数理解析研究所講究録 1287 巻 2002 年 90-98



# Motion of a graph by $R$-curvature

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## 1. Introduction.

In this talk we introduce our recent result:

H. Ishii and T. Mikami, Motion of a graph by $R$-vurvature, Hokkaido mathematical preprint series, No. 340.

Let us first introduce two definitions.

**Definition 1 ($R$-curvature)** *Let* $R \in L^1(\mathbf{R}^d : [0,\infty), dx)$. *For* $u \in C(\mathbf{R}^d : \mathbf{R})$, *we define the $R$-curvature of $u$ as the finite Borel measure* $w(R,u,dx)$ *on* $\mathbf{R}^d$ *given by*

$$w(R,u,A) \equiv \int_{\cup_{x\in A}\partial u(x)} R(y)dy \quad \textit{for all Borel } A \subset \mathbf{R}^d. \tag{0.1}$$

**Definition 2 (Motion by $R$-curvature)** *The graph of* $u \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$ *is called the motion by $R$-curvature if the following holds: for any* $\varphi \in C_o(\mathbf{R}^d : \mathbf{R})$ *and any* $t \geq 0$,

$$\int_{\mathbf{R}^d}\varphi(x)u(t,x)dx-\int_{\mathbf{R}^d}\varphi(x)u(0,x)dx \tag{0.2}$$
$$=\int_0^t ds\int_{\mathbf{R}^d}\varphi(x)w(R,u(s,\cdot),dx).$$

By the continuum limit of a class of infinite particle systems, we first show the existence of the motion by $R$-curvature, and then the uniqueness by the comparison theorem. We also show that the motion by $R$-curvature is a viscosity solution to

$$(PDE) \qquad \partial u(t,x)/\partial t=\chi(u,Du(t,x),t,x)\mathrm{Det}_+(D^2u(t,x))R(Du(t,x)),$$

where $Du(t,x)\equiv(\partial u(t,x)/\partial x_i)_{i=1}^d$, $D^2u(t,x)\equiv(\partial^2u(t,x)/\partial x_i\partial x_j)_{i,j=1}^d$,

$$\chi(u,p,t,x)\equiv\begin{cases}1 & \text{if } p\in\partial u(t,x),\\ 0 & \text{otherwise,}\end{cases}$$

$\partial u(t,x)$ denotes the subdifferential of the function $x\mapsto u(t,x)$, and for a real $d\times d$-symmetric matrix $X$,

$$\mathrm{Det}_+X\equiv\begin{cases}\mathrm{Det}X & \text{if } X \text{ is nonnegative definite,}\\ 0 & \text{otherwise.}\end{cases}$$

We introduce the definition of the viscosity solution to (PDE).

**Definition 3 (Viscosity solution)** *(Viscosity subsolution)* $u\in C((0,\infty)\times\mathbf{R}^d:\mathbf{R})$ *is a viscosity subsolution of (PDE) if whenever* $\varphi\in C^2((0,\infty)\times\mathbf{R}^d:\mathbf{R})$ *and* $u-\varphi\le(u-\varphi)(t_o,x_o)$,

$$\partial\varphi(t_o,x_o)/\partial t \leq \chi(u,D\varphi(t_o,x_o),t_o,x_o)Det_+(D^2\varphi(t_o,x_o))R(D\varphi(t_o,x_o)).$$

*(Viscosity supersolution)* $u \in C((0,\infty)\times \mathbf{R}^d : \mathbf{R})$ *is a viscosity supersolution of (PDE) if whenever* $\varphi \in C^2((0,\infty)\times \mathbf{R}^d : \mathbf{R})$ *and* $u-\varphi \geq (u-\varphi)(t_o,x_o)$,

$$\partial\varphi(t_o,x_o)/\partial t \geq \chi^-(u,D\varphi(t_o,x_o),t_o,x_o)Det_+(D^2\varphi(t_o,x_o))R(D\varphi(t_o,x_o)).$$

*Here* $\chi^-(v,p,t,x) = 1$ *if*

$$v(t,y) > v(t,x) + < p, y-x > \qquad (y \neq x)$$

*and if there exists* $\varepsilon > 0$ *such that for all* $(s,y) \in (0,\infty)\times \mathbf{R}^d$ *satisfying* $|y| > \varepsilon^{-1}$ *and* $|s-t| < \varepsilon$,

$$v(s,y) > p\cdot y + \varepsilon|y|,$$

*and* $\chi^-(v,p,t,x) = 0$, *otherwise.*

**Remark** 1 *If* $\chi^-(v,p,t,x) = 1$ *and* $s$ *is close to* $t$, *then* $p \in \partial v(s,y)$ *for some* $y$.

Finally we discuss under what condition the viscosity solution to (PDE) is the motion by $R$-curvature.

**2. Infinite particle systems and the motion by $R$-curvature.**

In this section we construct the motion by $R$-curvature by the continuum limit of infinite particle systems.

Fix $\varepsilon_n \downarrow 0$ as $n \to \infty$, and put

(A.1). $||R||_{L^1} \equiv \int_{\mathbf{R}^d} R(y)dy > 0,\ R \geq 0,\ h \in C(\mathbf{R}^d : \mathbf{R})$,

(A.2). $|\partial h(\mathbf{R}^d)(\equiv \cup_{x\in\mathbf{R}^d}\partial h(x))| > 0$,

$$\begin{aligned} S_n \equiv & \{v : \mathbf{Z}^d/n \mapsto \mathbf{R} | \sum_{z\in\mathbf{Z}^d/n} (v(z) - h(z)) < \infty, \\ & (v(z) - h(z))/\varepsilon_n \in \mathbf{N} \cup \{0\} \text{ for all } z \in \mathbf{Z}^d/n\}. \end{aligned}$$

Let $\{Y_n(k,\cdot)\}_{0\leq k}$ be a Markov chain on $S_n$ such that $Y_n(0,\cdot) = h(\cdot)$, and that

$$P(Y_n(k+1,\cdot) = v_{n,z}|Y_n(k,\cdot) = v) = w(R, \hat{v}, \{z\})/w(R, \hat{Y}_n(0,\cdot), \mathbf{R}^d),$$

where

$$v_{n,z}(x) \equiv \begin{cases} v(x) + \varepsilon_n & \text{if } x = z, \\ v(x) & \text{if } x \in (\mathbf{Z}^d/n)\backslash\{z\}. \end{cases}$$

Let $p_n(t)$ be a Poisson process, with parameter $n^d\varepsilon_n^{-1}w(R, \hat{Y}_n(0,\cdot), \mathbf{R}^d)$, which is independent of $Y_n$. Put

$$Z_n(t,z) \equiv Y_n(p_n(t), z),$$

$$X_n(t,x) \equiv \max(\hat{Z}_n(t,x), h(x)).$$

For $f$ and $g \in C(\mathbf{R}^d : \mathbf{R})$, we put

$d_{C(\mathbf{R}^d:\mathbf{R})}(f,g) \equiv \sum_{m\geq 1} 2^{-m} \min(\sup_{|x|\leq m} |f(x) - g(x)|, 1)$.

Then we show that $X_n(t,x)$ converges to the motion by $R$-curvature under the following additional conditions.

(A.3). The closure of the set $\{x \in \mathbf{R}^d : \hat{h}(x) < h(x)\}$ does not contain any line which is unbounded in two different directions.

(A.4). For any $p \notin \partial h(\mathbf{R}^d)$ and $C \in \mathbf{R}$,

$$\int_{\mathbf{R}^d} \max(< p, x > +C - h(x), 0) dx = \infty.$$

**Theorem 1** *Suppose that (A.1) and (A.3)-(A.4) hold. Then there exists a unique continuous solution $u$ to (1.2) with $u(0,\cdot) = h$. Suppose in addition that (A.2) holds. Then the following holds: for any $\gamma > 0$ and $T > 0$,*

$$\lim_{n\to\infty} P(\sup_{0\le t\le T} d_{C(\mathbf{R}^d:\mathbf{R})}(X_n(t,\cdot), u(t,\cdot)) \ge \gamma) = 0.$$

**Remark 2** *(A.3) holds when $d = 1$. If $h$ is convex, then (A.4) holds.*

We give the properties of the motion by $R$-curvature.

**Theorem 2** *Suppose that (A.1) holds. Let $u \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$ be the solution to (1.2) with $u(0,\cdot) = h$. Then:*

*(a) $t \mapsto u(t,x)$ is nondecreasing.*

*(b) $u = \max(\hat{u}, h)$*

*(c) $u(t,x) - \hat{u}(t,x) \le h(x) - \hat{h}(x)$. In particular, if $h(x) = \hat{h}(x)$, then $u(t,x) = \hat{u}(t,x)$.*

*Suppose in addition that (A.4) holds. Then:*

*(d) For any $t > 0$, $\partial u(t, \mathbf{R}^d) = \partial h(\mathbf{R}^d)$.*

$$\int_{\mathbf{R}^d} (u(t,x) - h(x)) dx = t \cdot w(R, h, \mathbf{R}^d).$$

*(e) Let* $\overline{u} \in C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$ *be the solution to (1.2) with* $u(0,\cdot) = \hat{h}$. *If* $u(s,\cdot) - \hat{u}(s,\cdot) \neq h - \hat{h}$ *for some* $s \in (0,\infty)$, *then* $\overline{u}(t,\cdot) - \hat{u}(t,\cdot) \neq 0$ *for all* $t \geq s$.

According to the above theorem, (a) any graph moves upward by $R$-curvature, (b) those points on any graph moving by $R$-curvature do not move as far as they stay in its cavities, (c) the height between any graph moving by $R$-curvature and its convex envelope is nonincreasing as it evolves, (d) any graph moving by $R$-curvature sweeps in time $t > 0$ a region with volume given by $t \cdot w(R, h, \mathbf{R}^d)$, and (e) for the motion of a graph by $R$-curvature, taking its convex envelope at time $t > 0$ and evolving up to time $t$ starting with the convex envelope of the initial graph give different graphs in general, if the initial graph is not convex.

## 3. Motion by $R$-curvature and the viscosity solution.

In this section we discuss the relation between the motion by $R$-curvature and the viscosity solution to (PDE).

(A.5). $R \in C(\mathbf{R}^d : [0,\infty))$.

**Theorem 3** *Suppose that (A.1) and (A.5) hold. Then a continuous solution* $u$ *to (1.2) with* $u(0,\cdot) = h$ *is a viscosity solution to (PDE).*

Theorem 3 means that the motion by $R$-curvature is the viscosity solution to (PDE). To discuss under what condition the reverse is true, we discuss the uniqueness of the viscosity solution to (PDE).

(A.6). $R(x) \geq R(rx)$ for any $r \geq 1$ and $x \in \mathbf{R}^d$.

(A.7). $\inf_{x \neq o} h(x)/|x| > 0$.

(A.8). There exists a constant $C > 0$ such that $h(x+y)+h(x-y)-2h(x) \leq C$ for all $(x, y) \in \mathbf{R}^d \times U_1(o)$, where $U_1(o) \equiv \{y \in \mathbf{R}^d : |y| < 1\}$.

**Theorem 4** *Suppose that (A.1) and (A.3)-(A.8) hold. Then there exists a unique continuous viscosity solution $u$ to (PDE) with $u(0, \cdot) = h$ in the space of continuous functions $v$ for which*

$$\sup\{|v(t,x) - h(x)| : (t,x) \in [0,T] \times \mathbf{R}^d\} < \infty \text{ for all } T > 0.$$

*$u$ is also a unique continuous solution to (1.2) with $u(0, \cdot) = h$.*

We restrict our attention to Gauss curvature flow and give a finer result. For $A \subset \mathbf{R}^d$ and $v : A \mapsto \mathbf{R}$, put

$$\mathrm{epi}(v) = \{(x, y) : x \in A, \ y \geq v(x)\}.$$

For $r > 0$, put

$$h^r(x) = \inf\{y \in \mathbf{R} \mid U_r((x,y)) \subset \mathrm{epi}(h)\} \quad (x \in \mathbf{R}^d).$$

Under the following condition, we give the comparison theorem for the continuous viscosity solution to (PDE).

(A.1)'. $R(y) = (1 + |y|^2)^{-(d+1)/2}$ and $h \in C(\mathbf{R}^d : \mathbf{R})$.

(A.2)'.

$$\liminf_{\theta \downarrow 1} \{\liminf_{r \to \infty} [\liminf_{|x| \to \infty} (h(\theta x) - h^r(x))]\} > 0,$$

$$\lim_{\theta \downarrow 1}\{\sup_{x \in \mathbf{R}^d}(h(x) - h(\theta x))\} = 0.$$

**Theorem 5** *Suppose that (A.1)'-(A.2)' hold. Then for any viscosity subsolution $u$ and supersolution $v$, of (PDE) in the space $C([0,\infty) \times \mathbf{R}^d : \mathbf{R})$, such that $u(0,\cdot) \leq h \leq v(0,\cdot)$, $u \leq v$.*

**Remark 3** *(A.2)' holds if there exists a convex function $h_0 : \mathbf{R}^d \mapsto \mathbf{R}$ such that $h_0(x) \to \infty$ as $|x| \to \infty$ and that*

$$\lim_{|x| \to \infty}[h(x) - h_0(x)] = 0.$$

*In fact, the following holds:*

$$\lim_{|x| \to \infty}[h(\theta x) - h^r(x)] = \infty \quad \textit{for all } \theta > 1, r > 0,$$

$$\lim_{\theta \downarrow 1}\{\sup_{x \in \mathbf{R}^d}[h(x) - h(\theta x)]\} = 0.$$

The following corollary is better than Theorem 4 in that we can consider the viscosity solution in the entire space $C(\mathbf{R}^d : \mathbf{R})$.

**Corollary 1** *Suppose that (A.1)'-(A.2)' and (A.3)-(A.4) hold. Then there exists a unique continuous viscosity solution $u$ to (PDE) with $u(0,\cdot) = h$. $u$ is also a unique continuous solution to (1.2) with $u(0,\cdot) = h$.*

Acknowledgement: We woule like to thank Prof. K. Ishii for informing us the following paper:

G. Barles, S. Biton and O. Ley, Quelque résultats d'unicité pour l'equation du mouvement par courbure moyenne dans $\mathbf{R}^N$, preprint, Theorem 4.1,

where they studied a similar result to Theorem 5 for the mean curvature flow with a convex coercive initial function.