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On fully nonlinear PDEs with quadratic nonlinearity

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1 Introduction

We are concerned with L^p-viscosity solutions of fully nonlinear, second order, uniformly elliptic PDEs:

$$F(x, Du(x), D^2u(x)) = f(x) \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $F : \Omega \times \mathbf{R}^n \times S^n \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$ are given functions. Here, S^n denotes the set of real-valued symmetric $n \times n$ matrices equipped with the standard ordering. We will use the notion $B_r = \{x \in \mathbf{R}^n \mid |x| < r\}$ for $r > 0$.

We refer [5] and [12] for the viscosity solution theory of fully nonlinear, second order, (possibly degenerate) elliptic PDEs.

Throughout this paper, we freeze the constants $0 < \lambda \leq \Lambda$. By using these, the uniform ellipticity means the following property:

$$(A1) \qquad \mathcal{P}^-_{\lambda,\Lambda}(X - Y) \leq F(x, q, X) - F(x, q, Y) \leq \mathcal{P}^+_{\lambda,\Lambda}(X - Y)$$

for any $(x, q, X, Y) \in \Omega \times \mathbf{R}^n \times S^n \times S^n$, where $\mathcal{P}^\pm_{\lambda,\Lambda} : S^n \to \mathbf{R}$ are given by

$$\mathcal{P}^-_{\lambda,\Lambda}(X) = \min_{\lambda I \leq A \leq \Lambda I}\{-\mathrm{trace}(AX)\} \quad \text{and} \quad \mathcal{P}^+_{\lambda,\Lambda}(X) = \max_{\lambda I \leq A \leq \Lambda I}\{-\mathrm{trace}(AX)\}$$

for $X \in S^n$. In what follows, we shall write $\mathcal{P}^\pm$ for $\mathcal{P}^\pm_{\lambda,\Lambda}$ since we have fixed λ and Λ.

We notice that the following relation holds: for X and $Y \in S^n$,

$$\mathcal{P}^-(X) + \mathcal{P}^-(Y) \leq \mathcal{P}^-(X + Y) \leq \mathcal{P}^-(X) + \mathcal{P}^+(Y) \leq \mathcal{P}^+(X + Y) \leq \mathcal{P}^+(X) + \mathcal{P}^+(Y).$$

When F does not have the divergence structure, less is known on the regularity of solutions of (1). Moreover, before the viscosity solution theory was born, we did not know what was the correct notion of weak solutions for (1).

Since there are many works when the mapping $x \to F(x, q, X)$ is supposed to be continuous, we shall forcus our attention to the case when

$$(A2) \qquad \text{the mapping } x \to F(x, q, X) \text{ is measurable for any fixed } (q, X) \in \mathbf{R}^n \times S^n.$$

We only refer Trudinger's works [29], [30], [31] for the regularity of viscosity solutions of (1) when the mapping $x \to F(x, q, X)$ is continuous.

Under these hypotheses, even when $(q, X) \to F(x, q, X)$ is linear, we only have a few results: Initiated by the pioneering work in terms of probability by Krylov-Safonov in [22], Trudinger gave a "PDE" proof of the Hölder estimates for strong solutions of (1) in [29].

When $(q, X) \to F(x, q, X)$ is fully nonlinear and linear growth, Caffarelli in [2] (see also the book [1]) showed the Hölder estimate and, moreover, assuming "VMO" continuity for coefficients, $W^{2,p}$-estimates for viscosity solutions. We also refer the book [23] for the regularity theory of strong solutions when coefficients are of VMO type.

However, by a technical reason, we have to suppose that the right hand side f is continuous. In fact, in various situations, we need to find solutions of

$$\mathcal{P}^{\pm}(D^2 u) = f \quad \text{in } \Omega, \tag{2}$$

under the Dirichlet condition $u = 0$ on $\partial\Omega$ as the so-called test functions in the viscosity solution theory. We refer Evans' works [10] and [11] for the existence of classical solutions of (2) when f is smooth.

Unfortunately, when $f \in L^p(\Omega)$, we can only expect the solution u of (2) belongs to $W^{2,p}(\Omega)$ but $C^2(\Omega)$. Recalling that the set of test functions in the standard viscosity solution theory is C^2, we need a bit wider class of test functions when we intend to study this case since we will have to use (strong) solutions of (2).

Here, we refer a series of works by L. Wang, [32], [33], [34], [35], for the parabolic case.

Recently, Caffarelli-Crandall-Kocan-Święch [3] introduced a new notion "L^p-viscosity solutions" (which is a bit stronger than the standard one) to be able to recover Caffarelli's results to the case of $f \in L^p(\Omega)$. We refer [3], [7], [4], [27], [15], [13], [6], [8], [16] for the recent development of L^p-viscosity solutions.

On the other hand, it is important to study uniformly elliptic PDEs with quadratic growth in Du. We only mention several applications such as risk-sensitive stochastic control problems, large deviation problems, etc.

For simplicity, we suppose that there is $\mu > 0$ such that

$$(A3) \qquad |F(x, q, O)| \leq \mu |q|^2 \quad \text{for } (x, q) \in \Omega \times \mathbf{R}^n.$$

When (A3) is switch to the case when $|F(x, q, O)| \leq \mu |q|^{2-\varepsilon}$ for $\varepsilon \in (0, 1)$, in [19], we verify that Caffarelli's argument works to get the Hölder estimate provided that the L^∞-bound of L^p-viscosity solutions is known. Later, in [21], we obtain the same result as in [19] even under assumption (A3).

Thus, our questions here are as follows: assuming (A1)-(A3),

(i) can we get the L^∞-bound for L^p-viscosity solutions ?
(ii) if not, under which condition, can we get the L^∞-estimate of L^p-viscosity solutions ?
(iii) how about the existence of L^p-viscosity solutions ?

Here, we recall the notion of L^p-viscosity solutions for $p > n/2$.

Definition $u \in C(\Omega)$ is called an L^p-viscosity subsolution (resp., supersolution) of (1) if for any $\phi \in W^{2,p}_{loc}(\Omega)$ such that $u - \phi$ attains its local maximum (resp., minimum) at $x \in \Omega$, it holds that

$$ess \liminf_{y \to x} \Big(F(y, D\phi(y), D^2\phi(y)) - f(y)\Big) \leq 0$$

$$\left(\text{resp.,} \quad ess \limsup_{y \to x} \Big(F(y, D\phi(y), D^2\phi(y)) - f(y)\Big) \geq 0\right).$$

Also, $u \in C(\Omega)$ is called an L^p-viscosity solution of (1) if it is an L^p-viscosity sub- and supersolution of (1).

2 Nagumo's results

In this section, we recall some known facts from [25].

In [25], Nagumo gave an existence result of classical solutions for "principally" linear (*i.e.* linear in the variable D^2u) PDEs with quadratic growth in Du. For this purpose, he supposed that there exist a "quasi" subsolution $\underline{\omega}$ and "quasi" supersolution $\overline{\omega}$ such that $\underline{\omega} < \overline{\omega}$ in Ω, and $||\underline{\omega}||_\infty, ||\overline{\omega}||_\infty \leq M$ for some $M > 0$ satisfying that

$$M\mu\Lambda < \frac{1}{16}. \tag{3}$$

In the above, a quasi subsolution $\underline{\omega}$ (resp., supersolution $\overline{\omega}$) means the point-wise maximum (resp., minimum) of a finite number of (local) classical subsolutions (resp., supersolutions); roughly speaking,

$$\underline{\omega}(x) = \max_{i=1,2,\ldots,k} u_i(x) \quad \text{such that} \quad F(x, Du_j(x), D^2u_j(x)) \leq 0 \quad j \in \{1, 2, \ldots, k\}$$

$$\left(\text{resp.,} \quad \overline{\omega}(x) = \min_{i=1,2,\ldots,k} u_i(x) \quad \text{such that} \quad F(x, Du_j(x), D^2u_j(x)) \geq 0 \quad j \in \{1, 2, \ldots, k\}\right).$$

We also recall a Nagumo's example in [25] for the non-existence of solutions when the growth order of the mapping $q \to F(x, q, O)$ is more than quadratic (although in the example below contains the u-dependence). We refer a similar example in [17] (p. 23), which indicates the non-existence of solutions for the super-quadratic case.

Example 1. Setting $\Omega = B_2 \setminus \overline{B}_1$, we consider the following PDE:

$$\begin{cases} -\triangle u + (n-1)|x|^{-2}\langle x, Du\rangle + u(1+|Du|^2)^{1+\varepsilon} = 0 & \text{in } \Omega, \\ u(x) = 0 & \text{for } |x| = 1, \\ u(x) = h & \text{for } |x| = 2, \end{cases}$$

where the constant $h > 0$ will be fixed later. Because of the uniqueness, the solution u is radial, and $v(|x|) := u(x)$ satisfies that

$$v'' = v(1+|v'|^2)^{1+\varepsilon} \quad \text{in } (1,2).$$

Thus, $(1+|v'|^2)^{-\varepsilon} = \varepsilon(C_h - v^2)$, where $C_h = C_h(\varepsilon)$ will be defined. Since $v^2(r) \le C_h$ for $r \in (1,2)$, we get

$$h^2 \le C_h.$$

Moreover, we have $\varepsilon^{\frac{1}{2\varepsilon}}(C_h - v^2)^{\frac{1}{2\varepsilon}} v' \le 1$. Hence, we have

$$1 \ge \varepsilon^{\frac{1}{2\varepsilon}} \int_0^h (C_h - v^2)dv \ge \varepsilon^{\frac{1}{2\varepsilon}} \int_0^{h/\sqrt{2}} (h^2 - v^2)dv \ge \varepsilon^{\frac{1}{2\varepsilon}} \frac{5h^3}{6\sqrt{2}}.$$

Therefore, for fixed $\varepsilon > 0$, by taking large $h > 0$ (*i.e.* C_h is also large), the above inequality does not hold true.

3 Maximum principle

In this section, we first give a counter-example for which the maximum principle does not hold when $\mu > 0$ in (A3).

Example 2. ([22]) Setting $\Omega = B_1$, for $\varepsilon \in (0,1)$, we define $u_\varepsilon \in C(\overline{\Omega}) \cap C^2(\Omega)$ by

$$u_\varepsilon(x) = \begin{cases} 2\log(2-(2-\varepsilon)|x|) - 2\log\varepsilon & \text{provided } x \in \overline{\Omega} \setminus B_{(2-\varepsilon)^{-1}}, \\ 1 - (2-\varepsilon)^2|x|^2 - 2\log\varepsilon & \text{provided } x \in B_{(2-\varepsilon)^{-1}}. \end{cases}$$

It is easy to check that u_ε is a classical subsolution of

$$-\triangle u_\varepsilon - n|Du_\varepsilon|^2 \le 8n =: f \quad \text{in } \Omega$$

with $u_\varepsilon(x) = 0$ for $x \in \partial\Omega$. However, we cannot find a universal constant $C > 0$ (*i.e.* independent of $\varepsilon > 0$) such that

$$\max_{\overline{\Omega}} u_\varepsilon \le C\|f\|_{L^n(\Omega)} \tag{4}$$

because $\max_{\overline{\Omega}} u_\varepsilon = 1 - 2\log\varepsilon \to \infty$ as $\varepsilon \to 0$.

Remark. It is not hard to construct a counter-example when $|F(x,q,O)| \leq \mu|q|^\alpha$ holds for any fixed $\alpha > 1$ instead of (A3). However, in the above example for instance, we do not know if we can find a universal $C > 0$ so that (4) holds true for solutions (not only subsolutions).

Setting d_0 =diam(Ω), we present our main result here:

Theorem 1. ([22]) *Assume that (A1), (A2) and (A3) hold. Fix $p > n$. Then, there are $\delta = \delta(\lambda, \Lambda, n, p) > 0$ and $C = C(\lambda, \Lambda, n, p) > 0$ such that if*

$$\mu d_0^{2-\frac{n}{p}} \|f^+\|_{L^p(\Omega)} \leq \delta \quad \left(\text{resp., } \mu d_0^{2-\frac{p}{n}} \|f^-\|_{L^p(\Omega)} \leq \delta\right), \tag{5}$$

and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution (resp., supersolution) of (1), then

$$\max_{\overline{\Omega}} u^+ \leq \max_{\partial\Omega} u^+ + C d_0^{2-\frac{n}{p}} \|f^+\|_{L^p(\Omega)}$$

$$\left(\text{resp., } \max_{\overline{\Omega}} u^- \leq \max_{\partial\Omega} u^- + C d_0^{2-\frac{n}{p}} \|f^-\|_{L^p(\Omega)}\right).$$

Remarks. We can extend this result to the case when $p \in (p_0, n]$, where $p_0 = p_0(\lambda, \Lambda, n) > n/2$ is a constant derived by Escauriaza [9] although the above estimate becomes a bit complicated. To prove the assertion for $p \in (p_0, n]$, we have to use the argument below for that of $p > n$ in a "bootsrap way".

We will obtain our existence result under assumption (5). Thus, since we construct a solution between $\underline{\omega}$ and $\overline{\omega}$ in [25], our sufficient condition (5) is similar to Nagumo's (3).

Sketch of proof of Theorem 1.

Step 1: Let us suppose that $0 \in \Omega$ and set $B = B_{2d_0}$.
To avoid the lack of L^∞-bound of the right hand side "f", we use the strong supersolution $w \in C(\overline{B}) \cap W^{2,p}_{loc}(B)$ of

$$\begin{cases} \mathcal{P}^-(D^2 w) = g & \text{in } B, \\ \quad w = 0 & \text{on } \partial B, \end{cases}$$

where

$$g(x) = \begin{cases} f^+(x) + d_0^{-\frac{n}{p}} \|f^+\|_{L^p(\Omega)} & \text{for } x \in \Omega, \\ 0 & \text{for } x \in B \setminus \Omega. \end{cases}$$

By [3], we can find a strong supersolution of the above satisfying that

$$0 \le w \le C d_0^{2-\frac{n}{p}} \|f^+\|_{L^p(\Omega)} \quad \text{in } B,$$

and, by remembering $W^{2,p}$ is imbedded in $C^{1,\alpha}$ for some $\alpha \in (0,1)$,

$$\|Dw\|_{L^\infty(\Omega)} \le d_0^{1-\frac{n}{p}} \|f^+\|_{L^p(\Omega)}.$$

Setting $v = u - w - \max_{\partial\Omega} u^+$, we easily (at least formally) verify that

$$\mathcal{P}^-(D^2 v) \le 2\mu |Dv|^2 + d_0^{-\frac{n}{p}} \|f^+\|_{L^p(\Omega)} \left(C\mu d_0^{2-\frac{n}{p}} \|f^+\|_{L^p(\Omega)} - 1 \right).$$

Thus, we can find $\delta > 0$ such that if (5) holds, then v is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2 v) - \mu |Dv|^2 = 0 \quad \text{in } \Omega,$$

with $\max_{\partial\Omega} v \le 0$.

Step 2: By [14], we find functions $\psi_m \in C^\infty$ such that

$$\lim_{m\to\infty} \psi_m = 0 \quad \text{uniformly in } \overline{\Omega},$$

and, by setting $v_m = v + \psi_m$, v_m is an L^p-visocosity subsolution of

$$\mathcal{P}^-(D^2 v_m) - \mu |Dv_m|^2 = -\frac{\lambda}{2} \quad \text{in } \Omega.$$

From the definition (with the test function $\phi \equiv 0$) of v_m, we have

$$\max_{\overline{\Omega}} v_m = \max_{\partial\Omega} v_m.$$

Sending $m \to \infty$, we have

$$\max_{\overline{\Omega}} v = \max_{\partial\Omega} v \ (\le 0),$$

which implies the assertion by (6).

4 Existence of L^p-viscosity solutions

For the existence result, we suppose the following continuity in Du-variable:

(A4) $|F(x,q,X) - F(x,q',X)| \le \mu(|q|+|q'|)|q-q'|$ for $(x,q,q',X) \in \Omega \times \mathbf{R}^n \times \mathbf{R}^n \times S$

Our existence result is as follows:

Theorem 2. *Assume that (A1), (A2), (A3) and (A4) hold. Fix $p > p_0$. Let $g \in C(\partial\Omega)$ be given.*

Then, there exists $\delta = \delta(\lambda, \Lambda, n, p) > 0$ such that if

$$\mu d_0^{2-\frac{n}{p}} \|f\|_{L^p(\Omega)} \leq \delta, \tag{7}$$

holds, then there exists an L^p-viscosity solution $u \in C(\overline{\Omega})$ of (1) with $u = g$ on $\partial\Omega$.

Sketch of proof of Theorem 2.

For simplicity, we give our strategy of the proof when g is smooth and $p > n$.

Step 1: Approximate F by F_j which satisfies linear growth (the rate depends on j) and (A3); for $q = (q_1, \ldots, q_n) \in \mathbf{R}^n$, we define $q^j = (q_1^j, \ldots, q_n^j)$ by

$$q_i^j = \begin{cases} j & \text{provided } q_i \geq j, \\ q_i & \text{provided } |q_i| < j, \\ -j & \text{provided } q_i \leq -j. \end{cases} \qquad \text{for } i \in \{1, \ldots, n\}.$$

Then, we set

$$F_j(x, q, X) = F(x, q^j, X) \quad \text{for } (x, q, X) \in \Omega \times \mathbf{R}^n \times S^n.$$

Using the result in [6], we then solve the L^p-viscosity solutions u_j of

$$F_j(x, Du, D^2u) = f_j \quad \text{in } \Omega, \tag{8}$$

under Dirichlet condition $u_j = g$ on $\partial\Omega$.

Step 2: In view of Theorem 1, we obtain the L^∞ estimate of u_j because (7) holds. Hence, we can apply the Hölder estimate in [21] to get the equi-continuity of u_j. Thus, we can find $u \in C(\overline{\Omega})$ such that u_j converges to u uniformly in $\overline{\Omega}$ by taking a subsequence u_{j_k} if necessary.

Step 3: Applying the following stability result, we verify that u is an L^p-viscosity solution of (1).

Theorem 3. *Assume that F, $F_j : \Omega \times \mathbf{R}^n \times S^n \to \mathbf{R}$ satisfy (A1), (A2), and (A3). Assume also that F_j satisfies (A4).*

Fix $p > p_0$. Let $f_j,\ f \in L^p(\Omega)$ be given. Let $u_j \in C(\Omega)$ be an L^p-viscosity subsolution (resp., supersolution) of (8).

Assume also that $u_j \to u$ locally uniformly in Ω, as $j \to \infty$, and that for $B_{2r}(x) \subset \Omega$ and $\phi \in W^{2,p}(B_r(x))$,

$$\|(F(\cdot, D\phi(\cdot), D^2\phi(\cdot)) - f(\cdot) - F_j(\cdot, D\phi(\cdot), D^2\phi(\cdot)) + f_j(\cdot))^+\|_{L^p(B_r(x))} \to 0 \qquad (9)$$

$$\left(\text{resp., } \|(F(\cdot, D\phi(\cdot), D^2\phi(\cdot)) - f(\cdot) - F_j(\cdot, D\phi(\cdot), D^2\phi(\cdot)) + f_j(\cdot))^-\|_{L^p(B_r(x))} \to 0\right),$$

as $j \to \infty$.

Then, u is an L^p-viscosity subsolution (resp., supersolution) of (1).

Remarks. It is not hard to verify that (9) holds when F_j is constructed by the above procedure.

It is well-known that the uniqueness of L^p-viscosity solutions does not hold under assumptions (A1), (A2), and the linear growth of the mapping $q \to F(x, q, O)$ instead of (A3) (*i.e.* even in a linear case). We refer [24] and [26] for the non-uniqueness of "weak" solutions of (1).

Idea of proof of Theorem 3.

We modify the proof in [3] (when $q \to F(x, q, O)$ is linear growth) using some ideas from the proof of Theorem 1.

We also need the maximum principle and $W^{2,p}$-estimates for L^p-viscosity solutions of

$$\mathcal{P}^{\pm}(D^2u) + \gamma(x)|Du| = f(x) \quad \text{in } \Omega,$$

for $\gamma \in L^p$, which was studied by Fok [13].

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