Harmonic Relations between Green's Functions and Green's Matrices for Boundary Value Problems II

1 Introduction

In previous papers [3], [4], we remarked that there is a harmonic relation between the Green functions \( G(x, \xi) \) for

\[
\begin{align*}
-\frac{d}{dx}(p(x)\frac{du}{dx}) &= f(x), \quad a < x < b \\
u(a) &= u(b) = 0, \quad p(x) > 0 \text{ in } [a, b]
\end{align*}
\]

and the Green matrix \( A_0^{-1} = (g_{ij}) \) for the discretized system

\[
\begin{align*}
a &= x_0 < x_1 < \cdots < x_n < x_{n+1} = b, \quad x_{i+\frac{1}{2}} &= \frac{1}{2}(x_i + x_{i+1}) \\
p_{i+\frac{1}{2}} &- p_{i-\frac{1}{2}} \quad \frac{U_{i+1} - U_i}{h_{i+1}} - \frac{U_i - U_{i-1}}{h_i} = f_i, \quad h_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n \\
U_0 &= U_{n+1} = 0
\end{align*}
\]

or

\[ HA_0 U = f \]

with

\[ H = \begin{pmatrix} \frac{2}{h_1 + h_2} & & \\ & \ddots & \\ & & \frac{2}{h_n + h_{n+1}} \end{pmatrix}, \]
$$A_0 = \begin{pmatrix} a_1 + a_2 & -a_2 \\ -a_2 & a_2 + a_3 & -a_3 \\ & \ddots & \ddots & \ddots \\ -a_n & a_n + a_{n+1} \end{pmatrix}, \quad a_i = \frac{1}{h_i}, \quad p_i = \frac{1}{h_i}$$  \tag{1.4}

$$U = (U_1, \cdots, U_n)^t, f = (f_1, \cdots, f_n)^t$$

In fact, we have

$$G(x_i, x_j) = \begin{cases} \left( \int_a^b \frac{ds}{p(s)} \right)^{-1} \int_a^x \frac{ds}{p(s)} \int_x^b \frac{ds}{p(s)} & (x \leq \xi) \\ \left( \int_a^b \frac{ds}{p(s)} \right)^{-1} \int_a^\xi \frac{ds}{p(s)} \int_\xi^b \frac{ds}{p(s)} & (x \geq \xi) \end{cases}$$

and

$$g_{ij} = \begin{cases} \left( \sum_{k=1}^{n+1} \frac{h_k}{p_k} \right)^{-1} \left( \sum_{k=1}^i \frac{h_k}{p_k} \right) \left( \sum_{k=j+1}^{n+1} \frac{h_k}{p_k} \right) & i \leq j \\ \left( \sum_{k=1}^{n+1} \frac{h_k}{p_k} \right)^{-1} \left( \sum_{k=1}^j \frac{h_k}{p_k} \right) \left( \sum_{k=i+1}^{n+1} \frac{h_k}{p_k} \right) & i \geq j \end{cases}$$

Hence,

$$G(x_i, x_j) = g_{ij} + O(h^2) \quad \forall i, j, \quad h = \max_i h_i,$$

if $p \in C^{1,1}[a, b]$. On the other hand, the finite element approximation $v_n = \sum_{i=1}^{n} U_i \phi_i$ with piecewise linear polynomials is determined by solving

$$\sum_{j=1}^{n} \left( \int_a^b p(x) \phi'_i \phi'_j dx \right) \hat{U}_j = \int_a^b f(x) \phi_i(x) dx, \quad i = 1, 2, \cdots, n \tag{1.5}$$

with respect to $\hat{U}_j$, where $\phi_i, i = 1, 2, \cdots, n$ are piecewise linear polynomials satisfying $\phi_i(x_j) = \delta_{ij}$. The equations (1.5) can be written in the matrix-vector form

$$\hat{A} \hat{U} = \hat{f},$$

where $\hat{A} = A_0$ is obtained by replacing $a_i$ in (1.4) by

$$\hat{a}_i = \frac{1}{h_i}, \quad \rho_i = \frac{1}{h_i}, \quad p_i = \int_{x_{i-1}}^{x_i} p(x) dx, \quad f = (\hat{f}_1, \cdots, \hat{f}_n), \quad \hat{f}_i = \int_{x_{i-1}}^{x_i} f(x) \phi_i(x) dx.$$
Then it can also be shown that $A^{-1} = (g_{ij})$ satisfies
\[
\hat{g}_{ij} = \left\{\begin{array}{ll}
\left(\sum_{k=1}^{n+1} \frac{h_k}{\rho_k}\right)^{-1} \left(\sum_{k=1}^{i} \frac{h_k}{\rho_k}\right) \left(\sum_{k=j+1}^{n+1} \frac{h_k}{\rho_k}\right) & i \leq j \\
\left(\sum_{k=1}^{n+1} \frac{h_k}{\rho_k}\right)^{-1} \left(\sum_{k=1}^{j} \frac{h_k}{\rho_k}\right) \left(\sum_{k=i+1}^{n+1} \frac{h_k}{\rho_k}\right) & i \geq j,
\end{array}\right.
\]
which indicates a similar harmony between the Green function $G(x, \xi)$ and the corresponding discrete Green function:
\[
G(x_i, x_j) = \hat{g}_{ij} + O(h^2) \quad \forall i, j,
\]
if $p \in C^{1,1}[a, b]$.

The purpose of this paper is to establish a similar relation for the Green function $G(x, \xi)$ for
\[
Lu \equiv -\frac{d}{dx} (p(x) \frac{du}{dx}) + q(x) \frac{du}{dx} + r(x)u = f(x), a < x < b \quad (1.6)
\]
\[
u(a) = u(b) = 0
\]
and the discrete Green function $G_h(x_i, x_j)$ (Green matrix) for the discretized system
\[
L_h U \equiv \frac{p_{i+\frac{1}{2}} U_{i+1} - U_i}{h_{i+1}} - \frac{p_{i-\frac{1}{2}} U_i - U_{i-1}}{h_i} + q_i \frac{U_{i+1} - U_{i-1}}{2h_{i+1} + h_i} + r_i U_i = f_i, \quad i = 1, 2, \ldots, n \quad (1.7)
\]
\[
U_0 = U_{n+1} = 0,
\]
provided that $p(x) \in C^{3,1}, q(x), r(x) \in C^{1,1}[a, b], p(x) > 0, r(x) \geq 0$ in $[a, b]$

2 Results

The discrete Green function $G_h(x_i, x_j)$ for the operator $L_h$ is defined as the solution of the linear system
\[
L_h G_h(x_i, x_j) = \frac{2}{h_{j+1} + h_j} \delta_{ij}, \quad i, j = 1, 2, \ldots, n
\]
\[
G_h(x_i, x_j) = 0, i = 0, n + 1, \quad 1 \leq j \leq n,
\]
where $\delta_{ij}$ stands for the Kronecker symbol. This means that the $n \times n$ matrix $(G_h(x_i, x_j))$ is the inverse of the matrix $A = A_0 + Q + D$, where $A_0$ is defined by (1,3),

$$Q = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\frac{q_1}{2} & \frac{q_2}{2} & \cdots & \frac{q_{n-1}}{2} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{q_n}{2} & 0 & \cdots & 0 \end{pmatrix}, \quad D = \begin{pmatrix} r_1(h_1+h_2) \frac{2}{2} \\ r_2(h_2+h_3) \frac{2}{2} \\ \cdots \\ r_{n}(h_{n}+h_{n+1}) \frac{2}{2} \end{pmatrix}.$$

We first prove the following lemma:

**Lemma 2.1.** Given positive integers $N_a$ and $N_b$, we have

$$\sum_{j=1}^{n} G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} = \begin{cases} O(h) & \text{if } i \leq N_a \text{ or } i \geq n+1-N_b, \\ O(1) & \text{otherwise.} \end{cases}$$

**Proof.** Let $\phi(x) \in C^2[a, b]$ be the solution of the problem $Lu = 1$ in $(a, b)$ and $u(a) = u(b) = 0$. Then we have

$$\sum_{j=1}^{n} G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} \leq 2\phi(x_i) \quad \forall i$$

(cf.Matsunaga-Yamamoto[2]), which proves Lemma 2.1. $\square$

Then we have the following result.

**Theorem 2.2.** If $p \in C^{3,1}[a, b]$, $q(x), r(x) \in C^{11}[a, b]$, $p(x) > 0$, $r(x) \geq 0$ in $[a, b]$, then

$$G_h(x_i, x_j) - G(x_i, x_j) = \begin{cases} O(h^3) & (i \leq N_a \text{ or } i \geq n+1-N_b), \\ O(h^2) & \text{(otherwise).} \end{cases}$$

**Proof.** Let $\{V_i\}$ be any mesh function defined on $I = \{x_0, x_1, \cdots, x_n, x_{n+1}\}$ such that $V_0 = V_{n+1} = 0$. Then it is easy to see

$$V_i = \sum_{j=1}^{n} G_h(x_i, x_j) \frac{h_j + h_{j+1}}{2} L_h V_j, \quad i = 1, 2, \cdots, n$$

Hence

$$G(x_i, x_j) = \sum_{k=1}^{n} G_h(x_i, x_k) \frac{h_k + h_{k+1}}{2} L_h G(x_k, x_j), \quad i, j = 1, 2, \cdots, n \quad (2.1)$$
Furthermore, a careful computation leads to

\[ L_h G(x_k, x_j) = \begin{cases} \frac{2}{h_{k+1}}[(h_k^2 - h_{k+1}^2)\phi_{kj} + O(h_k^3 + h_{k+1}^3)] & (k \neq j) \\ \frac{2}{h_{j+1}}[1 + (h_{j+1}^2 \phi_j^+ - h_j^2 \phi_j^-) + O(h_j^3 + h_{j+1}^3)] & (k = j), \end{cases} \]

where

\[
\phi_{kj} = \frac{1}{6} p_k \frac{\partial^3 G(x_k, x_j)}{\partial x^3} + \frac{1}{4} (p'_k - q_k) \frac{\partial^2 G(x_k, x_j)}{\partial x^2} + \frac{1}{8} p'_k \frac{\partial G(x_k, x_j)}{\partial x}, \\
\phi_j^+ = \frac{1}{4} q_j \frac{\partial^2 G(x_j + 0, x_j)}{\partial x^2} - \frac{1}{8} p_j' \frac{\partial G(x_j + 0, x_j)}{\partial x} - \frac{1}{4} p_j' \frac{\partial^2 G(x_j + 0, x_j)}{\partial x^2} - \frac{1}{6} p_j \frac{\partial^3 G(x_j + 0, x_j)}{\partial x^3}, \\
\phi_j^- = \frac{1}{4} q_j \frac{\partial^2 G(x_j - 0, x_j)}{\partial x^2} - \frac{1}{8} p_j' \frac{\partial G(x_j - 0, x_j)}{\partial x} - \frac{1}{4} p_j' \frac{\partial^2 G(x_j - 0, x_j)}{\partial x^2} - \frac{1}{6} p_j \frac{\partial^3 G(x_j - 0, x_j)}{\partial x^3}.
\]

Substituting this relation into (2.1) yields

\[ G(x_i, x_j) = \sum_{k \neq j}^{n} G_h(x_i, x_k) \{(h_k^2 - h_{k+1}^2)\phi_{kj} + O(h_k^3 + h_{k+1}^3)\} + G_h(x_i, x_j)\{1 + (\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h_{j+1}^3 + h_j^3)\} \]

or

\[ G(x_i, x_j) - G_h(x_i, x_j) = \sum_{k \neq j}^{n} G_h(x_i, x_j)\{(h_k^2 - h_{k+1}^2)\phi_{kj} + O(h_k^3 + h_{k+1}^3)\} + G_h(x_i, x_j)\{(\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h_{j+1}^3 + h_j^3)\} \quad (2.2) \]

Hence there exists a constant \( C_1 > 0 \) such that

\[ |G(x_i, x_j) - G_h(x_i, x_j)| \leq C_1 h \sum_{k \neq j}^{n} G_h(x_i, x_k)(h_k + h_{k+1}) + O(h^2), \]

and, by Lemma 2.1, we have

\[ G_h(x_i, x_j) = G(x_i, x_j) + O(h). \quad (2.3) \]
Substituting this into (2.2) we have

\[
G(x_i, x_j) - G_h(x_i, x_j) = \sum_{k \neq j} G(x_i, x_k)(h_k^2 - h_{k+1}^2)\phi_{kj} + G(x_i, x_j)(\phi_j^+ h_{j+1}^2 - \phi_j^- h_j^2) + O(h^2)
\]

\[
= h_1^2 G(x_i, x_1)\phi_{1j} + \sum_{k=1}^{j-2} h_{k+1}^2 \left[ G(x_i, x_{k+1})\phi_{k+1j} - G(x_i, x_k)\phi_{kj} \right] 
- h_j^2 \left[ G(x_i, x_{j-1})\phi_{j-1j} + G(x_i, x_j)\phi_j^- \right] 
+ h_{j+1}^2 \left[ G(x_i, x_{j+1})\phi_{j+1j} + G(x_i, x_{j+1})\phi_{j+1j} \right] 
+ \sum_{k=j+1}^{n-1} h_{k+1}^2 \left[ G(x_i, x_{k+1})\phi_{k+1j} - G(x_i, x_k)\phi_{kj} \right] 
- h_{n+1}^2 \left[ G(x_i, x_n)\phi_{nj} \right] + O(h^2)
= O(h^2) \quad \text{(an improvement of (2.3))}
\]

since

\[
G(x_i, x_{k+1})\phi_{k+1j} - G(x_i, x_k)\phi_{kj} = [G(x_i, x_{k+1}) - G(x_i, x_k)]\phi_{k+1j} + G(x_i, x_k)[\phi_{k+1j} - \phi_{kj}]
= O(h_{k+1})\phi_{k+1j} + G(x_i, x_k)h_{k+1} = O(h_{k+1}), \quad \text{etc.}
\]

Replacing \(O(h)\) in (2.3) by \(O(h^2)\) and repeating similar argument as above, we obtain for \(i \leq N_a\) or \(i \geq n + 1 - N_b\)

\[
G(x_i, x_j) - G_h(x_i, x_j) = O(h^3).
\]

This proves Theorem 2.2. \(\square\)

We can apply Theorem 2.2 to derive the superconvergence of the Shortley-Weller approximation applied to the semilinear problem

\[
\begin{aligned}
&-\frac{d}{dx}(p(x)\frac{du}{dx}) + q(x)\frac{du}{dx} + f(x, u) = 0, \quad a < x < b \\
u(a) = \alpha, \quad u(b) = \beta
\end{aligned}
\]

with any nodes (1.2):

**Theorem 2.3.** In addition to the assumptions of Theorem 2.2, assume that \(f\) is continuous on \(\mathcal{R} : a \leq x \leq b, -\infty < u < +\infty\). Furthermore, assume that \(f\) is continuously differentiable with respect to \(u\) on \(\mathcal{R}\) and \(f_u \geq 0\). Then the finite difference method

\[
\begin{aligned}
\left\{ & -\frac{p_{i+\frac{1}{2}}}{h_{i+1}} \frac{U_{i+1} - U_i}{h_{i+1}} - \frac{p_{i-\frac{1}{2}}}{h_i} \frac{U_i - U_{i-1}}{h_i} + q_i \frac{U_{i+1} - U_{i-1}}{h_i + h_{i+1}} + f(x_i, U_i) = 0, \quad i = 1, 2, \ldots, n, \quad (2.6) \\
U_0 = \alpha, \quad U_{n+1} = \beta
\end{aligned}
\]


for solving (2.4)-(2.5) is superconvergent with any nodes (1.2):

\[ u_i - U_i = \begin{cases} 
O(h^3), & i \in \Gamma = \{1, 2, \cdots, N_a, n - N_b + 1, n - N_b + 2, \cdots, n\} \\
O(h^2), & i \notin \Gamma
\end{cases} \]

as \( h \to 0 \), where \( N_a \) and \( N_b \) are arbitrary given positive integers.

Remark. If the boundary conditions (2.5) are replaced by

\[ \alpha_1 u(a) + \alpha_2 u'(a) = \alpha \quad \text{and} \quad \beta_1 u(b) + \beta_2 u'(b) = \beta, \]

where \( \alpha_2 \beta_2 \neq 0 \), \( \alpha_1 \alpha_2 \geq 0 \) and \( \beta_1 \beta_2 \geq 0 \), then it can be shown that the corresponding Shortley-Weller approximation (2.6) is quadratic convergent with any nodes (1.2):

\[ u_i - U_i = O(h^2) \quad \forall i \]

as \( h \to 0 \). However, superconvergence can not be expected in general.

References


