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ON THE UNIQUENESS OF EQUIVARIANT ORIENTATION CLASSES

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INTRODUCTION

Let $p : E \to B$ be a G-vector bundle. We would like to define an orientation class in the equivariant setting, so that each of the fixed-point-set bundles are compatibly oriented by that orientation class for all orbit type subgroups of G. As the paper of S. Costenoble, J. P. May and S. Waner [CMW] writes, there is no satisfactory answer in the literature except under rather restrictive hypotheses.

Here is an illustrative example of one of the difficulties around this problem.

Definition 0.1. *Let V be a G-representation. A G-vector bundle is called to be of dimension V if each of its fiber representations V_x is isomorphic to V as a G_x-representation. A naive orientation class is defined to be a compatible collection of homotopy classes $\phi(x)$ of G_x-linear isometries between V_x and V, for each $x \in B$.*

Example 0.2. *Let $G = S^1$ and let it act on $B = S^2$ by the axis rotation. There are two fixed points, which we denote by n and s. The tangent bundle of B should obviously be equivariantly orientable, but it is difficult to define a satisfactory orientation class for it. In fact, there cannot exist any naive orientation class, because for x in the G-free part of B there is only one homotopy class of S^1-isometries $V \to V$ but if we connect the two fixed points n and s with a path and compare the corresponding fiber representations the induced pullback isometry between V_n and V_s is necessarily orientation reversing. Therefore there cannot be any compatible way to construct a naive orientation class.*

In order to overcome this difficulty, S. Costenoble, J. P. May and S. Waner [CMW] have constructed a new, categorical definition of orientation for any G-vector bundle. We will briefly outline their definition in Section 1 below.

We would like to algebraically define orientation classes in cohomology theories, and compare those definitions with Costenoble-May-Waner's categorical definition. We will define equivariantly orientable equivariant cohomology theories, and get some uniqueness theorem of equivariant orientation classes in such theories, thus proving that those algebraically defined equivariant orientation classes are equivalent to Costenoble-May-Waner's categorically defined classes, under some circumstances. The basic reference for equivariant cohomology theories and equivariant homotopy theory is J. Peter May's book [M].

The main tool here is the notion of G-$CW(V,\gamma)$-complex studied in the author's earlier paper [N].

The notion of a G-$CW(\gamma)$-complex was defined in [CMW] in order to construct a natural notion of a "G-orientation", and in order to construct a Poincarè Duality on spaces that are not "G-connected" (See [CW 2]). It was impossible to determine a natural notion of a "dual G-cell" under the traditional notion of G-CW-complexes, but if we use a new notion of a "G-cell", as explained below, it is now possible to obtain a natural "dual cell" and "dual decomposition". The new building-block of a "G-cell" is:

$$\gamma(x) = \left[G/H \times \mathbb{R}^{\ell} \longrightarrow G/H\right]$$

that is, some object that "assembles" an $\mathbb{R}^{\ell}$-trivial bundle for each orbit $x : G/H \to X$ in X, functorially. ($\ell \geq 0, H < G$.)

The notion of a G-$CW(V)$-complex was defined by L. G. Lewis in [L] in order to construct a generalized "equivariant suspension theorem". In there, a building-block "G-cell" is:

$$\Sigma^{V+k} G/K^{+},$$

which is the compactification of the $(V \oplus \mathbb{R}^k)$-trivial bundle over an orbit G/K ($k \geq -1, K < G$), for a *fixed* G-representation space V. (Here he assumes that $|V^G| \geq 1$.) By assigning a common representation component V to all of the cells, he tries to grasp the contribution of a V-suspension onto the total space.

In [L], Lewis has constructed a natural "G-Eilenberg-MacLane space", constructed a natural obstruction theory from there, based on the above G-$CW(V)$-complexes, still assuming that $|V^G| \geq 1$. He then proved the following theorem:

Suspension Theorem ([L], Theorem 2.5). *Assuming that $|V^G| \geq 1$, if Y is a $(|V^*| - 1)$-connected, based G-CW-complex, then the morphism*

$$\widetilde{\sigma} : s_* \underset{\sim}{\pi}{}_V^G Y \longrightarrow \underset{\sim}{\pi}{}_{V+W}^G \Sigma^W Y$$

is a natural isomorphism for any representation W.

Here, s_* is constructed as a left adjoint of the natural functor derived from a forgetful functor

$$s : \mathcal{B}_G(V) \longrightarrow \mathcal{B}_G(V + W)$$

between the Burnside categories, and $\underset{\sim}{\pi}{}_V^G(Y)$ is a Mackey functor assigning:

$$\underset{\sim}{\pi}{}_V^G(Y)(G/K) = \left[\Sigma^V G/K^+, Y\right]_G = \left[\Sigma^V, Y\right]_K$$

to each orbit G/K in Y. The details will be explained below.

In [N], we extended Lewis' construction under the more general situation where the condition $|V^G| \geq 1$ is removed:

Suspension Theorem ([N], Theorem B). *With the s_* as defined in the above, for any $(|V^G|-1)$-connected G-CW-complex Y, the morphism:*

$$\tilde{\sigma} : s_* \underset{\sim}{\pi}{}_V^G Y \longrightarrow \underset{\sim}{\pi}{}_{V+W}^G \left(\Sigma^W Y\right).$$

is a natural isomorphism of G-$(V+W)$-Mackey functors.

Using this result, we re-construct orientation classes for "equivariantly orientable cohomology theories", by patching together classes over orbit bundles, thus we get some characterization result on equivariant orientation classes on such cohomology theories.

Section 1. Costenoble-May-Waner's orientation

Let G be a finite group, and let V be a finite dimensional G-representation over $\mathbb{R}$. Let $\mathcal{O}_G$ be the category of G-orbits as objects and G-maps between them as morphisms. Let $\mathcal{V}_G(n)$ be the homotopy category of n-dimensional orthogonal G-bundles over G-orbits. A morphism in $\mathcal{V}_G(n)$ is a G-homotopy class of bundle maps.

For a G-space B, let $\pi_G B$ be the category consisting of G-maps $x : G/H \longrightarrow B$, where $H < G$, as objects, and any pair (σ, ω):

$$\begin{array}{ll} \sigma : G/H \longrightarrow G/K & G\text{-map} \\ \omega : G/H \times I \longrightarrow B & G\text{-homotopy between } x \text{ and } y \circ \sigma \end{array}$$

We will call this $\pi_G B$ "the equivariant fundamental groupoid of B."

Definition 1.1 ([CMW], Definition 7.1). *The two categories $\pi_G B$ and $\mathcal{V}_G(n)$ both have a natural projection functor ϕ onto the category $\mathcal{O}_G$. We will call any functor*

$$\gamma : \pi_G B \longrightarrow \mathcal{V}_G$$

which is compatible with the projections (that is, $\phi \circ \gamma = \phi$) with the name "an n-dimensional representation of the groupoid $\pi_G X$." If $p : E \to B$ is a G-vector bundle, then a representaion

$$p^* : \pi_G B \longrightarrow \mathcal{V}_G$$

is naturally defined via bundle pullback. More generally, any functor

$$R : \mathcal{E} \longrightarrow \mathcal{R}$$

over $\pi_G B$ is called a representaion, where $\mathcal{R}$ is a skeletal groupoid (e.g. $\mathcal{V}_G$) over $\pi_G B$ and $\mathcal{E}$ is any groupoid (e.g. $\pi_G B$) over $\pi_G B$.

Definition 1.2 ([CMW], Definition 2.8). *A G-vector bundle $p : E \to B$ is called orientable if the functor $p^* : \pi_G B \longrightarrow \mathcal{V}_G$ satisfies $p^*(\omega, \alpha) = p^*(\omega', \alpha)$ for every pair of morphisms (ω, α) and (ω', α) with the same source and target and the same image in $\mathcal{O}_G$. That is, $p^*(\omega, \alpha)$ is independent of the choice of the path class ω. For example, for a representation V of G, the projection $B \times V \longrightarrow B$ is orientable.*

Now the point is that there is not any straightforward way to define an orientation *class* for an orientable G-bundle. As Example 0.2 shows, a naive orientation class is undefinable. In order to overcome this situation, S. Costenoble, J. P. May and S. Waner made the following construction in [CMW]:

Definition 1.3 ([CMW], Definition 7.6). *One can construct a natural "universally saturated" representation $(\mathcal{SR}, S)$:*

$$S : \mathcal{SR} \longrightarrow \mathcal{R}$$

such that every faithful representation $(\mathcal{E}, R)$:

$$R : \mathcal{E} \longrightarrow \mathcal{R}$$

*maps into it. In the case where $\mathcal{R} = \mathcal{V}_G$, $\mathcal{E} = \pi_G B$ and $R = p^*$ for a G-vector bundle $p : E \to B$, its **orientation** is defined to be a map from the bundle-pullback representation $p^* : \pi_G B \longrightarrow \mathcal{V}_G$ to the "universal saturation" of p^*, $S : \mathcal{SV}_G \longrightarrow \mathcal{V}_G$:*

$$(F, \phi) : (\pi_G, p^*) \longrightarrow (\mathcal{SV}_G, S).$$

Consult Section 7 of [CMW] for the details. As the main feature of the definition, they have the following result:

Corollary 7.7 of [CMW]. *A representation is orientable if and only if it has an orientation.*

A G-bundle map $(\tilde{f}, f) : (E \to B) \longrightarrow (E' \to B')$ is orientation preserving if the natural map

$$(\pi_G B, p^*) \longrightarrow (\pi_G B', q^*)$$

is compatible with the orientations $(F, \phi) : (\pi_G B, p^*) \longrightarrow (\mathcal{SV}_G, S)$ and $(F', \phi') : (\pi_G B', q^*) \longrightarrow (\mathcal{SV}_G, S)$.

That is, the orientation class is not based on a single representation space, but rather based on the "universally saturated" ***system***: $(\mathcal{SV}_G, S)$.

SECTION 2. G-$CW(V,\gamma)$-COMPLEXES

Now we define a G-$CW(V,\gamma)$-complex. Let X be a G-space, V be a G-representation and $\gamma : \pi_G X \to \mathcal{V}_G(n)$ be as above.

Definition 1.1. *A G-$CW(V,\gamma)$-structure on X is a filtration*

$$X = \operatorname{colim} X^n$$

which satisfies the following two conditions:

(1) $X^0 = \coprod \left(G/H \xrightarrow{x} X\right)$ *: a disjoint union of G-orbits such that*

$$\gamma(G/H \xrightarrow{x} X) = \left[G/H \times V \times \mathbb{R}^{\ell} \longrightarrow G/H\right]$$

(a trivial bundle over G/H);

(2) $X^n = X^{n-1} \bigcup_{\varphi_n} \left(\coprod_m e^n_m\right)$, *where a "core orbit" $x : G/H \longrightarrow e^n_m$ is specified to each e^n_m, and a G-homeomorphism*

$$e^n_m \cong D\left(\gamma(x) \oplus \mathbb{R}^{n-\ell}\right)$$

and a sub-representation as a split summand

$$\begin{cases} V \overset{\oplus}{\hookrightarrow} \gamma(x) & \text{if } n \geq \ell \\ V \oplus \mathbb{R}^{\ell-n} \overset{\oplus}{\hookrightarrow} \gamma(x) & \text{if } n < \ell \end{cases}$$

are also specified.

In other words, this definition adds an extra requirement that all of the $\gamma(x)$'s include V as a direct sum component, into the original definition of G-$CW(\gamma)$-complexes by Costenoble-May-Waner in [CMW].

Next, we construct our basic notion of equivariant homotopy set objects, and define the Burnside category, on which all of the algebraic constructions will be based.

For a G-representation V and G-spaces X and Y, $[X,Y]_G$ will denote the set of G-homotopy classes of G-maps $f : X \to Y$ (where the base points are not taken into account). Let $S^V = DV/SV$, the usual one-point compactification of V, and let $\Sigma^V X^+ = S^V \wedge X^+$, the smash product of S^V and the space X^+ that is X attached with a disjoint base point.

Definition. *$\mathcal{B}_G(V)$ denotes the **Burnside category** of V, where the objects are of the form*

$$\coprod_j G/K_j,$$

a finite disjoint union of G-orbits, and for any two objects A and B, the morphism set is:

$$\mathcal{B}_G(V)(A,B) = \left[\Sigma^V A^+, \Sigma^V B^+\right]_G$$

Note that $\mathcal{B}_G(V)(A,B)$ is a group when $|V^G| \geq 1$.

Definition. *A "G-V-Mackey functor" is a contravariant functor of the form*

$$M : \mathcal{B}_G(V) \longrightarrow \mathsf{Sets}_*$$

from the Burnside category to the category of based sets, which satisfies the condition that "converts finite coproducts into finite products," that is, there is a natural isomorphism between sets:

$$M(A \sqcup B) \cong M(A) \times M(B).$$

Definition. *A G-V-Mackey functor M is called "groupoid-valued", if the image of M lies in the subcategory of groupoids (i.e., all morphisms are invertible).*

Definition. *Let $\mathcal{M}_G(V)$ denote the category of all "groupoid-valued" G-V-Mackey functors from $\mathcal{B}_G(V)$ to Sets_*.*

The basic example of a G-V-Mackey functor is the following.

Definition. *Define a G-V-Mackey functor $\underset{\sim}{\pi}{}_V^G Y : \mathcal{B}_G(V) \longrightarrow \mathsf{Sets}_*$ to be:*

$$\underset{\sim}{\pi}{}_V^G Y(A) = \left[\Sigma^V A^+, Y\right]_G$$

on objects, and

$$\underset{\sim}{\pi}{}_V^G Y(f) = f^* : \left[\Sigma^V B^+, Y\right]_G \longrightarrow \left[\Sigma^V A^+, Y\right]_G,$$

on morphisms, for an $f : \Sigma^V A^+ \longrightarrow \Sigma^V B^+$ in $\mathcal{B}_G(V)$.

It is obvious that $\underset{\sim}{\pi}{}_V^G Y$ is always a groupoid-valued Mackey functor.

Definition of an Eilenberg-MacLane space. *For a G-V-Mackey functor M, a space $K_V^G M$ is called a "G-V-Eilenberg-Maclane space, if it is a G-space which is $(|V^*|-1)$-connected, is G-homotopic to a G-CW-complex, and if it satisfies:*

$$\underset{\sim}{\pi}{}_V^G \left(K_V^G M\right) = M, \qquad \underset{\sim}{\pi}{}_{V+k}^G \left(K_V^G M\right) = 0 \qquad \text{for any } k > 0.$$

We have proved the following theorem in our previous paper [N]:

Theorem A ([N]). *For any V and any groupoid-valued G-V-Mackey functor M, a G-V-Eilenberg-MacLane space $K_V^G M$ exists. Moreover, the assignment from M to $K_V^G M$ is the categorical right adjoint of the "homotopy groupoid" construction $\underset{\sim}{\pi}{}_V^G$, that is, there is a natural isomorphism of sets:*

$$\left[X, K_V^G M\right] \xrightarrow[\cong]{} \mathcal{M}_G(V)\left[\underset{\sim}{\pi}{}_V^G(X), \underset{\sim}{\pi}{}_V^G\left(K_V^G M\right)\right] = \mathcal{M}_G(V)\left[\underset{\sim}{\pi}{}_V^G(X), M\right].$$

Section 3. The Suspension Theorem and representation of cohomology theories

The traditional suspension theorem on (non-equivariant) homotopy sets obviously fails in the equivariant situation. For the suspension map:

$$[X,Y]_G \longrightarrow [\Sigma^W X, \Sigma^W Y]_G$$

to be an isomorphism, we need a very strong (and non-natural) restriction about the dimensions of fixed-point sets X^H, Y^H for *all* subgroups H of G. For example, the suspension map $[S^n, S^n]_G \to [S^{W+n}, S^{W+n}]_G$ can *never* be surjective, if W contains a regular representation.

Therefore, we need a different formulation of an "equivariant suspension theorem." Recall that $\mathcal{M}_G(V)$ was the category of all groupoid-valued G-V-Mackey functors. We have proved the following results:

Lemma ([N], Lemma 2.1). *If $V \subset U$ is a sub-G-representation, the natural "forgetful functor" $s^* : \mathcal{M}_G(U) \longrightarrow \mathcal{M}_G(V)$ (that is induced by the canonical functor $s : \mathcal{B}_G(V) \longrightarrow \mathcal{B}_G(U)$) has a left adjoint, that is, there is a natural isomorphism:*

$$\mathcal{M}_G(U)(M, s_*N) = \mathcal{M}_G(U)(s_*s^*M, s_*N) \cong \mathcal{M}_G(V)(s^*M, s_*N).$$

Definition. *For a representation space W, define a "suspension morphism":*

$$\sigma : \underset{\sim}{\pi}_V^G Y \longrightarrow s^* \underset{\sim}{\pi}_{V+W}^G (\Sigma^W Y)$$

by assigning the suspension map:

$$\sigma_A : [\Sigma^V A^+, Y]_G \longrightarrow [\Sigma^{W+V} A^+, \Sigma^W Y]$$

to each object A in $\mathcal{B}_G(V)$. Since this σ is naturally a transformation of Mackey functors, it is a functor from $\mathcal{M}_G(V)$ to $\mathcal{M}_G(V+W)$. Therefore, by Lemma 2.1, we have its left adjoint:

$$\tilde{\sigma} : s_* \underset{\sim}{\pi}_V^G Y \longrightarrow \underset{\sim}{\pi}_{V+W}^G (\Sigma^W Y).$$

Lemma ([N], Lemma 2.2). *Consider the suspension functor $s : \mathcal{B}_G(V) \longrightarrow \mathcal{B}_G(V+W)$. For any $(V+W)$-Mackey functor N, we have a G-homotopy equivalence*

$$\theta : \Omega^W K_{V+W}^G N \longrightarrow K_V^G(s^*N)$$

which makes the following diagram of natural isomorphisms to commute:

$$\begin{array}{ccc}
\underset{\sim}{\pi}_V^G (\Omega^W K_{V+W}^G N) & \xrightarrow{\theta_*} & \underset{\sim}{\pi}_V^G (K_V^G(s^*N)) \\
\varphi \big\downarrow & & \big\downarrow \\
\left(s^* \underset{\sim}{\pi}_{V+W}^G\right)(K_{V+W}^G N) & \longrightarrow & s^*N.
\end{array}$$

Theorem B (Suspension Theorem, [N]). *With the s_* as defined in the above, for any $(|V^G|-1)$-connected G-CW-complex Y, the morphism:*

$$\tilde{\sigma} : s_* \underset{\sim}{\pi}{}_V^G Y \longrightarrow \underset{\sim}{\pi}{}_{V+W}^G (\Sigma^W Y).$$

is a natural isomorphism of G-(V + W)-Mackey functors.

Remark that the morphism is based on the category of groups if $|V^G| \geq 1$ (and of abelian groups if $|V^G| \geq 2$), but it is merely on the category of sets, if $|V^G| = 0$.

In Theorem A above, we have constructed a G-V-Eilenberg-MacLane space using the explicit cell structure of G-$CW(V,\gamma)$-complexes. Therefore, the G-V-Eilenberg-MacLane space $K_V^G M$ satisfies the standard obstruction theory properties. (cf. [N], Section 3)

With the presence of the Suspension Theorem, the existence of the equivariant Eilenberg-MacLane spaces and the standard obstruction theory techinques, the representation theorem of Brown type ([Bro]) for generalized equivariant cohomology theories can be proved by the standard argument:

Theorem C. *(cf. Waner's Definition 4.2 in Chapter X of [M]) For a G-V-Mackey functor M, an $RO(G)$-graded ordinary cohomology theory $H_G^*(X;M)$ can be defined by*

$$H_G^{V+n}(X;M) = H^{|V|+n}\operatorname{Hom}_{\mathcal{O}_G}(C_*^V(X), M)$$

for G-CW-complexes X, using the G-$CW(V,\gamma)$-cell structures.

Theorem D (Brown Representability Theorem, [Bro]). *(Theorem 3.1 in Chapter XIII of [M]) A contravariant set-valued functor k on the homotopy category of G-connected based G-$CW(V,\gamma)$-complexes is representable in the form $k(X) \cong [X,K]_G$ for a based G–$CW(V,\gamma)$-complex K if and only if k satisfies the wedge and Mayer-Vietoris axioms: k takes wedges to products and takes homotopy pushouts to weak pullbacks.*

Using these theorems, together with the above-mentioned Suspension Theorem and the existence of Eilenberg-MacLane spaces for general G-V-Mackey functors, we see that there is a compatible method for the construction of classifying space for G-V-Mackey functors, that is, such G-V-Mackey functors are classified as an $RO(G)$-graded system, as a system indexed by the representation V.

Now that we have the basic tools readily available, we can proceed to investigate the comparison of generalized equivariant cohomology theories.

Section 4. Equivariantly oriented cohomology theories

We define a generalized equivariant cohomology theory *orientable* when it admits an orientation class for any orientable G-vector bundle in the sense of Definition 1.2. Let $BU(V,\gamma,S)$ be the classifying space for oriented (V,γ)-bundles constructed in Theorem 22.4 of [CMW].

Definition 4.1. *Let E_G^* be a generalized $RO(G)$-graded cohomology theory in the sense of Chapter XIII of Peter May's book* [M]. *E_G^* is called orientable if there is a class $\sigma(V,\gamma,S) \in E_G^*(BU(V,\gamma,S))$ that maps to an orientation class when restricted to any H-fixed point sets of all subgroups H of G. This class $\sigma(V,\gamma,S)$ is called the E_G^*-orientation class.*

For any orientable G-vector bundle $p : E \to B$ and its classifying map $B \longrightarrow BU(V,\gamma,S)$, we call $p^*(\sigma(V,\gamma,S)) \in E_G^*(B)$ be the orientation class of p.

Proposition 4.2. *The equivariant K-theory, K_G^*, is orientable.*

Proof. For the definition and basic properties of the equivariant K-theory, consult Chapter XIV of [M]. To the universal oriented bundle over the classifying space $BU(V,\gamma,S)$, we assign the K-cohomology classes that consist of the building-blocks for the class $\sigma(V,\gamma,S)$ inductively, by the categorical definition and construction performed in the original method of Costenoble-May-Waner in [M]. The construction is purely straightforward, following the non-equivariant method in each step, using the functorial properties of bundle lifting. Since the construction of the orientation is purely categorical, this can be done by using only formal arguments. Now the proof is just the standard routine work.

In the case when B is a smooth G-manifold, the definition reduces to the classical definition of the orientation class via the Thom complex. (cf. Chapter XVI of [M])

Definition 4.3. *If $p : E \to B$ is an n-plane G-bundle, then an E_G^*-orientation class of p is defined to be an element $\mu \in E_G^\alpha(Tp)$ for some $\alpha \in RO(G)$ of virtual dimension n such that, for each inclusion $i : G/H \longrightarrow B$, the restriction of μ to the Thom complex of the pullback $i^*\mu$ is a generator of the free $E^*(S^0)$-module $E_G^*(Ti^*p)$.*

In this situation, Costenoble and Waner have proved a variant of Thom Isomorphism Theorem and the Poincarè Duality:

Thom Isomorphism (Theorem 9.2, Chapter XVI of [M]). *Let $\mu \in E_G^\alpha(Tp)$ be an orientation of the G-vector bundle p over B. Then*

$$\cup\mu : E_G^\beta(B_+) \longrightarrow E_G^{\alpha+\beta}(Tp)$$

is an isomorphism for all β.

Poincarè Duality (Definition 9.3, Chapter XVI of [M]). *If M is a closed smooth manifold such that its tangent bundle τ is oriented via $\mu \in E_G^V(T\tau)$, then one can define the composite of the Thom and Spanier-Whitehead duality isomorphisms*

$$D : E_G^\beta(M_+) \longrightarrow E_G^{V-\alpha+\beta}(T\nu) \longrightarrow E^G_{\alpha-\beta}(M)$$

where ν is the normal bundle, and $[M] = D(1) \in E^G_\alpha(M)$ is called the fundamental class associated with the orientation.

Now we specialize to the case where the base space B is a single orbit G/H. Since

$$[\Sigma^V G/H^+, Y]_G = [\Sigma^V, Y]_H,$$

We have $E^V_G(G/H) = E^V_H(pt)$, and this is where the orientation class lives in this case. Next, for any morphism $\sigma : G/H \longrightarrow G/K$ in the category $\pi_G B$, we get a map $\sigma^* : E^V_K(pt) \longrightarrow E^V_H(pt)$, so we get a straight covariant representation from the category $\pi_G B$ into the system $\{E^V_H(pt)\}$ with all subgroups H of G.

Proposition 4.4. *For any orientable generalized $RO(G)$-graded cohomology theory E^*_G and any oriented G-vector bundle $p : E \to B$, the restriction $i^*p^*(\sigma(V,\gamma,S)) \in E^*_G(G/H)$ of the orientation class $p^*(\sigma(V,\gamma,S)) \in E^*_G(B)$ is uniquely defined, and must coincide with the one defined in Definition 4.1, in order that it gives a (non-equivariant) orientation class in the sense of Thom.*

Proof. Definition 4.3 and the Thom Isomorphism Theorem determines those classes at the $E^V_H(pt)$ level. Since the orientation of Costenoble-May-Waner is naturally constructed by the purely categorical construction, the result follows in the same way as in the proof of Proposition 4.2, making use of the naturality provided by our Suspension Theorem (Theorem B) for generalized equivariant cohomology theories.

We now discuss the effect of the change of groups regarding the orientation classes for the more general base space B (cf. Section 17 of [CMW].) Let $H \subset G$. We have the functor

$$i_* : \mathcal{O}_H \longrightarrow \mathcal{O}_G$$

given by $i_*(H/K) = G \times_H (H/K) \cong G/K$ on objects and $i_*(\alpha) = G \times_H \alpha$ on morphisms. Then, any representation $\gamma : \pi_G B \longrightarrow \mathcal{V}_G$ naturally pulls back to

$$i^*\gamma : \pi_H B \cong i^*\pi_G X \longrightarrow \mathcal{V}_G \cong \mathcal{V}_H$$

where i^* simply restricts everything to those orbits G/K in $\pi_G B$ such that K is a subgroup of H.

For a G-bundle $p : E \to B$, we can take its H-fixed point bundle $p^H : E^H \to B^H$, and it becomes a $WH = NH/H$-equivariant bundle. In the situation with the projection map $q : G \to J = G/N$, for N a normal subgroup of a group G, we have the functor

$$q^* : \mathcal{O}_J \longrightarrow \mathcal{O}_G$$

given by $q^*(J/K) = G/H$ on objects, where $H = q^{-1}K$. Then, G/H and J/K are isomorphic as G-spaces via q, and so we also have the functor

$$q_* : \mathcal{O}_G \longrightarrow \mathcal{O}_J$$

that sends a G-orbit G/H to the J-orbit $J \times_G G/H \cong G/HN \cong J/K$, where $K = HN/N$.

A routine check shows that $q_*\pi_G X$ and $\pi_J X^N$ are isomorphic over $\mathcal{O}_J$, and that $q^*q_*\pi_G X$ and $\pi_G X^N$ are isomorphic over $\mathcal{O}_G$, for any G-space X.

Proposition 4.5 (Proposition 17.6 of [CMW]). *Let $p : E \to B$ be a G-bundle and let p_N be the complementary G-bundle to the N-fixed point J-bundle p^N over B^N, so that $p^N \oplus p_N \cong p|_{B^N}$ as G-bundles. The representation $(p^N)^* : \pi_J B^N \longrightarrow \mathcal{V}_J$ is isomorphic to the composite*

$$\pi_J B^N \cong q_* \pi_G B \xrightarrow{q_* p^*} q_* \mathcal{V}_G \longrightarrow \mathcal{V}_J.$$

The representation $(p_N)^* : \pi_G B^N \longrightarrow \mathcal{V}_G$ is isomorphic to the composite

$$\pi_G B^N \cong q^* q_* \pi_G B \xrightarrow{q^* q_* p^*} q^* q_* \mathcal{V}_G \xrightarrow{q^* \Phi_N} q^* q_* \mathcal{V}_G \xrightarrow{\Gamma} \mathcal{V}_G,$$

where

$$\Phi_N : q_* \mathcal{V}_G \longrightarrow q_* \mathcal{V}_G$$

sends $G \times_H V$ to $G \times_H V_N$, and

$$\Gamma : q^* q_* \mathcal{V}_G \longrightarrow \mathcal{V}_G$$

sends an object $(G/H, G \times_{HN} V \to G/HN)$ to the pullback $G \times_H V \to G/H$ along the quotient G-map $\gamma : G/H \to G/HN$.

Now that we have the necessary change-of-groups information available, we can proceed to investigate the comparison of orientation classes.

Section 5. Uniqueness of equivariant orientation classes

Let us recall that the author's earlier work [N 1], [N 2] characterized certain classifying spaces via some equivariant surgery exact sequences, and the key step there was a construction of an explicit characteristic class (called "the structure invariant" there) which lived in a certain equivariant cohomology group. Even earlier result of Ib Madsen and M. Rothenberg [MR 2] characterized the equivariant homotopy type of a related classifying space, and their key step was the construction of a certain class in the equivariant K-theory. Here we will extend their methods to fit into the current situation, that is, we will try to relate the construction of the orientation class for an orientable G-vector bundle in equivariant generalized cohomology with the categorical characterization of the orientation.

When we say "uniqueness" of orientation, we do not mean that the choice of the cohomology orientation class is unique. In fact, there are multiple choices of orientation (two, in the non-equivariant case, and more, in general, in the equivariant case) for a single G-bundle. By "uniqueness", we mean the following: For the coefficient module $E_G^*(pt)$ of the generalized $RO(G)$-graded cohomology, the cohomology orientation class is naturally determined directly from the categorical definition of the orientation, in the methods of Costenoble-May-Waner [CMW], because they live in the system of bundles over one-point, that is, over the equivariant vector spaces (representations) themselves. We say that the cohomology orientation class is "unique" when a choice of the orientation classes in this system of $E_G^*(pt)$ can be uniquely extended to a class in $E_G^*(B)$ for any orientable G-bundle $p : E \to B$.

Hereafter we assume that G is a finite group, and we proceed by the induction on the orbit types, in the same way as in [N 1] and [N 2]. First, we apply the slice theorem to the bundle $p : E \to B$, to single out a maximal orbit

$$G \times_{G_x} V \longrightarrow G/G_x.$$

Then let $J = G/G_x$ and push things down via

$$q_* : \mathcal{O}_G \longrightarrow \mathcal{O}_J.$$

Using Proposition 4.5, all cohomological information in the J-level can be recovered back onto the G-level, and so we can construct a cohomology class in the maximal orbit piece $p : E| \to B_{max}$.

On the other hand, the complementary pieces $p : E| \to (B - B_{max})$ can be uniquely patched together, due to the induction hypotheses, and Proposition 4.4 provides the uniquely extended orientation class there.

As the last step, we re-construct a class on $p : E \to B$ from the above two pieces, using the Mayer-Vietoris exact sequence in the E_G^*-cohomology (Theorem D). The technical background tools are described in Chapter XVIII of Peter May's book [M]. The double-coset formula in the G-V-Mackey functor provides the relationship between the global class in the cohomology of B and the local classes coming from the fiber-direction and transferred back from the J-equivariant cohomology classes via the map q_* above. The E_G^*-cohomology Mayer-Vietoris exact sequence now provides a uniquely determined global class, in the formal way which is similar to the arguments in [N 1] and [N 2], and therefore ensures both the existence and the uniqueness of the global orientation class in the cohomology group $E_G^*(B)$. Thus we have obtained the following theorem:

Theorem 5.1. *Let G is a finite group, B is a compact G-manifold and $p : E \to B$ is an oriented G-vector bundle in the sense of Costenoble-May-Waner (Definition 1.2). If E_G^* is an orientable equivariant generalized cohomology in the sense of Definition 4.1, then for any choice of a compatible system of local pointwise orientation classes (as discussed in the above), there exists a unique global orientation class in the cohomology group $E_G^*(B)$ which extends the given local data.*

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