_0)$ lie on a line for any fixed integer $m_0$. Then it is locally of the form*

$$(n, m) \mapsto {}^t(n, m, nm + \Phi(n)),$$

*where $\Phi$ is an arbitarary sequence of $n$. Moreover it becomes the continuous graph $z = xy + \varphi(x)$ by taking an appropriate continuum limit.*

**Proof** We show that the difference systems (10)-(13) provide the theorem. From the Gauss equation (11), the vector $F(n, m)$ is of the form

$$F(n,m) = \begin{cases} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Omega(i,j)\ \Xi + F(n,0) + F(0,m) - F(0,0), & n, m > 0, \\ -\sum_{i=n}^{-1} \sum_{j=0}^{m-1} \Omega(i,j)\ \Xi + F(n,0) + F(0,m) - F(0,0), & n < 0, m > 0, \\ F(n,m), & nm = 0, \\ \sum_{i=n}^{-1} \sum_{j=m}^{-1} \Omega(i,j)\ \Xi + F(n,0) + F(0,m) - F(0,0), & n, m < 0, \\ -\sum_{i=0}^{n-1} \sum_{j=m}^{-1} \Omega(i,j)\ \Xi + F(n,0) + F(0,m) - F(0,0), & n > 0, m < 0. \end{cases}$$

If $AB \neq 0$, a discrete indefinite improper affine sphere $F$ cannot be ruled. Hence we can assume $B = 0$ without a loss of generality. The function $\Omega(n, m)$ is of the form

$$\Omega(0,0)\Omega(n,m) = P(n)Q(m),$$

where $P(n) = \Omega(n, 0)$ and $Q(m) = \Omega(0, m)$ are arbitrary one variable functions.

We can assume that the initial value $\Omega(0, 0)$ is equal to 1, and we set formally the summation $\sum_{k=k_2}^{k_1}$ to be always zero for $k_1 < k_2$. From the discrete Gauss equations (10) and (12), we have

$$F(0,m) = \begin{cases} \sum_{j=0}^{m-1} Q(j)\,(F(0,1) - F(0,0)) + F(0,0), & m > 0, \\ -\sum_{j=m}^{-1} Q(j)\,(F(0,1) - F(0,0)) + F(0,0), & m \leq 0, \end{cases}$$

$$F(n,0) = \begin{cases} \sum_{l=0}^{n-1} P(l)\,(F(1,0)-F(0,0)) \\ \quad +\sum_{l=0}^{n-1} P(l) \sum_{k=0}^{l-1} \big(A(k+1)/P(k)P(k+1)\big)\,(F(0,1)-F(0,0)) \\ \quad +\sum_{l=0}^{n-1} P(l) \sum_{k=0}^{l-1} \big(A(k+1)/P(k)P(k+1)\big) \sum_{i=0}^{k} P(i)\Xi + F(0,0), & n>0, \\ \\ -\sum_{l=n}^{-1} P(l)\,(F(1,0)-F(0,0)) \\ \quad +\sum_{l=n}^{-1} P(l) \sum_{k=l}^{-1} \big(A(k+1)/P(k)P(k+1)\big)\,(F(0,1)-F(0,0)) \\ \quad -\sum_{l=n}^{-1} P(l) \sum_{k=l}^{-1} \big(A(k+1)/P(k)P(k+1)\big) \sum_{i=k+1}^{-1} P(i)\Xi + F(0,0), & n\le 0. \end{cases}$$

Then we obtain the following expressions: in the case that $n \geq 0$ and $m \geq 0$,

$$\begin{aligned} F(n,m) &= \sum_{i=0}^{n-1} P(i)(F(1,0)-F(0,0)) \\ &+ \left(\sum_{j=0}^{m-1} Q(j) + \sum_{i=0}^{n-1} P(i) \sum_{k=0}^{i-1} \frac{A(k+1)}{P(k+1)P(k)}\right)(F(0,1)-F(0,0)) \\ &+ \left(\sum_{i=0}^{n-1} P(i) \sum_{j=0}^{m-1} Q(j) + \sum_{i=0}^{n-1} P(i) \sum_{k=0}^{i-1} \frac{A(k+1)}{P(k+1)P(k)} \sum_{l=0}^{k} P(l)\right) \Xi + F(0,0). \end{aligned}$$

In the case that $n \leq 0$ and $m \geq 0$,

$$\begin{aligned} F(n,m) &= -\sum_{i=n}^{-1} P(i)(F(1,0)-F(0,0)) \\ &+ \left(\sum_{j=0}^{m-1} Q(j) + \sum_{i=n}^{-1} P(i) \sum_{k=i}^{-1} \frac{A(k+1)}{P(k+1)P(k)}\right)(F(0,1)-F(0,0)) \\ &- \left(\sum_{i=n}^{-1} P(i) \sum_{j=0}^{m-1} Q(j) + \sum_{i=n}^{-1} P(i) \sum_{k=i}^{-1} \frac{A(k+1)}{P(k+1)P(k)} \sum_{l=k+1}^{-1} P(l)\right) \Xi + F(0,0). \end{aligned}$$

In the case that $n \leq 0$ and $m \leq 0$,

$$\begin{aligned} F(n,m) &= -\sum_{i=n}^{-1} P(i)(F(1,0)-F(0,0)) \\ &- \left(\sum_{j=m}^{-1} Q(j) - \sum_{i=n}^{-1} P(i) \sum_{k=i}^{-1} \frac{A(k+1)}{P(k+1)P(k)}\right)(F(0,1)-F(0,0)) \\ &+ \left(\sum_{i=n}^{-1} P(i) \sum_{j=m}^{-1} Q(j) - \sum_{i=n}^{-1} P(i) \sum_{k=i}^{-1} \frac{A(k+1)}{P(k+1)P(k)} \sum_{l=k+1}^{-1} P(l)\right) \Xi + F(0,0). \end{aligned}$$

In the case that $n \geq 0$ and $m \leq 0$,

$$
\begin{aligned}
F(n,m) &= \sum_{i=0}^{n-1} P(i)(F(1,0)-F(0,0)) \\
&\quad - \left(\sum_{j=m}^{-1} Q(j) - \sum_{i=0}^{n-1} P(i) \sum_{k=0}^{i-1} \frac{A(k+1)}{P(k+1)P(k)}\right)(F(0,1)-F(0,0)) \\
&\quad - \left(\sum_{i=0}^{n-1} P(i) \sum_{j=m}^{-1} Q(j) - \sum_{i=0}^{n-1} P(i) \sum_{k=0}^{i-1} \frac{A(k+1)}{P(k+1)P(k)} \sum_{l=0}^{k} P(l)\right) \Xi + F(0,0).
\end{aligned}
$$

Thus a ruled discrete indefinite improper affine sphere is locally the graph $(n,m) \longmapsto {}^t(n, m, nm + \Phi(n))$, where $\Phi$ is an arbitrary sequence of $n$.

Moreover, by regarding the functions $P, Q$ and $A$ as approximations of smooth functions $p, q$ and $a$, respectively, via

$$
\begin{aligned}
P(n) &= \exp\left(p\left(u_0 + n\frac{u-u_0}{k-1}\right)\right)\frac{u-u_0}{k-1}, \\
Q(m) &= \exp\left(q\left(v_0 + m\frac{v-v_0}{k-1}\right)\right)\frac{v-v_0}{k-1}, \\
A(n) &= a\left(u_0 + n\frac{u-u_0}{k-1}\right)\left(\frac{u-u_0}{k-1}\right)^3,
\end{aligned}
$$

we obtain

$$
\lim_{k\to\infty} \sum_{n=0}^{k-1} P(n) = \int_{u_0}^{u} e^{p(\sigma)} d\sigma, \quad \lim_{k\to\infty} \sum_{n=0}^{k-1} Q(m) = \int_{v_0}^{v} e^{q(\sigma)} d\sigma,
$$

and

$$
\lim_{k\to\infty} \sum_{n=0}^{k-1} \frac{A(n+1)}{P(n+1)P(n)} = \int_{u_0}^{u} d\sigma.
$$

Thus, $F$ becomes the smooth graph $z = xy + \varphi(x)$ as $k$ tends to infinity. □

# 4 Examples

We illustrate examples of discrete indefinite improper affine spheres.

**Example 4.1 (discrete hyperbolic paraboloid)** The graph $z = (x^2 - y^2)/2$ is called *hyperbolic paraboloid*. The Gauss equations are $f_{xx} = \xi$, $f_{xy} = 0$ and $f_{yy} = -\xi$, where

$\xi = {}^t(0,0,1)$. Hence a hyperbolic paraboloid is an indefinite improper affine sphere. We choose an asymptotic coordinate system $(u,v)$ and obtain $f(u,v) = {}^t(u+v, u-v, 2uv)$, where the Gauss equations are

$$f_{uu} = 0, \quad f_{uv} = 2\xi, \quad f_{vv} = 0.$$

We call the map

$$F(n,m) = {}^t(n+m, n-m, 2nm)$$

*discrete hyperbolic paraboloid.* The discrete Gauss equations are

$$(F_1 - F) - (F - F_{\bar{1}}) = 0, \quad F_{12} + F - F_1 - F_2 = 2\xi, \quad (F_2 - F) - (F - F_{\bar{2}}) = 0.$$

This is one of the simplest examples of discrete indefinite improper affine spheres.

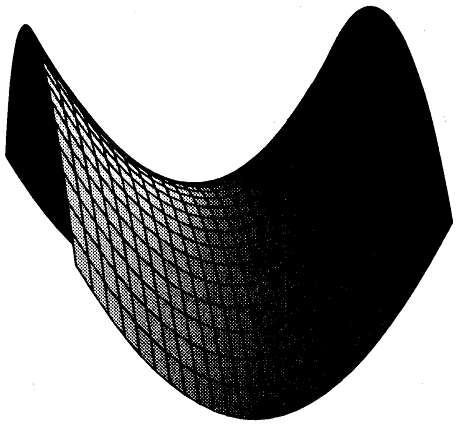


Figure 1: hyperbolic paraboloid

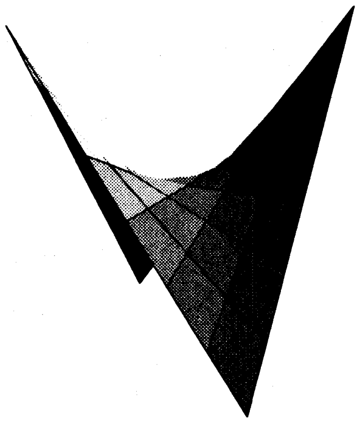


Figure 2: discrete hyperbolic paraboloid

**Example 4.2 (discrete Cayley surface)** The graph $z = xy - x^3/3$ is called *Cayley surface.* There is a simple characterization of the Cayley surface, namely, if the cubic form $C = \nabla h$ is not 0 and parallel relative to $\nabla$, a Blaschke immersion is affinely congruent to the Cayley surface. We call the map

$$F(n,m) = {}^t\left(n, \frac{n^2 - m^2}{2}, \frac{n^3 - 3nm^2}{6}\right)$$

*discrete Cayley surface.* This is a ruled discrete indefinite improper affine sphere.

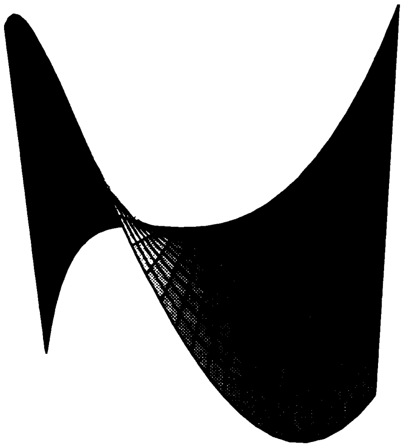


Figure 3: Cayley surface

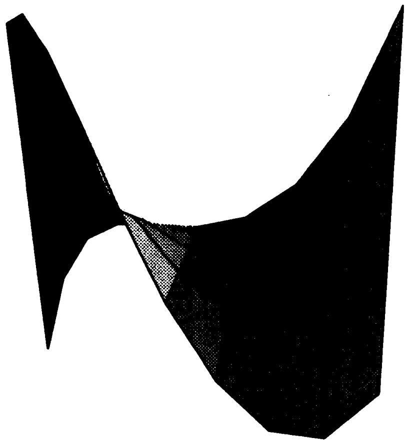


Figure 4: discrete Cayley surface

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