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VISCOSITY SOLUTIONS OF FULLY NONLINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

The purpose of this note is to provide a brief introduction to the new theory of fully nonlinear, first- and second-order, stochastic partial differential equations, which the authors have been developing during the last few years, ([LS1], [LS2], [LS3] and [LS4]). The goal here is to provide a general motivation in terms of the possible applications and to discuss some of the mathematical difficulties. We will also list a number of open problems.

The class of equations we consider is

$$du = F(D^2u, Du, u, x, t, \omega)\, dt + H(Du, u, x, t, \omega) \cdot dB_t \quad \text{in } \mathbb{R}^N \times (0, \infty)\ , \tag{1}$$

with initial datum

$$u = u_0 \ \text{ on } \ \mathbb{R}^N \times (0, \infty)\ . \tag{2}$$

Here $u_o \in BUC(\mathbb{R}^N)$, the space of bounded uniformly continuous functions on $\mathbb{R}^N$. The nonlinear function $F : S^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times (0, \infty) \times \Omega \to \mathbb{R}$, where S^N is the space of $N \times N$ symmetric matrices, is assumed to be *degenerate elliptic*, i.e., to satisfy, for all $(p, u, x, t, \omega) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times (0, \infty) \times \Omega$ and $X, Y \in S^N$,

$$F(X, p, u, x, t, \omega) \leqq F(Y, p, u, x, t, \omega)\ , \quad \text{if } \ X \leqq Y\ . \tag{3}$$

The functions F and H need, of course, to satisfy a number of other assumptions, which we omit here to simplify the presentation.

The stochastic character of (1) is due to the presence of the term dB_t, which denotes the differential, in the Itô sense, of an N-dimensional standard Brownian motion B_t — in this case H is assumed to take values in $\mathbb{R}^N$. Below we list a number of applications where equations like (1) arise. In Section 3 we discuss some of the mathematical difficulties. In Section 4 we state a couple of typical results.

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2. APPLICATIONS

Below we indicate briefly a number of applications where one finds equations likes (1).

(i) Stochastic analysis and linear and semilinear pde filtering and stochastic control with partial observation.

A classical example in stochastic analysis of equations like (1) (see, for example, Pardoux [P1], Watanabe [W], etc.) is the case where H is linear in Du and F is linear in D^2u and *uniformly elliptic*, i.e., there exists $\nu > 0$ such that for all (X, p, u, x, t, ω)

$$F(X, p, u, x, t, \omega) \geqq \nu \operatorname{tr}(X) .$$

The simplest example of such equations is

$$du = Du \cdot dB_t + \frac{1}{2}\Delta u \, dt , \tag{4}$$

whose solution is given, by the Itô formula, by

$$u(x,t) = u_0(x + B_t) . \tag{5}$$

It is well known that the Zakai equation plays a fundamental role in nonlinear filtering and in stochastic control with partial observation. This equation, which governs the time evolution of the conditional density of a diffusion, is one of the form (1) with H linear in Du, and F linear in Du and D^2u, but independent of u. We refer for details to Zakai [Z], Rozovsky [R], Pardoux [P2], Krylov and Rozovsky [KR], Kunita [K], etc..

(ii) Pathwise stochastic control.

In the classical stochastic control theory the value function is characterized as the unique viscosity solution of a deterministic Hamilton-Jacobi-Bellman equation, i.e., an equation like (1) with $H \equiv 0$ and $F(D^2u, Du, u, t)$ convex in (D^2u, Du). It is, however, natural to consider another type of stochastic control problem where the pay-off is optimized over all stochastic paths instead of the mean. In this case is turns out (see [LS1], [LS2]) that the value function satisfies a stochastic pde like (1). Stochastic control models of this type have been introduced in mathematical finance to study the evolution of prices (see, for example, Cont [C], Musiela [M], etc.)

(iii) Front propagation with stochastic normal velocity.

There are several models in Physics and Material Sciences, for example in the study of nucleations, which involve moving fronts (interfaces) with stochastic normal velocity. A canonical such example is the motion in two dimensions of closed, smooth curves with white noise normal velocity.

Recall that the level set approach to describe such evolutions past singularities consists of characterizing the evolving front as the zero-level set of a function, which solves a certain nonlinear geometric pde. In the case, for example, of the evolution of a front with constant normal velocity c, the corresponding geometric pde is

$$u_t = c|Du| \ .$$

It is therefore natural to extend this definition to normal velocities equal to white noise, in which case, the geometric pde will be of the form

$$du = |Du|\, dB \ . \tag{6}$$

More generally, if the normal velocity is given by

$$v = \alpha \frac{dW}{dt} - \beta\kappa$$

where κ denotes the curvature of the front ($\beta > 0$), the same heuristic argument leads to the equation

$$du = \beta \operatorname{tr}\left[I - \frac{Du \otimes Du}{|\nabla u|^2} D^2 u\right] + \alpha |Du|\, dB \ . \tag{7}$$

For the details in the deterministic case we refer to Barles and Souganidis [BS], Souganidis [BCESS] and the references therein.

(iv) Asymptotic problems with random tune oscillating coefficients.

Equations like (1) arise as limits of equations with rapidly oscillating in time coefficients. Until now only linear equations of this type have been studied (see, for example Kushner and Huang [KH], Watanabe [W], etc.). Other asymptotic problems deal with phase transitions in stochastic environments. A canonical example in this direction is equations of Allen-Cahn type, with a stochastic, rapidly oscillating in time force. The study of the asymptotics of such equations leads to equations like (1). This has been analyzed rigorously in $N = 2$ and for convex regular surfaces by Funaki [F].

3. Mathematical difficulties

Even in the case where u is regular in x, the mathematical formulation of (1) is not clear. To explain the difficulties, we consider the following two examples.

Example 1: The level set approach to define the global in time evolution of the fronts is based on the fundamental property that the resulting pde is geometric, i.e., invariant under increasing changes of the unknown. In other words, if u is a solution, then $\beta(u)$ is also a solution, if, for example, β is smooth and strictly increasing on $\mathbb{R}$. This yields, in particular, that u and $\beta(u)$ have the same level sets.

Consider now (7) and assume that u is smooth. A straightforward application of Itô's formula yields

$$d\beta(u) = \beta'(u)\,du + \frac{1}{2}\beta''(u)|Du|^2\,dt = |D\beta(u)|\,dB + \frac{1}{2}\beta''(u)|Du|^2\,dt\ .$$

The invariance of the equation is therefore not true.

Example 2: Consider for $N = 1$ the simple linear equation

$$du = u_x\,dB + \lambda u_{xx}\,dt \qquad (\lambda \geqq 0)\ .$$

Once again Itô's formula applied to $v(x,t) = u(x - B_t, t)$ yields the equation

$$dv = du - u_x\,dB + \frac{1}{2}u_{xx}\,dt - u_{xx}\,dt = \Big(\lambda - \frac{1}{2}\Big)u_{xx}\,dt = \Big(\lambda - \frac{1}{2}\Big)v_{xx}\,dt\ ,$$

which is ill posed for $\lambda \in (0, 1/2)$. Similarly (1) may be ill posed unless it is assumed that, for $p = Du$ and $X = D^2u$,

$$F(X,p) \geqq \frac{1}{2}(XDH, DH) \ \text{ for all } \ (X,p)\ .$$

The difficulties described above can be overcome, if the Itô's differential in (1) is replaced by the Stratonovich differential, which is denoted by $\circ dB_t$. In this case (1) takes the form

$$du = F(D^2u, Du, u, x, t, \omega)\,dt + H(Du, u, x, t, \omega) \circ dB_t\ . \tag{8}$$

In the first example, (6) is replaced then by

$$du = |Du| \circ dB\ . \tag{9}$$

It then follows, using the Stratonovich integral that

$$d\beta(u) = \beta'(u)|Du| \circ dB = |D\beta(u)| \circ dB\ .$$

Similarly, if, for $\lambda \geqq 0$,

$$du = u_x \circ dB + \lambda u_{xx}\,dt\ , \tag{10}$$

then $v = u(x - B_t, t)$ solves

$$dv = \lambda v_{xx}\,dt\ ,$$

which is well posed for all $\lambda \geqq 0$.

Finally we remark that (8), (9) and (10) can be rewritten, using the relationship between the Itô's and Stratonovich's integrals, in the case that H depends on Du but not u as, respectively

$$du = \Big(F + \frac{1}{2}(D^2uD_pH, D_pH)\Big)\,dt + HdB\ , \tag{11}$$

$$du = |Du|dB + \frac{1}{2}(D^2u, Du, Du)|Du|^{-2}\,dt\ , \tag{12}$$

and

$$du = u_x dB + \Big(\lambda + \frac{1}{2}\Big) u_{xx}\, dt\ . \tag{13}$$

It turns out that the correct setting for the equations under consideration is the one involving Stratonovich's integral. This, of course, leads to serious mathematical difficulties due to the lack of regularity for u, since, in general, we cannot expect u to be more regular than Lipschitz on x. Indeed even in the deterministic case, i.e., when $H \equiv 0$, and in the case that F only depends on Du, i.e., when (1) reduces to a Hamilton-Jacobi equation, it is well known that "shocks", i.e., discontinuities in Du, appear. To overcome such difficulties, it is necessary to introduce the notion of viscosity solutions (see Crandall and Lions [CL], the "User's Guide" by Crandall, Ishii and Lions [CIL], as well as the books [BCESS], Barles [B], Fleming and Soner [FS], Bardi and Capuzzo-Dolceta [BC], etc.).

In order to give (8) a pathwise meaning, it is necessary to come up with a viscosity formulation. For a typical Brownian trajectory $(B_t,\, t \geqq 0)$ one does not have but some Hölder regularity with exponent $\theta < 1/2$. The classical theory of viscosity solutions requires absolute continuity ($W^{1,1}(0,T)$ for all $T > 0$) dependence in time — see Lions and Perthame [LP] and Ishii [I], which is never satisfied for the Brownian motion.

One may, of course, try to adapt the notion introduced in [LP] by considering, for all smooth functions ϕ, the quantities

$$\overline{m}(t) = \sup_x \big[u(x,t) - \phi(x)\big] \quad \text{and} \quad \underline{m}(t) = \inf_x \big[u(x,t) - \phi(x)\big]$$

and asking that they satisfy the inequalities

$$d\overline{m} \leqq F\big(D^2\phi(x_t), D\phi(x_t)\big)dt + H\big(D\phi(x_t)\big) \circ dB_t \tag{14}$$

and

$$d\underline{m} \geqq F\big(D^2\phi(x_t), D\phi(x_t)\big)dt + H\big(D\phi(x_t)\big) \circ dB_t\ , \tag{15}$$

where x_t is a maximum or minimum point of $u(x,t) - \phi(x)$. To simplify the presentation here we assume that H only depends on p and F on X, p. There are two issues which make (14) and (15) not the correct definition, namely the question of selection of x_t but, more fundamentally, the meaning of the term $H\big(D\phi(x_t)\big) \circ dB_t$.

In the particular case that H depends only on p and is regular and $F \equiv 0$, it is possible to construct for $u_0 \in C_b^2(\mathbb{R}^N)$, using the method of characteristics, on a time interval $[t_0, t_0 + \tau]$ $(\tau > 0)$ a solution of

$$\begin{cases} du = H(Du) \circ dB & \big(t \in [t_0, t_0 + \tau]\big) \\ u = u_0 \text{ on } \mathbb{R}^N \times \{t_0\}\ . \end{cases} \tag{16}$$

Indeed it suffices to solve for x, for all $(y,t) \in \mathbb{R}^n \times [t_0, t_0+\tau]$,

$$x = \big(B(t) - B(t_0)\big) \cdot DH\big(Du_0(x)\big) = y$$

and to define

$$Du(y,t) = Du_0(x) \ ,$$

and

$$u(y,t) = u_0(x) + \big[B(t) - B(t_0)\big]\big[H\big(Du_0(x)\big) - D_pH\big(u_0(x)\big)du_0(x)\big] \ .$$

This construction is clearly possible for all $t_o \in [0,\tau]$, provided $\tau = \tau(\omega)$ is sufficiently small so that

$$\Big(\max_{\substack{0 \leqq s \leqq T \\ |s'-s| \leqq \tau}} |B(s) - B(s')|\Big)\Big(\max_{|p| \leqq \|Du_0\|_{L^\infty}} \|D^2H(p)\|\Big)\|D^2u_0\|_{L^\infty} < 1 \ .$$

Observe that only the continuity of B plays a role in this construction. Moreover, the solution u is C^2 in x, uniformly on $t_0 \in [0,T]$, $t \in [t_0, t_0+\tau]$, if $u_0 \in C^2$.

4. Some results

We present here some typical results obtained in [LS1], [LS2], [LS3] and [LS4]. To simplify the presentation we only consider here the equation

$$du = F(D^2u, Du)\,dt + H(Du) \circ dB \tag{17}$$

with initial datum $u_0 \in BUC(\mathbb{R}^N)$. Moreover we assume that H is Lipschitz continuous and C^2 and that F satisfies (4).

We study (17) in a pathwise sense, i.e., we consider a trajectory $(B(t),\ t \geqq 0)$. As a matter of fact we show that we may consider an arbitrary continuous trajectory $(B(t),\ t \geqq 0)$.

We proceed now with the definition of the stochastic viscosity solution. To this end, we denote by $S^0(t,t_0)\phi$ the short time smooth in x solution of (16) with initial datum $u_0 = \phi$.

We have

Definition. *The function $u \in BUC(\mathbb{R}^N \times [0,T])$ is a viscosity subsolution (resp. supersolution) of (17) if, for all $\phi \in (C^2 \cap C^{0,1})(\mathbb{R}^N)$, all $g \in C^1\big([0,+\infty)\big)$ and all $t \in [0,T]$, if $u(\cdot, t+\cdot) - S^0(t+\cdot, t)\phi(\cdot) - g(\cdot)$ admits a maximum (respectively minimum) at $x_0, h_0 \in (0,\tau)$, then*

$$g'(h_0) \leqq F\big(D^2S(t+t_0,t)\varphi(x_0), DS(t+t_0,t)\phi(x_0)\big) \tag{18}$$

respectively,

$$g'(h_0) \geqq F\big(D^2S(t+h_0,t)\varphi(x_0), DS(t+t_0,t)\phi(x_0)\big) \ . \tag{19}$$

The previous results allow us to consider (17) for $H \in C^2 \cap C^{0,1}(\mathbb{R}^N)$. It is, however, possible to eliminate the Lipschitz assumption, if we assume, for example, $u_0 \in C^{0,1}(\mathbb{R}^N)$. The assumption $H \in C^2(\mathbb{R}^N)$ seems, however, to be more essential, but it can be relaxed to H being the difference of two convex functions. In this case, it turns out, it is still possible to define $S(H, B, t)$. There is also a real interplay between the regularity of H and B. The assumption that H is the difference of two convex functions is necessary if we consider arbitrary paths $(B_t)_{t \geqq 0}$. If the paths are Brownian, the only requirement on H is that $H \in C^{0,1}(\mathbb{R}^N)$.

A typical result is

Theorem. *Assume that $H \in C^{0,1}(\mathbb{R}^N)$ is the difference of two convex functions and that $F \in C(S^N \times \mathbb{R}^N)$ satisfies* (4). *Fix a path $(B_t)_{t \geqq 0}$. Then, for each $u_0 \in BUC(\mathbb{R}^N)$, there exists a unique solution of* (17).

We conclude with a brief discussion about open problems. Although there are results in the case that H depends on x, much more needs to be done to reduce the complexity of the assumptions. It is also necessary to develop efficient numerical schemes, representation formulæ to understand the possible regularity effects of (17) and, finally, the stochastic properties of the solution.

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