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Nonlinear second order elliptic equations with subdifferetial terms

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1. Introduction This is a brief report of my joint work [6] with Prof. N. Yamada (Fukuoka Univ.).

We consider the following second order elliptic partial differential equation (PDE) with subdifferential

$$\begin{cases} -\Delta u + u - f + \partial\Phi(x,u) \ni 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subset \mathcal{R}^N$ is a bounded domain, f is a given function and $\partial\Phi(x,r)$ denotes the subdifferential with respect to r for a proper, convex and lower semicontinuous function $\Phi(x,r)$. An example for (1.1) is the following obstacle problem

$$\begin{cases} u \leqq \psi & \text{in} \quad \overline{\Omega}, \\ -\Delta u + u - f = 0 & \text{in} \quad \Omega \quad \text{if} \quad u(x) < \psi(x), \\ -\Delta u + u - f \leqq 0 & \text{in} \quad \Omega \quad \text{if} \quad u(x) = \psi(x), \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

We define Φ by

$$\Phi(x,r) = \begin{cases} 0 & \text{if} \;\; r \leqq \psi(x), \\ +\infty & \text{otherwise.} \end{cases}$$

Then its subdifferential $\partial\Phi(x,r)$ is

$$\partial\Phi(x,r) = \begin{cases} 0 & \text{if} \;\; r < \psi_1(x), \\ [0,+\infty) & \text{if} \;\; r = \psi_1(x), \\ \emptyset & \text{otherwise,} \end{cases}$$

and (1.2) turns to (1.1).

(1.2) has been studied from various viewpoints.

1.1 Variational inequality Find $u \in K$ satisfying

$$\int_\Omega \langle Du, D(u-v)\rangle dx + \int_\Omega u(u-v)dx \geqq \int_\Omega f(u-v)dx \quad (\forall v \in K). \tag{1.3}$$

Here $\langle \cdot,\cdot \rangle$ is the inner product in $\mathcal{R}^N$ and $K = \{u \in H_0^1(\Omega) \mid u \leqq \psi \text{ a.e. in } \Omega\}$. This is a weak form of (1.2). We refer D. Kinderleher - G. Stampacchia [7] for an introdution to variational inequalities and applications.

1.2 Subdifferential equation Consider the follwing inclusion.

$$u - f \in -\partial\Psi(u), \;\; u \in K, \tag{1.4}$$

$$\Psi(u) = \frac{1}{2}\|Du\|^2_{L^2(\Omega)} + I_K(u), I_K(u) = 0\ (u \in K), = +\infty\ (u \notin K),$$
$$\partial\Psi(u) = -\Delta u + \partial I_K(u), \partial I_K(u) = \text{subdifferential of } I_K(u),$$
$$K = \{u \in H^1_0(\Omega) \mid u \leq \psi \text{ a.e. in } \Omega\}.$$

The existence and uniqueness of solutions of (1.4) was discussed by applying the theory of subdifferential operators. See H. Brézis [2] etc.

1.3 Degenerate elliptic equation (1.2) is the same as the following equation.

$$(1.5) \qquad \begin{cases} \max\{-\Delta u + u - f, u - \psi\} = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

See A. Benssousan-J.-L. Lions [1] etc. for the treatments of (1.5) and the relation of (1.5) to stochastic control problems.

These problems are equivalent to (1.2) in some sense, although their derivations are different from each other. Hence it seems to us intuitively that their solutions should coincide with each other. It is obvious that (1.3) is equivalent to (1.4) in L^2-sense. Since the subdifferential $\partial\Psi(\cdot)$ is defined in $L^2(\Omega)$ and it is a maximal monotone operator in $L^2(\Omega)$, we want to understand $\partial\Psi(\cdot)$ in the sense of pointwise. If we can do so, we think that we can make the equivalence between (1.4) and (1.5) clearer.

Motivated by these considerations, N. Yamada [8] has given a notion of viscosity solutions of nonlinear first order PDE's with subdifferential and proved the comparison principle. Our aim of this article is to extend the result of [8] and to propose a notion of weak solutions of second order multi-valued PDE's such as (1.1).

Our plan is the following. In Section 2 we state our assumptions and give our definition of viscosity solutions. In Section 3 we present the comparison principle and existence of solutions of (1.1). Section 4 is devoted to the stability of viscosity solutions and the convergence of Yosida approximation for (1.1).

In the following we suppress the term "viscosity" since we are mainly concerned with viscosity sub-, super- and solutions.

2. Preliminaries In this section we state our assumptions and give the definitions of solutions of (1.1).

We make the following assumptions.

(A.1) $\Omega \subset \mathcal{R}^N$ is a bounded domain with smooth boundary.
(A.2) $f \in C(\overline{\Omega})$.
(A.3) For each $x \in \overline{\Omega}$, $\Phi(x, \cdot)$ is proper, convex and lower semicontinuous in $\mathcal{R}$.
(A.4) Let $E(x) = \{r \in \mathcal{R} \mid \Phi(x,r) < +\infty\}$. The set-valued function $x \to E(x)$ is "continuous" on $\overline{\Omega}$ (see Remark 2.1 (2) below).
(A.5) For any (x,r) with $r \in E(x)$, Φ satisfies

$$\lim_{\substack{(y,s)\to(x,r) \\ s\in E(y)}} \Phi(y,s) = \Phi(x,r).$$

(A.6) $0 \in E(x)$ for all $x \in \partial\Omega$.

Remark 2.1. (1) If (A.3) holds, then, for each $x \in \overline{\Omega}$, $E(x)$ is a closed interval and $\Phi(x, \cdot)$ is continuous in $\operatorname{int} E(x)$.
(2) Set $e^+(x) = \sup\{r \mid r \in E(x)\}$ and $e^-(x) = \inf\{r \mid r \in E(x)\}$. Then (A.4) means that the interval $[e^+(x), e^-(x)]$ varies continuously with respect to $x \in \overline{\Omega}$. Thus it follows that either $e^+ \in C(\overline{\Omega})$ or $e^+(x) \equiv +\infty$ on $\overline{\Omega}$ holds. Similarly, either $e^- \in C(\overline{\Omega})$ or $e^-(x) \equiv -\infty$ on $\overline{\Omega}$ holds.

To give the definition of solutions of (1.1), we prepare some notations. Let $u : \overline{\Omega} \to \mathcal{R}$. For each $x \in \overline{\Omega}$, we define

$$u^*(x) = \lim_{r\to 0} \sup\{u(y) \mid |y-x| < r, y \in \overline{\Omega}\}, u_*(x) = \lim_{r \to 0} \inf\{u(y) \mid |y-x| < r, y \in \overline{\Omega}\}.$$

Definition 2.2. *Let* $u : \overline{\Omega} \to \mathcal{R}$.

(1) *We say* u *is a subsolution of* (1.1) *if and only if* $u^*(x) < +\infty$, $\Phi(x, u^*(x)) < +\infty$ *on* $\overline{\Omega}$ *and for any* $\phi \in C^2(\Omega)$, $x \in \Omega$ *and* $r < u^*(x)$, *we have*

$$\Phi(x,r) - \Phi(x, u^*(x)) \geqq -(-\Delta\phi(x) + u^*(x) - f(x))(r - u^*(x))$$

provided $u^* - \phi$ *takes its maximum at* x.

(2) *We say* u *is a supersolution of* (1.1) *if and only if* $u_*(x) > -\infty$, $\Phi(x, u_*(x)) < +\infty$ *on* $\overline{\Omega}$ *and for any* $\phi \in C^2(\Omega)$, $x \in \Omega$ *and* $r > u_*(x)$, *we have*

$$\Phi(x,r) - \Phi(x, u_*(x)) \geqq -(-\Delta\phi(x) + u_*(x) - f(x))(r - u_*(x)).$$

provided $u_* - \phi$ *takes its minimum at* x.

(3) *We say* u *a solution of* (1.1) *if* u *is both a subsolution and a supersolution of* (1.1).

Remark 2.3. If $\partial\Phi(x,r)$ is singleton, then the above definition is the same as the usual one (cf. [5, Section 2]).

3. Comparison principle and existence of solutions

In this section we prove the comparison principle and existence of solutions of (1.1).

The comparison principle is stated as follows:

Theorem 3.1. *Assume* (A.1)-(A.5). *Let* u, v *be, respectively, a subsolution and a supersolution of* (1.1). *If* $u^* \leqq v_*$ *on* $\partial\Omega$, *then* $u^* \leqq v_*$ *on* $\overline{\Omega}$.

Outline of Proof. We assume $u \in C(\overline{\Omega})$ and $v \in C^2(\Omega) \cap C(\overline{\Omega})$ for simplicity. Suppose $\sup_{\overline{\Omega}}(u - v) = u(z) - v(z) = \theta > 0$ and we shall get a contradiction. Then $z \in \Omega$ because $u \leqq v$ on $\partial\Omega$.

Since u is a subsolution of (1.1) and v is a supersolution of (1.1), for any $r_1 < u(z)$ and $r_2 > v(z)$, we have the following inequalities.

$$\Phi(z, r_1) - \Phi(z, u(z)) \geqq -(-\Delta v(z) + u(z) - f(z))(r_1 - u(z)), \tag{3.1}$$

$$\Phi(z, r_2) - \Phi(z, v(z)) \geqq -(-\Delta v(z) + v(z) - f(z))(r_2 - v(z)). \tag{3.2}$$

Hence, substituting $r_1 = v(z)$ in (3.1) and $r_2 = u(z)$ in (3.2) and summing up these inequalities, we get $0 \geqq (u(z) - v(z))^2$, which is a contradiction. □

Next we establish the existence of a unique solution of (1.1). We use Perron's method to show the existence of solutions (cf. [5, Section 4]). For simplicity we assume $e^{\pm} \in C(\overline{\Omega})$ and $e^{\pm} = 0$ on $\partial\Omega$, which are defined in Remark 2.1 (2). Then e^- (resp., e^+) is a subsolution (resp., a supersolution) of (1.1). We set

$$\begin{aligned} &\mathcal{S} = \{v \mid v : \text{subsolution of } (1.1), v^* \leqq 0 \text{ on } \partial\Omega\}(\neq \emptyset), \\ (3.3) \qquad &u(x) = \sup\{v(x) \mid v \in \mathcal{S}\}. \end{aligned}$$

We have the following theorem.

Theorem 3.2. *Assume* (A.1)-(A.6). *Let u be defined by* (3.3). *Then u is a unique solution of* (1.1) *satisfying $u = 0$ on $\partial\Omega$. Moreover, $u \in C(\overline{\Omega})$.*

Perron's methods is divided into two lemmas. We assume (A.1)-(A.6) in the following lemmas.

Lemma 3.3. *u is a subsolution of* (1.1).

Lemma 3.4. *Assume $v \in \mathcal{S}$ satisfies $\Phi(x, v_*(x)) < +\infty$ on $\overline{\Omega}$. If v is not a supersolution of* (1.1), *then there exists a $w \in \mathcal{S}$ such that $v(y) < w(y)$ for some $y \in \Omega$.*

We admit Lemmas 3.3 and 3.4 and prove Theorem 3.2. After doing so, we give their proofs.

Proof of Theorem 3.2 We note that $e^- = u_* = u^* = e^+ = 0$ on $\partial\Omega$. It follows from Lemma 3.2 that u is a subsolution of (1.1) and therefore $u \in \mathcal{S}$.

It is easily seen by the facts $e^- \leqq u$ on $\overline{\Omega}$ and $e^- \in C(\overline{\Omega})$, we get $e^- \leqq u_* \leqq u^*$ on $\overline{\Omega}$ and $\Phi(x, u_*(x)) < +\infty$ on $\overline{\Omega}$.

Suppose u is not a supersolution of (1.1). By Lemma 3.4 we can find a $w \in \mathcal{S}$ such that $u(y) < w(y)$ for some $y \in \Omega$. This is a contradiction to the maximality of u. Hence u is a supersolution of (1.1).

We use $u^* = u_* = 0$ on $\partial\Omega$ and Theorem 3.1 to have $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. The uniqueness also follows from Theorem 3.1. □

Put $E \equiv \cup_{x \in \overline{\Omega}}(\{x\} \times E(x))$.

Outline of Proof of Lemma 3.3. *Step 1.* We prove $\Phi(x, u^*(x)) < +\infty$ on $\overline{\Omega}$.

Fix $x_0 \in \overline{\Omega}$. By the definition of u^*, there exists a sequence $\{x_n\} \subset \overline{\Omega}$ and $\{v_n\} \subset \mathcal{S}$ such that

$$(3.4) \qquad x_n \to x_0, v_n^*(x_n) \to u^*(x_0) \quad (n \to +\infty).$$

Since $(x_n, v_n^*(x_n)) \in E$ and E is closed in $\mathcal{R}^{N+1}$ by (A.4), we get $(x_0, u^*(x_0)) \in E$. Therefore we have $\Phi(x_0, u^*(x_0)) < +\infty$ on $\overline{\Omega}$.

Step 2. Let $\phi \in C^2(\Omega)$ and let $x_0 \in \Omega$ be a maximum point of $u^* - \phi$. We show

$$(3.5) \qquad \Phi(x_0, r) - \Phi(x_0, u^*(x_0)) \geqq -(-\Delta\phi(x_0) + u^*(x_0) - f(x_0))(r - u^*(x_0)).$$

for all $r < u^*(x_0)$.

By a slight modification of ϕ we may consider

$$u^*(x_0) - \phi(x_0) = 0, u^*(x) - \phi(x) \leqq -|x - x_0|^4 \quad \text{on } \overline{\Omega}. \tag{3.6}$$

The definitions of u^* and u imply there exist $\{x_n\} \subset \overline{\Omega}$ and $\{v_n\} \subset \mathcal{S}$ satisfying (3.4).

Let y_n be a maximum point of $v_n^* - \phi$ on $\overline{\Omega}$. Then we have, by (3.4), (3.6) and some calculations,

$$y_n \to x_0, v_n^*(y_n) \to u^*(x_0) \quad (n \to +\infty). \tag{3.7}$$

Fix $r < u^*(x_0)$. If $\Phi(x_0, r) = +\infty$, then we have nothing to prove and thus we assume $\Phi(x_0, r) < +\infty$. We restrict our attention to the case $(x_0, r) \in \operatorname{int} E$ because the case of $(x_0, r) \notin \operatorname{int} E$ can be proved similarly, by using some perturbations. It is easily seen that $(y_n, v_n^*(y_n)) \in \operatorname{int} E$, $r < v_n^*(y_n)$ for large $n \gg 1$. Since v_n is a subsolution of (1.1), we obtain the following inequality

$$\Phi(y_n, r) - \Phi(y_n, v_n^*(y_n)) \geqq -(-\Delta\phi(y_n) + v_n^*(y_n) - f(y_n))(r - v_n^*(y_n)).$$

Letting $n \to +\infty$, we get (3.5) by (3.7), (A.2) and (A.5). □

Outline of Proof of Lemma 3.4. Suppose v is not a supersolution of (1.1). Then, there exist a $\phi \in C^2(\Omega)$, an $x_0 \in \Omega$ and an $r_0 > v_*(x_0)$ such that $v_* - \phi$ takes its minimum at x_0 and

$$\Phi(x_0, r_0) - \Phi(x_0, v_*(x_0)) + 4\delta \leqq -(-\Delta\phi(x_0) + v_*(x_0) - f(x_0))(r_0 - v_*(x_0)) \tag{3.8}$$

for some $\delta > 0$. We note $\Phi(x_0, r_0) < +\infty$ and we may assume $v_*(x_0) = \phi(x_0)$. Moreover we observe

$$\begin{aligned} v(x) \geqq v_*(x) \geqq \phi(x) = \phi(x_0) + \langle D\phi(x_0), x - x_0\rangle + \frac{1}{2}\langle D^2\phi(x_0)(x - x_0), x - x_0\rangle \\ + o(|x - x_0|^2) \qquad (\forall x \in B(x_0, \eta_0)) \end{aligned}$$

for small $\eta_0 > 0$. We define

$$\psi(x) = \phi(x_0) + \langle D\phi(x_0), x - x_0\rangle + \frac{1}{2}\langle D^2\phi(x_0)(x - x_0), x - x_0\rangle - \gamma|x - x_0|^2.$$

(3.8), (A.2) and (A.5) yield that there exists $0 < \alpha, \eta_1 \ll 1$ such that

$$\begin{aligned} &\psi(x) + \alpha < r_0 - \alpha \\ &\Phi(x, r_0 - \alpha) - \Phi(x, \psi(x) + \alpha) \\ &\quad \leqq -(-\Delta\psi(x) + (\psi(x) + \alpha) - f(x))((r_0 - \alpha) - (\psi(x) + \alpha)) - \delta. \end{aligned}$$

for any $x \in B(x_0, \eta_1)$. By these inequalities and the convexity of Φ we see

$$\Phi(x, r) - \Phi(x, \psi(x) + \alpha) \geqq -(-\Delta\psi(x) + (\psi(x) + \alpha) - f(x))(r - (\psi(x) + \alpha)).$$

for all $r < \psi(x) + \alpha$. Thus we conclude that $\psi(x) + \alpha$ is a C^2-subsolution of (1.1) in $B(x_0, \eta_1)$. We set

$$\widetilde{v}(x) = \begin{cases} \max\{v(x), \psi(x) + \alpha\} & \text{in } \; B(x_0, \eta_1), \\ v(x) & \text{otherwise.} \end{cases}$$

Then we can show $\widetilde{v} \in \mathcal{S}$ by a similar argument to the proof of Lemma 3.2.

We notice by the choice of ψ that $\psi(x) + \alpha < v(x)$ $(\eta_1/2 \leqq |\forall x - x_0| \leqq \eta_1)$ for $\alpha \leqq (\gamma\eta_1^2)/8$. The definition of v_* implies that we can extract a sequence $\{x_n\} \subset \Omega$ satisfying $(x_n, v(x_n)) \to (x_0, v_*(x_0))$ as $n \to +\infty$. Thus we have $\psi(x_n) + \alpha > v(x_n)$ for all $n \gg 1$ since $\psi(x_n) + \alpha \to v_*(x_0) + \alpha$ as $n \to +\infty$. □

4. Convergence properties of solutions

4.1. Stability of solutions In this subsection we discuss the stability of solutions under some perturbations on Φ. Our arguments are based on [4, Section 6].

We consider the following problems

$$(4.1)_n \qquad \begin{cases} -\Delta u + u - f + \partial\Phi_n(x,u) \ni 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We make the following assumptions.

(A.7) $\lim_{n\to+\infty} d_H(E_n, E) = 0$. Here we denote $E_n(x) = \{r \mid \Phi_n(x,r) < +\infty\}$ and $E_n = \cup_{x\in\overline{\Omega}}(\{x\}\times E_n(x))$.

(A.8) For each $x \in \overline{\Omega}$, $r \in E(x)$,

$$\limsup_{\substack{(y,s)\to(x,r), s\in E_n(y) \\ n\to+\infty}} \Phi_n(y,s) = \liminf_{\substack{(y,s)\to(x,r), s\in E_n(y) \\ n\to+\infty}} \Phi_n(y,s) = \Phi(x,r)$$

Let $u_n \in C(\overline{\Omega})$ be a unique solution of $(4.1)_n$. Then we have the stability of solutions.

Theorem 4.1. *Assume* (A.1)-(A.2). *Moreover assume that Φ and Φ_n satisfy* (A.3)-(A.8). *Then u_n converges to u uniformly on $\overline{\Omega}$ as $n \to +\infty$. Here u is a unique solution of* (1.1) *satisfying $u = 0$ on $\partial\Omega$.*

Outline of Proof. At first, by the barrier construction argument and the comparison principle, we get $\sup_{n\geqq 1}\|u_n\|_{L^\infty(\Omega)} < +\infty$. We define

$$(4.2) \qquad \overline{u}(x) = \lim_{k\to+\infty} \sup\{u_n(y) \mid |y-x| < k^{-1}, y \in \overline{\Omega}, n > k\},$$

$$(4.3) \qquad \underline{u}(x) = \lim_{k\to+\infty} \inf\{u_n(y) \mid |y-x| < k^{-1}, y \in \overline{\Omega}, n > k\}.$$

We prove only that $\overline{u}$ is a subsolution of (1.1) because we can prove similarly that $\underline{u}$ is a supersolution of (1.1).

It is easily seen by (A.8) and (A.4) that $\Phi(x, \overline{u}(x)) < +\infty$ on $\overline{\Omega}$.

Next, we show $\overline{u}$ is a subsolution of (1.1).

For any $\phi \in C^2(\Omega)$, let $\overline{u} - \phi$ take its maximum at $x_0 \in \Omega$. By a suitable modification of ϕ, we may consider

$$\overline{u}(x_0) = \phi(x_0), \overline{u}(x) - \phi(x) \leqq -|x - x_0|^4 \quad \text{on } \overline{\Omega}$$

By (4.2) there exists a sequence $\{(n_k, x_{n_k})\} \subset \mathcal{N} \times \Omega$ satisfying $x_{n_k} \to x_0$, $u_{n_k}(x_{n_k}) \to \overline{u}(x_0)$ as $k \to +\infty$. Set $n_k = k$. Let $y_k \in \overline{\Omega}$ be a maximum point of $u_k^* - \phi$ on $\overline{\Omega}$. Then, by some calculations we observe

$$(4.4) \qquad y_k \to x_0, u_k^*(y_k) \to \overline{u}(x_0) \quad (k \to +\infty).$$

Fix $r < \overline{u}(x_0)$. We may assume $\Phi(x_0, r) < +\infty$. We consider only the case of $(x_0, r) \in \operatorname{int} E$ because the case of $(x_0, r) \notin \operatorname{int} E$ can be proved similarly.

It follows from (4.4) and (A.8) that $r < u_k(y_k)$ and $\Phi_k(y_k, r) < +\infty$ for large k. Since u_k is a subsolution of $(4.1)_k$, we have the following inequality

$$\Phi_k(y_k, r) - \Phi_k(y_k, u_k(y_k)) \geq -(-\Delta\phi(y_k) + u_k(y_k) - f(x_k))(r - u_k(y_k)).$$

Sending $k \to +\infty$, we obtain by (4.4), (A.2) and (A.8).

$$\Phi(x_0, r) - \Phi(x_0, \overline{u}(x_0)) \geq -(-\Delta\phi(x_0) + \overline{u}(x_0) - f(x_0))(r - \overline{u}(x_0)).$$

We can show $\overline{u} = \underline{u} = 0$ on $\partial\Omega$ by the barrier construction arguments and the comparison principle and therefore we apply Theorem 3.1 to have $\overline{u} = \underline{u}(\equiv u)$. on $\overline{\Omega}$. The uniform convergence is derived from the same argument as in [4, Section 6]. □

4.2. Convergence of Yosida approximation This subsection is devoted to the convergence of solutions of Yosida appoximation for (1.1). Yosida approximation of Φ is defined by

$$\Phi_n(x, r) = \inf_{s \in \mathcal{R}} \left\{ \Phi(x, s) + \frac{n}{2}(r - s)^2 \right\} \quad (x \in \overline{\Omega}, r \in \mathcal{R}, n \in \mathcal{N}).$$

We consider the following problems.

$$(4.5)_n \qquad \begin{cases} -\Delta u + u - f + \partial\Phi_n(x, u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We show that a solution of $(4.5)_n$ converges to that of (1.1). As to the notion of viscosity solutions of $(4.5)_n$, we adopt the usual one (cf. [4, Definition 2.2]).

Before discussing the convergence of Yosida approximation, we recall some properties of Φ_n and $\partial\Phi_n$.

Proposition 4.3. *Assume* (A.5) *and fix $x \in \overline{\Omega}$. Then we have the following properties.*

(1) *There exists a unique minimizer $s_0 \in E(x)$ for $\Phi_n(x, r)$. Set $s_0 = J_n(x, r)$.*
(2) *$J_n(x, \cdot)$ is nonexpansive and $\lim_{n \to +\infty} J_n(x, r) = r$ if $r \in E(x)$.*
(3) *$\Phi_n(x, r)$ is nondecreasing with respect to $n \in \mathcal{N}$ and $\lim_{n \to +\infty} \Phi_n(x, r) = \Phi(x, r)$.*
(4) *$\Phi_n(x, \cdot)$ is differentiable and convex. Moreover, it holds*

$$\partial\Phi_n(x, r) = \frac{\partial\Phi_n}{\partial r}(x, r) = n(r - J_n(x, r)),$$

and $\partial\Phi_n(x, \cdot)$ is Yosida approximation of $\partial\Phi(x, \cdot)$ for eaxh $x \in \overline{\Omega}$.
(5) *$\partial\Phi_n(x, \cdot)$ is nondecreasing.*
(6) *$\lim_{n \to +\infty} J_n(x, r) = \mathrm{Proj}_{E(x)} r$. Here $\mathrm{Proj}_{E(x)} r$ is the projection of r onto $E(x)$.*

See H. Brezis [3] for the proof.

Proposition 4.4. *Assume* (A.3)-(A.5). *Then Φ_n, $J_n \in C(\overline{\Omega} \times \mathcal{R})$ for all $n \in \mathcal{N}$.*

This proposition can be proved by the convexity of Φ and lengthy calculations, so we omit the proof.

Under the assumptions (A.1)-(A.5), for each $n \in \mathcal{N}$, there exists a unique solution $u_n \in C(\overline{\Omega})$ of $(4.5)_n$ satisfying $u_n = 0$ on $\partial\Omega$. We have the following theorem.

Theorem 4.5. *Assume* (A.1)-(A.6). *Let u_n be a solution of* $(4.7)_n$ *satisfying $u_n = 0$ on $\partial\Omega$. Then u_n converges to u uniformly on $\overline{\Omega}$ as $n \to +\infty$. Here u is a unique solution of* (1.1) *satisfying $u = 0$ on $\partial\Omega$.*

Outline of Proof. By the barrier construction argument and the comparison principle, we get $\sup_{n \geqq 1} \|u_n\|_{L^\infty(\Omega)} < +\infty$. Let $\overline{u}$, $\underline{u}$ be defined by (4.2), (4.3), respectively.
Step 1. We show $\Phi(x, \underline{u}(x))$, $\Phi(x, \overline{u}(x)) < +\infty$ on $\overline{\Omega}$.

Let $x_0 \in \Omega$ be a point satisfying $J^{2,+}\overline{u}(x_0) \neq \emptyset$. Then there exists a $\phi \in C^2(\Omega)$ such that $\overline{u} - \phi$ takes its maximum at $x_0 \in \Omega$. Thus we can find a sequence $\{(n_k, y_{n_k})\} \subset \mathcal{N} \times \overline{\Omega}$ satisfying

$$(4.6) \qquad \begin{cases} y_{n_k} \in \Omega \ : \ \text{maximum point of } u^*_{n_k} - \phi, \\ n_k \to +\infty, y_{n_k} \to x_0, u_{n_k}(y_{n_k}) \to \overline{u}(x_0) \quad (k \to +\infty). \end{cases}$$

Set $n_k = k$ for the sake of simplicity. Since u_k is a subsolution of $(4.5)_k$, we get

$$(4.7) \qquad -\Delta\phi(y_k) + u_k(y_k) - f(y_k) + \partial\Phi_k(y_k, u_k(y_k)) \leqq 0.$$

We can see $\{J_k(y_k, u_k(y_k))\}$ is bounded. Therefore we may consider $J_k(y_k, u_k(y_k)) \to \exists\alpha_0 (= \alpha_0(x_0))$ as $k \to +\infty$ by taking a subsequence if necessary. Hence, by (4.6), (4.7) and (A.2), we obtain $\overline{u}(x_0) \leqq \alpha_0$. We note $(x_0, \alpha_0) \in E (= \cup_{x \in \overline{\Omega}} \{x\} \times E(x))$ because $(y_k, J_k(y_k, u_k(y_k))) \in E$ and E is closed in $\mathcal{R}^{N+1}$.

For any $x_0 \in \Omega$, there exists a sequence $\{x_n\} \subset \Omega$ satisfying

$$x_n \to x_0, \overline{u}(x_n) \to \overline{u}(x_0) \quad (n \to +\infty), J^{2,+}\overline{u}(x_n) \neq \emptyset \quad (\forall n \in \mathcal{N}).$$

It follows from the above observation that, for each $n \in \mathcal{N}$, there exists an $\alpha_n \in \mathcal{R}$ such that $(x_n, \alpha_n) \in E$ and $\overline{u}(x_n) \leqq \alpha_n$. Since $\{u_n\}$ is uniformly bounded on $\overline{\Omega}$, we may consider $\{\alpha_n\}$ is bounded. Hence we can extract a subsequence $\{\alpha_{n_k}\}$ satisfying $\alpha_{n_k} \to \exists\overline{\alpha}$ as $k \to +\infty$. Since $(x_{n_k}, \alpha_{n_k}) \in E$ and E is closed in $\mathcal{R}^{N+1}$, we have $(x_0, \overline{\alpha}) \in E$ and $\overline{u}(x_0) \leqq \overline{\alpha}$.

Similarly we can show that, for any $x_0 \in \overline{\Omega}$, there exists an $\underline{\alpha} \in \mathcal{R}$ such that $(x_0, \underline{\alpha}) \in E$ and $\underline{u}(x_0) \geqq \underline{\alpha}$. Therefore we obtain $\underline{\alpha} \leqq \underline{u}(x_0) \leqq \overline{u}(x_0) \leqq \overline{\alpha}$. Using $(x_0, \overline{\alpha})$, $(x_0, \underline{\alpha}) \in E$ and this, we conclude $\Phi(x_0, \overline{u}(x_0))$, $\Phi(x_0, \underline{u}(x_0)) < +\infty$ for all $x_0 \in \overline{\Omega}$.
Step 2. We prove that $\overline{u}$ is a subsolution of (1.1).

Assume that, for any $\phi \in C^2(\Omega)$, $\overline{u} - \phi$ takes its maximum at x_0. We can find a sequence $\{(n_k, y_{n_k})\} \subset \mathcal{N} \times \overline{\Omega}$ satisfying (4.6). Put $n_k = k$ for simplicity. Since $J^{2,+}\overline{u}(x_0) \neq \emptyset$, we may consider $J_k(y_k, u_k(y_k)) \to \exists\alpha_0$ as $k \to +\infty$ by the argument in *Step 1.* On the other hand, using

$$\Phi_k(y_k, u_k(y_k)) \leqq \Phi(y_k, \mathrm{Proj}_{E(x_k)} u_k(y_k)) + k(u_k(y_k) - \mathrm{Proj}_{E(x_k)} u_k(y_k))^2$$

and (A.5), we obtain

$$|\overline{u}(x_0) - \alpha_0| \leqq |\overline{u}(x_0) - \mathrm{Proj}_{E(x_0)}\overline{u}(x_0)|.$$

Thus we have

$$J_k(y_k, u_k(y_k)) \to \overline{u}(x_0) \quad (k \to +\infty) \tag{4.8}$$

by means of $\overline{u}(x_0) \in E(x_0)$.

Fix $r < \overline{u}(x_0)$. We may consider $\Phi(x_0, r) < +\infty$. For simlicity, we assume $(x_0, \overline{u}(x_0))$, $(x_0, r) \in \operatorname{int} E$. Since we can see by (A.4) that $(y_k, u_k(y_k)) \in \operatorname{int} E$ for large $k \in \mathcal{N}$, we get

$$\Phi(y_k, J_k(y_k, u_k(y_k))) \to \Phi(x_0, \overline{u}(x_0)), k(u_k(y_k) - J_k(y_k, u_k(y_k)))^2 \to 0,$$
$$\Phi_k(y_k, u_k(y_k)) \to \Phi(x_0, \overline{u}(x_0)).$$

as $k \to +\infty$, by using (4.8) and (A.5).

We can prove that $\underline{u}$ is a supersolution of (1.1) by the same argumne as above. The remainder is similar to the proof of Theorem 4.1. □

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