Motion of a graph by $R$-curvature

北海道大学・理学研究科 三上 敏夫 (Toshio Mikami)
Department of Mathematics, Hokkaido University

1. Introduction.

In this talk we introduce our recent result:

H. Ishii and T. Mikami, Motion of a graph by $R$-curvature, Hokkaido mathematical preprint series, No. 340.

Let us first introduce two definitions.

**Definition 1 ($R$-curvature)** Let $R \in L^1(\mathbb{R}^d : [0, \infty), dx)$. For $u \in C(\mathbb{R}^d : \mathbb{R})$, we define the $R$-curvature of $u$ as the finite Borel measure $w(R, u, dx)$ on $\mathbb{R}^d$ given by

$$w(R, u, A) \equiv \int_{\cup_{x \in A} \partial u(x)} R(y)dy \quad \text{for all Borel } A \subset \mathbb{R}^d. \quad (0.1)$$

**Definition 2 (Motion by $R$-curvature)** The graph of $u \in C([0, \infty) \times \mathbb{R}^d : \mathbb{R})$ is called the motion by $R$-curvature if the following holds: for any $\varphi \in C_o(\mathbb{R}^d : \mathbb{R})$ and any $t \geq 0$, 

...
By the continuum limit of a class of infinite particle systems, we first show the existence of the motion by $R$-curvature, and then the uniqueness by the comparison theorem. We also show that the motion by $R$-curvature is a viscosity solution to

\[(PDE) \quad \partial u(t, x)/\partial t = \chi(u, Du(t, x), t, x) \text{Det}_+(D^2 u(t, x)) R(Du(t, x)),\]

where $Du(t, x) \equiv (\partial u(t, x)/\partial x_i)_{i=1}^d$, $D^2 u(t, x) \equiv (\partial^2 u(t, x)/\partial x_i \partial x_j)_{i,j=1}^d$,  

\[\chi(u, p, t, x) \equiv \begin{cases} 
1 & \text{if } p \in \partial u(t, x), \\
0 & \text{otherwise},
\end{cases}\]

$\partial u(t, x)$ denotes the subdifferential of the function $x \mapsto u(t, x)$, and for a real $d \times d$-symmetric matrix $X$,  

\[\text{Det}_+ X \equiv \begin{cases} 
\text{Det} X & \text{if } X \text{ is nonnegative definite}, \\
0 & \text{otherwise}.
\end{cases}\]

We introduce the definition of the viscosity solution to (PDE).

**Definition 3 (Viscosity solution)** (Viscosity subsolution) $u \in C((0, \infty) \times \mathbb{R}^d : \mathbb{R})$ is a viscosity subsolution of (PDE) if whenever $\varphi \in C^2((0, \infty) \times \mathbb{R}^d : \mathbb{R})$ and $u - \varphi \leq (u - \varphi)(t_o, x_o)$,
\[
\frac{\partial \varphi(t_o, x_o)}{\partial t} \leq \chi(u, D\varphi(t_o, x_o), t_o, x_o) \text{Det}_+(D^2\varphi(t_o, x_o)) R(D\varphi(t_o, x_o)).
\]

(Viscosity supersolution) \( u \in C((0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) is a viscosity supersolution of (PDE) if whenever \( \varphi \in C^2((0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) and \( u - \varphi \geq (u - \varphi)(t_o, x_o) \),

\[
\frac{\partial \varphi(t_o, x_o)}{\partial t} \geq \chi^-(u, D\varphi(t_o, x_o), t_o, x_o) \text{Det}_+(D^2\varphi(t_o, x_o)) R(D\varphi(t_o, x_o)).
\]

Here \( \chi^-(v, p, t, x) = 1 \) if

\[ v(t, y) > v(t, x) + \langle p, y - x \rangle \quad (y \neq x) \]

and if there exists \( \varepsilon > 0 \) such that for all \( (s, y) \in (0, \infty) \times \mathbb{R}^d \) satisfying \( |y| > \varepsilon^{-1} \) and \( |s - t| < \varepsilon \),

\[ v(s, y) > p \cdot y + \varepsilon|y|, \]

and \( \chi^-(v, p, t, x) = 0 \), otherwise.

**Remark** 1 If \( \chi^-(v, p, t, x) = 1 \) and \( s \) is close to \( t \), then \( p \in \partial v(s, y) \) for some \( y \).

Finally we discuss under what condition the viscosity solution to (PDE) is the motion by \( R \)-curvature.

**2. Infinite particle systems and the motion by \( R \)-curvature.**

In this section we construct the motion by \( R \)-curvature by the continuum limit of infinite particle systems.
Fix $\epsilon_n \downarrow 0$ as $n \to \infty$, and put

(A.1). $\|R\|_{L^1} \equiv \int_{\mathbb{R}^d} R(y)dy > 0$, $R \geq 0$, $h \in C(\mathbb{R}^d : \mathbb{R})$,

(A.2). $|\partial h(\mathbb{R}^d)(\equiv \cup_{x \in \mathbb{R}^d} \partial h(x))| > 0$,

\begin{align*}
S_n & \equiv \{ v : \mathbb{Z}^d/n \mapsto \mathbb{R} | \sum_{z \in \mathbb{Z}^d/n} (v(z) - h(z)) < \infty, \\
& \quad (v(z) - h(z))/\epsilon_n \in \mathbb{N} \cup \{0\} \text{ for all } z \in \mathbb{Z}^d/n\}.
\end{align*}

Let $\{Y_n(k, \cdot)\}_{0 \leq k}$ be a Markov chain on $S_n$ such that $Y_n(0, \cdot) = h(\cdot)$, and that

$$P(Y_n(k + 1, \cdot) = v_{n,z} | Y_n(k, \cdot) = v) = w(R, \hat{v}, \{z\})/w(R, \hat{Y}_n(0, \cdot), \mathbb{R}^d),$$

where

$$v_{n,z}(x) \equiv \begin{cases} 
    v(x) + \epsilon_n & \text{if } x = z, \\
    v(x) & \text{if } x \in (\mathbb{Z}^d/n)\backslash\{z\}.
\end{cases}$$

Let $p_n(t)$ be a Poisson process, with parameter $n^d\epsilon_n^{-1}w(R, \hat{Y}_n(0, \cdot), \mathbb{R}^d)$, which is independent of $Y_n$. Put

$$Z_n(t, z) \equiv Y_n(p_n(t), z),$$

$$X_n(t, x) \equiv \max(\hat{Z}_n(t, x), h(x)).$$

For $f$ and $g \in C(\mathbb{R}^d : \mathbb{R})$, we put

$$d_{C(\mathbb{R}^d : \mathbb{R})}(f, g) \equiv \sum_{m \geq 1} 2^{-m} \min(\sup_{|x| \leq m} |f(x) - g(x)|, 1).$$

Then we show that $X_n(t, x)$ converges to the motion by $R$-curvature under the following additional conditions.
The closure of the set \( \{ x \in \mathbb{R}^d : \hat{h}(x) < h(x) \} \) does not contain any line which is unbounded in two different directions.

For any \( p \not\in \partial h(\mathbb{R}^d) \) and \( C \in \mathbb{R} \),

\[
\int_{\mathbb{R}^d} \max(<p, x> + C - h(x), 0) dx = \infty.
\]

**Theorem 1** Suppose that (A.1) and (A.3)-(A.4) hold. Then there exists a unique continuous solution \( u \) to (1.2) with \( u(0, \cdot) = h \). Suppose in addition that (A.2) holds. Then the following holds: for any \( \gamma > 0 \) and \( T > 0 \),

\[
\lim_{n \to \infty} P( \sup_{0 \leq t \leq T} d_{C(\mathbb{R}^d; \mathbb{R})}(X_n(t, \cdot), u(t, \cdot)) \geq \gamma) = 0.
\]

**Remark 2** (A.3) holds when \( d = 1 \). If \( h \) is convex, then (A.4) holds.

We give the properties of the motion by \( R \)-curvature.

**Theorem 2** Suppose that (A.1) holds. Let \( u \in C([0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) be the solution to (1.2) with \( u(0, \cdot) = h \). Then:

(a) \( t \mapsto u(t, x) \) is nondecreasing.

(b) \( u = \max(\hat{u}, h) \)

(c) \( u(t, x) - \hat{u}(t, x) \leq h(x) - \hat{h}(x) \). In particular, if \( h(x) = \hat{h}(x) \), then \( u(t, x) = \hat{u}(t, x) \).

Suppose in addition that (A.4) holds. Then:

(d) For any \( t > 0 \),\( \partial u(t, \mathbb{R}^d) = \partial h(\mathbb{R}^d) \).

\[
\int_{\mathbb{R}^d} (u(t, x) - h(x)) dx = t \cdot w(R, h, \mathbb{R}^d).
\]
(e) Let \( \overline{u} \in C([0, \infty) \times \mathbb{R}^d : \mathbb{R}) \) be the solution to (1.2) with \( u(0, \cdot) = \hat{h} \). If 
\[ u(s, \cdot) - \hat{u}(s, \cdot) \neq h - \hat{h} \]
for some \( s \in (0, \infty) \), then \( \overline{u}(t, \cdot) - \hat{u}(t, \cdot) \neq 0 \) for all \( t \geq s \).

According to the above theorem, (a) any graph moves upward by \( R \)-curvature, (b) those points on any graph moving by \( R \)-curvature do not move as far as they stay in its cavities, (c) the height between any graph moving by \( R \)-curvature and its convex envelope is nonincreasing as it evolves, (d) any graph moving by \( R \)-curvature sweeps in time \( t > 0 \) a region with volume given by \( t \cdot w(R, h, \mathbb{R}^d) \), and (e) for the motion of a graph by \( R \)-curvature, taking its convex envelope at time \( t > 0 \) and evolving up to time \( t \) starting with the convex envelope of the initial graph give different graphs in general, if the initial graph is not convex.

3. Motion by \( R \)-curvature and the viscosity solution.

In this section we discuss the relation between the motion by \( R \)-curvature and the viscosity solution to (PDE).

(A.5). \( R \in C(\mathbb{R}^d : [0, \infty)) \).

**Theorem 3** Suppose that (A.1) and (A.5) hold. Then a continuous solution \( u \) to (1.2) with \( u(0, \cdot) = h \) is a viscosity solution to (PDE).

Theorem 3 means that the motion by \( R \)-curvature is the viscosity solution to (PDE). To discuss under what condition the reverse is true, we discuss the uniqueness of the viscosity solution to (PDE).

(A.6). \( R(x) \geq R(rx) \) for any \( r \geq 1 \) and \( x \in \mathbb{R}^d \).

(A.7). \( \inf_{x \neq 0} h(x)/|x| > 0 \).
(A.8). There exists a constant $C > 0$ such that $h(x+y) + h(x-y) - 2h(x) \leq C$ for all $(x, y) \in \mathbb{R}^d \times U_1(o)$, where $U_1(o) \equiv \{y \in \mathbb{R}^d : |y| < 1\}$.

**Theorem 4** Suppose that (A.1) and (A.3)-(A.8) hold. Then there exists a unique continuous viscosity solution $u$ to (PDE) with $u(0, \cdot) = h$ in the space of continuous functions $v$ for which

$$\sup\{|v(t, x) - h(x)| : (t, x) \in [0, T] \times \mathbb{R}^d\} < \infty \text{ for all } T > 0.$$ 

$u$ is also a unique continuous solution to (1.2) with $u(0, \cdot) = h$.

We restrict our attention to Gauss curvature flow and give a finer result.

For $A \subset \mathbb{R}^d$ and $v : A \mapsto \mathbb{R}$, put

$$\text{epi}(v) = \{(x, y) : x \in A, \ y \geq v(x)\}.$$ 

For $r > 0$, put

$$h^r(x) = \inf\{y \in \mathbb{R} \mid U_r((x, y)) \subset \text{epi}(h)\} \quad (x \in \mathbb{R}^d).$$

Under the following condition, we give the comparison theorem for the continuous viscosity solution to (PDE).

(A.1)'. $R(y) = (1 + |y|^2)^{-(d+1)/2}$ and $h \in C(\mathbb{R}^d : \mathbb{R})$.

(A.2)'.

$$\lim_{\theta \downarrow 1} \liminf_{r \to \infty} \left\{\liminf_{|x| \to \infty} (h(\theta x) - h^r(x))\right\} > 0,$$
Theorem 5 Suppose that (A.1)'-(A.2)' hold. Then for any viscosity sub-solution $u$ and supersolution $v$, of (PDE) in the space $C([0,\infty) \times \mathbb{R}^d : \mathbb{R})$, such that $u(0,\cdot) \leq h \leq v(0,\cdot)$, $u \leq v$.

Remark 3 (A.2)' holds if there exists a convex function $h_0 : \mathbb{R}^d \mapsto \mathbb{R}$ such that $h_0(x) \to \infty$ as $|x| \to \infty$ and that

$$\lim_{|x| \to \infty} [h(x) - h_0(x)] = 0.$$ 

In fact, the following holds:

$$\lim_{|x| \to \infty} [h(\theta x) - h^r(x)] = \infty \quad \text{for all } \theta > 1, r > 0,$$

$$\lim_{\theta \downarrow 1} \{\sup_{x \in \mathbb{R}^d} [h(x) - h(\theta x)]\} = 0.$$

The following corollary is better than Theorem 4 in that we can consider the viscosity solution in the entire space $C(\mathbb{R}^d : \mathbb{R})$.

Corollary 1 Suppose that (A.1)'-(A.2)' and (A.3)-(A.4) hold. Then there exists a unique continuous viscosity solution $u$ to (PDE) with $u(0,\cdot) = h$. $u$ is also a unique continuous solution to (1.2) with $u(0,\cdot) = h$.

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G. Barles, S. Biton and O. Ley, Quelque résultats d’unicité pour l’équation du mouvement par courbure moyenne dans $\mathbb{R}^N$, preprint, Theorem 4.1,

where they studied a similar result to Theorem 5 for the mean curvature flow with a convex coercive initial function.