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NONCOMMUTATIVE METRIC DIMENSION AND DYNAMICAL ENTROPY

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ABSTRACT. We give a summary of some recent work in which notions of dimension and dynamical entropy for metrized C^*-algebras were introduced and computed for examples involving the UHF algebras M_{p^∞} and noncommutative tori. The metric framework here is that determined by Rieffel's concept of a Lip-norm, which plays the role of a Lipschitz seminorm on functions over a compact metric space.

Motivated by the observation of Connes [1, 2, 3] that Dirac operators naturally give rise to metrics on state spaces, Rieffel initiated a line of investigation [11, 12] that led to the notion of a compact quantum metric space [13]. A compact quantum metric space is a pair (A, L) consisting of an order-unit space A equipped with a certain type of seminorm L, called a Lip-norm, which is a generalization of a Lipschitz seminorm on functions over a compact metric space. The key requirement in the definition of a Lip-norm is that the metric

$$\rho_L(\mu,\nu) = \sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\}$$

on the state space of A give rise to the weak* topology. Since the real linear subspace of self-adjoint elements in a unital C^*-algebra can be viewed as an order-unit space, Rieffel's theory applies in particular to the C^*-algebraic setting, and indeed C^*-algebras provide the fundamental motivating examples of the subject (see for instance Example 1.6). In [6] we introduced notions of dimension and dynamical entropy for metrized C^*-algebras within this framework using approximation by finite-dimensional subspaces, and carried out computations for some examples involving the UHF algebras M_{p^∞} and noncommutative tori. We also showed that for usual Lipschitz seminorms our noncommutative metric dimension coincides with the Kolmogorov dimension. The aim of the present article is to give a description of these results and an indication of the techniques involved in their proofs. In Section 1 we define our metric C^*-algebraic framework in precise terms and present some examples, and then devote Sections 2 and 3 to dimension and "product entropy," respectively.

1. cLIP-NORMS ON UNITAL C^*-ALGEBRAS

Recall that an *order-unit space* is a real partially ordered vector space A with distinguished element e, called the *order unit*, such that, for every $a \in A$,

(1) there exists an $r \in \mathbb{R}$ with $a \leq re$, and
(2) if $a \leq re$ for all $r \in \mathbb{R}_{>0}$ then $a \leq 0$.

Under the norm

$$\|a\| = \inf\{r \in \mathbb{R} : -re \leq a \leq re\}$$

A becomes a normed vector space, and we can recover the order from the norm using the fact that $0 \leq a \leq e$ if and only if $\|a\| \leq 1$ and $\|e - a\| \leq 1$. A *state* on A is defined as a norm-bounded linear functional on A whose dual norm and value on e are both 1 (this automatically implies positivity). We denote by $S(A)$ the state space of A. The space of self-adjoint elements in a unital C^*-algebra is a prime example of an order-unit space, an in fact every order-unit space can be realized as a unital subspace of self-adjoint operators on a Hilbert space (see [13, Appendix 2]). A *Lip-norm* on an order-unit space A is a seminorm L on A such that

(1) for all $a \in A$ we have $L(a) = 0$ if and only if $a \in \mathbb{R}e$, and
(2) the metric ρ_L defined on $S(A)$ by
$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in A \text{ and } L(a) \leq 1\}$$
gives rise to the weak* topology.

A *compact quantum metric space* is a pair (A, L) consisting of an order-unit space A equipped with a Lip-norm L.

In particular we can consider a Lip-norm on a real unital subspace of self-adjoint elements in a unital C^*-algebra, but in this case we wish our generalized Lipschitz seminorm to be meaningfully defined on the C^*-algebra as a vector space over the complex numbers, and so we introduce the notion of a ${}_{\mathrm{c}}$Lip-norm.

Notation 1.1. Given a seminorm L on a unital C^*-algebra A which is permitted to take the value $+\infty$, we denote the sets $\{a \in A : L(a) < \infty\}$ and $\{a \in A : L(a) \leq r\}$ (for a given $r > 0$) by $\mathfrak{L}$ and $\mathfrak{L}_r$, respectively. For $r > 0$ the norm ball $\{a \in A : \|a\| \leq r\}$ will be denoted by A_r.

Definition 1.2. A ${}_{\mathrm{c}}$*Lip-norm* on a unital C^*-algebra A is a seminorm L on A, possibly taking the value $+\infty$, such that

(i) $L(a^*) = L(a)$ for all $a \in A$,
(ii) for every $a \in A$ we have $L(a) = 0$ if and only if $a \in \mathbb{C}1$, and
(iii) $\rho_L(\sigma, \omega) = \sup_{a \in \mathfrak{L}_1} |\sigma(a) - \omega(a)|$ defines a metric on the state space $S(A)$ which gives rise to the weak* topology.

We say that a ${}_{\mathrm{c}}$Lip-norm L is a *Leibniz* ${}_{\mathrm{c}}$*Lip-norm* if it satisfies the Leibniz rule
$$L(ab) \leq L(a)\|b\| + \|a\|L(b)$$
for all $a, b \in \mathfrak{L}$.

The Leibniz rule plays a crucial role in our theory of dynamical entropy (Section 3). On the other hand, our definition of metric dimension (Definition 2.3) applies to any ${}_{\mathrm{c}}$Lip-norm on a unital C^*-algebra, and in fact is easily adapted to the general order-unit context of compact quantum metric spaces.

It is readily seen that if L is a ${}_{\mathrm{c}}$Lip-norm on a unital C^*-algebra then the restriction L' of L to the order-unit space $\mathfrak{L} \cap A_{\mathrm{sa}}$ is a Lip-norm, and the restriction map from $S(A)$ to $S(\mathfrak{L} \cap A_{\mathrm{sa}})$ is a weak* homeomorphism which is isometric relative to the metrics ρ_L and $\rho_{L'}$ defined via L and L', respectively.

Theorem 1.8 of [11] implies the following proposition. Our definitions of dimension and dynamical entropy will be based on the precompactness condition (4).

Proposition 1.3. Let L be a seminorm on a unital C^*-algebra A, possibly taking the value $+\infty$. Then L a $_{\mathrm{c}}$Lip-norm if and only if it separates $S(A)$ (equivalently, $\rho_L(\sigma,\omega) = \sup_{a\in\mathcal{L}_1} |\sigma(a)-\omega(a)|$ defines a metric on $S(A)$) and satisfies

(1) $L(a^*) = L(a)$ for all $a \in A$,
(2) for every $a \in A$ we have $L(a) = 0$ if and only if $a \in \mathbb{C}1$,
(3) $\sup\{|\sigma(a)-\omega(a)| : \sigma,\omega \in S(A) \text{ and } a \in \mathcal{L}_1\} < \infty$, and
(4) the set $\mathcal{L}_1 \cap A_1$ is totally bounded in A for $\|\cdot\|$.

The essential maps between $_{\mathrm{c}}$Lip-normed unital C^*-algebras are those which satisfy a Lipschitz condition:

Definition 1.4. Let A and B be unital C^*-algebras with $_{\mathrm{c}}$Lip-norms L_A and L_B, repectively. A positive unital linear map $\phi : A \to B$ is said to be *Lipschitz* if there exists a $\lambda \geq 0$ such that

$$L_B(\phi(a)) \leq \lambda L_A(a)$$

for all $a \in \mathcal{L}$, in which case the least such λ is defined to be the *Lipschitz number* of ϕ. If ϕ is invertible and both ϕ and ϕ^{-1} are Lipschitz and positive, then we say that ϕ is *bi-Lipschitz*, and if furthermore

$$L_B(\phi(a)) = L_A(a)$$

for all $a \in A$ then we say that ϕ is *Lipschitz isometric.* We denote by $\mathrm{Aut}_L(A)$ the collection of bi-Lipschitz *-automorphisms of A.

It can be shown that if L is a lower semicontinuous Leibniz $_{\mathrm{c}}$Lip-norm on a unital C^*-algebra A and u is a unitary in $\mathcal{L}$, then the inner *-automorphism $\mathrm{Ad}\, u$ is Lipschitz, with the Lipschitz numbers of $\mathrm{Ad}\, u$ and its inverse $\mathrm{Ad}\, u^*$ bounded by $2(1+2L(u)\mathrm{diam}(S(A)))$, where $\mathrm{diam}(S(A))$ is the diameter of $S(A)$ under the metric ρ_L defined by L.

Next we turn to a some examples.

Example 1.5. Let (X,d) be a compact metric space. The Lipschitz seminorm

$$L(f) = \sup\{|f(x)-f(y)|/d(x,y) : x,y \in X \text{ and } x \neq y\}.$$

on $C(X)$ yields an example of a Leibniz $_{\mathrm{c}}$Lip-norm.

Example 1.6 (ergodic compact group actions)**.** The most important examples for us will be those arising from ergodic actions of compact groups. Let γ be an ergodic action of a compact group G on a unital C^*-algebra A, with e denoting the identity element of G. Let ℓ be a length function on G, that is, a continuous function $\ell : G \to \mathbb{R}_{\geq 0}$ such that, for all $g,h \in G$,

(1) $\ell(gh) \leq \ell(g) + \ell(h)$,
(2) $\ell(g^{-1}) = \ell(g)$, and
(3) $\ell(g) = 0$ if and only if $g = e$.

From the length function ℓ and the group action γ we obtain the seminorm

$$L(a) = \sup\{\|\gamma_g(a) - a\|/\ell(g) : g \neq e\},$$

on A. Evidently L is adjoint invariant and satisfies the Leibniz rule, while $L(a) = 0$ if and only if $a \in \mathbb{C}1$. It follows from [11, Thm. 2.3] that the metric ρ_L gives rise to the weak* topology on $S(A)$, and so L is Leibniz $_{\mathrm{c}}$Lip-norm.

In [6] we also showed how a ${}_{\mathrm{c}}$Lip-norm can be defined on a crossed product by a bi-Lipschitz *-automorphism by means of the dual action, following the approach suggested by Example 1.6.

2. Metric dimension

We begin with some notation to describe our approximation framework.

Notation 2.1. Let $(X, \|\cdot\|)$ be a normed linear space (which will be either a C^*-algebra or a Hilbert space in our case). The collection of finite-dimensional subspaces of X will be denoted by $\mathcal{F}(X)$. Given subsets $Y, Z \subset X$ and $\delta > 0$ we will write $Y \subset_\delta Z$ if for every $y \in Y$ there exists an $x \in Z$ with $\|y - x\| < \delta$. With dim denoting vector space dimension, we set, for any subset $Z \subset A$ and $\delta > 0$,

$$D(Z, \delta) = \inf\{\dim X : X \in \mathcal{F}(A) \text{ and } Z \subset_\delta X\}$$

(or $D(Z, \delta) = \infty$ if the set on the right is empty) and if $\sigma \in S(A)$ then we set

$$D_\sigma(Z, \delta) = \inf\{\dim X : X \in \mathcal{F}(\mathcal{H}_\sigma) \text{ and } \pi_\sigma(Z)\xi_\sigma \subset_\delta X\}$$

(or $D_\sigma(Z, \delta) = \infty$ if the set on the right is empty). Here $\pi_\sigma : A \to \mathcal{B}(\mathcal{H}_\sigma)$ is the GNS representation associated to σ, with canonical cyclic vector ξ_σ.

The following proposition provides a tool for obtaining lower bounds for $D(Z, \delta)$ (and hence also for $D_\sigma(Z, \delta)$) which will be crucial in our computations of both dimension and dynamical entropy.

Proposition 2.2 ([15, Lemma 7.8]). Let B be an orthonormal set of vectors in a Hilbert space $\mathcal{H}$ and $\delta > 0$. Then

$$\inf\{\dim X : X \in \mathcal{F}(\mathcal{H}) \text{ and } X \subset_\delta B\} \geq (1 - \delta^2)\mathrm{card}(B).$$

Let A be a unital C^*-algebra with ${}_{\mathrm{c}}$Lip-norm L. The key to our definition of metric dimension is the fact that $D(\mathcal{L}_1, \delta)$ is finite for every $\delta > 0$. This finiteness follows from the observation that $D(\mathcal{L}_1, \delta)$, while not totally bounded, is the set of translations of a totally bounded set by scalar multiples of the identity (see [6, Prop. 3.2]).

Definition 2.3. The *metric dimension* of A with respect to L is defined by

$$\mathrm{Mdim}_L(A) = \limsup_{\delta \to 0^+} \frac{\log D(\mathcal{L}_1, \delta)}{\log \delta^{-1}}.$$

Note the formal similarity with Kolmogorov dimension, whose definition we recall in the next paragraph. The metric dimension is invariant under bi-Lipschitz positive unital maps, and decreases with respect to quotient ${}_{\mathrm{c}}$Lip-norms under surjective positive unital linear maps. Also, the metric dimension of a direct sum of two ${}_{\mathrm{c}}$Lip-normed unital C^*-algebras is the maximum of the metric dimensions of the summands.

Let (X, d) be a compact metric space. We denote by $N(\delta, d)$ the minimal cardinality of a cover of X by δ-balls. The *Kolmogorov dimension* of (X, d) is defined by

$$\mathrm{Kdim}_d(X) = \limsup_{\delta \to 0^+} \frac{\log N(\delta, d)}{\log \delta^{-1}}.$$

This also goes by various other names in the literature, such as box dimension and limit capacity. It can also be expressed as $\limsup_{\delta\to 0^+} \log \operatorname{sep}(\delta,d)/\log\delta^{-1}$ where $\operatorname{sep}(\delta,d)$ denotes the largest cardinality of a δ-separated set, or as $\limsup_{\delta\to 0^+} \log \operatorname{spn}(\delta,d)/\log\delta^{-1}$ where $\operatorname{spn}(\delta,d)$ denotes the smallest cardinality of a δ-spanning set (see [7, 8]).

Proposition 2.4. Let (X,d) be a compact metric space with associated Lipschitz seminorm

$$L(f) = \sup\{|f(x)-f(y)|/d(x,y) : x,y \in X \text{ and } x \neq y\}$$

on $C(X)$. Then

$$\mathrm{Mdim}_L(C(X)) = \mathrm{Kdim}_d(X).$$

To prove Proposition 2.4, we first obtain the inequality $\mathrm{Mdim}_L(C(X)) \leq \mathrm{Kdim}_d(X)$ by a straightforward partition of unity argument. The reverse inequality is established by means of the following idea. Given a δ-separated set E of maximum cardinality, we consider the probability measure μ uniformly supported on E. We then construct unitaries in $C(X)$ with suitably small Lipschitz seminorm which, when viewed inside $L^2(X,\mu)$, form an orthonormal basis dual to the standard basis. We can then use Proposition 2.2 to get a lower bound for $\log D(\mathcal{L}_1,\delta)/\log\delta^{-1}$ which is asymptotically sharp under taking the limit supremum.

Example 2.5 (the UHF algebra $M_p^{\otimes\mathbb{Z}}$). Consider the infinite tensor product $M_p^{\otimes\mathbb{Z}}$ (also denoted by M_{p^∞}) of $p\times p$ matrix algebras M_p over $\mathbb{C}$ with the infinite tensor product of Weyl actions. We recall that the Weyl action is the unique ergodic action of $G = \mathbb{Z}_p \times \mathbb{Z}_p$ on a simple C^*-algebra up to conjugacy [9], and it is defined on the clock and shift unitary generators u and v of M_p by the specifications

$$\gamma_{(r,s)}(u) = \rho^r u,$$
$$\gamma_{(r,s)}(v) = \rho^s v.$$

where ρ is the pth root of unity $e^{2\pi i/p}$. We can then define the infinite product action $\gamma^{\otimes\mathbb{Z}}$ of the product group $G^{\mathbb{Z}}$ on $M_p^{\otimes\mathbb{Z}}$.

Let ℓ be the length function on G induced by the Euclidean metric $\mathbb{R}^2$, viewing G as a subgroup of $\mathbb{R}^2/\mathbb{Z}^2$. While this length function may be regarded as standard, there is no canonical length function on $G^{\mathbb{Z}}$. Perhaps the simplest ones are those obtained by geometrically weighting ℓ on the factors with respect to a parameter $\lambda \in (0,1)$:

$$\ell_\lambda((g_j,h_j)_{j\in\mathbb{Z}}) = \sum_{j\in\mathbb{Z}} \lambda^{|j|}\ell((g_j,h_j)).$$

We denote by L the ${}_{\mathrm{c}}$Lip-norm on $M_p^{\otimes\mathbb{Z}}$ which arises from the action $\gamma^{\otimes\mathbb{Z}}$ and the length function ℓ_λ following Example 1.6.

Proposition 2.6. We have

$$\mathrm{Mdim}_L(M_p^{\otimes\mathbb{Z}}) = \frac{4\log p}{\log\lambda^{-1}}.$$

To establish the inequality $\mathrm{Mdim}_L(M_p^{\otimes\mathbb{Z}}) \le 4\log p/\log\lambda^{-1}$ we consider for each n the conditional expectation E_n of $M_p^{\otimes\mathbb{Z}}$ onto the subalgebra $M_p^{\otimes[-n,n]}$ given by

$$E_n(a) = \int_{G^{\mathbb{Z}\setminus[-n,n]}} \gamma_g^{\otimes\mathbb{Z}}(a)\, dg,$$

with dg denoting Haar measure on $G^{\mathbb{Z}}$ and $G^{\mathbb{Z}\setminus[-n,n]}$ the subgroup of $G^{\mathbb{Z}}$ of elements which are the identity at the coordinates in the interval $[-n,n]$. For $a \in \mathcal{L}_1$ we estimate $\|E_n(a) - a\|$ in terms of $L(a)$ and λ, so that we can approximate a inside a subalgebra $M_p^{\otimes[-n,n]}$ in a sufficiently controlled way with respect to n that we obtain an asymptotically sharp upper bound for the metric dimension using the linear dimensions $\dim M_p^{\otimes[-n,n]} = p^{2(2n+1)}$.

For the reverse inequality we consider for each n the subset

$$U_n = \left\{u^{i_{-n}}v^{j_{-n}} \otimes u^{i_{-n+1}}v^{j_{-n+1}} \otimes \cdots \otimes u^{i_n}v^{j_n} : 0 \le i_k, j_k \le p-1 \text{ for } k = -n,\dots,n\right\}$$

of $M_p^{\otimes[-n,n]}$ where the factors in the elementary tensors are products of powers of the clock and shift unitaries. We can view each U_n as an orthonormal set of vectors in the Hilbert space associated with the unique trace on M_{p^∞} via the GNS construction, so that we can appeal to Proposition 2.2, which, along with the observation that the ${}_{\mathrm{C}}$Lip-norms of elements in U_n are at most $(4n+2)\lambda^n$, yields the desired lower bound for the metric dimension.

Example 2.7 (noncommutative tori). Given an antisymmetric bicharacter $\rho : \mathbb{Z}^p \times \mathbb{Z}^p \to \mathbb{T}$, for $1 \le i,j \le k$ we set

$$\rho_{ij} = \rho(e_i, e_j),$$

where $\{e_1,\dots,e_p\}$ is the standard basis for $\mathbb{Z}^p$, and we define A_ρ to be the universal C^*-algebra generated by unitaries $u_1,\dots,u_p$ satisfying

$$u_j u_i = \rho_{ij} u_i u_j.$$

We refer to A_ρ as a *noncommutative p-torus.* Let $\gamma : \mathbb{T}^p \cong \mathbb{R}^p/(2\pi\mathbb{Z})^p \to \mathrm{Aut}(A_\rho)$ be the ergodic action determined by

$$\gamma_{(t_1,\dots,t_p)}(u_j) = e^{it_j}u_j$$

(see [9]), and consider the length function arising from the Euclidean metric, viewing $\mathbb{T}^p$ as the quotient $\mathbb{R}^p/(2\pi\mathbb{Z})^p$. Following Example 1.6 we thereby obtain a ${}_{\mathrm{C}}$Lip-norm L on A_ρ.

Proposition 2.8. We have

$$\mathrm{Mdim}_L(A_\rho) = p.$$

This computation is formally similar to that of Proposition 2.6, and indeed the inequality $\mathrm{Mdim}_L(A_\rho) \le p$ is established in the same way, only now with respect to sets of products of the generating unitaries, which form orthonormal sets in the Hilbert space associated to the canonical trace

$$\tau(a) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \gamma_{(t_1,\dots,t_p)}(a)\, dt_1 \cdots dt_p$$

via the GNS construction. For the reverse inequality the conditional expectations E_n in the M_{p^∞} case must now be replaced by the operation σ_n of taking a Cesàro mean, and some Fourier analysis is required to obtain the desired bounds for $\|a - \sigma_n(a)\|$ for $a \in \mathcal{L}_1$ as a function of n.

We also showed in [6] that, with respect to a natural $_{\mathrm{c}}$Lip-norm, the metric dimension of a crossed product of a noncommutative p-tori by a bi-Lipschitz *-automorphism is $p+1$.

3. Product entropy

The "product entropy" to which this section is devoted has two versions, one topological and the other measure-theoretic. The appropriate framework for our definitions is that of unital C^*-algebras with Leibniz $_{\mathrm{c}}$Lip-norms, with the dynamics given by bi-Lipschitz *-automorphisms. Our approximation approach is formally similar to that of Voiculescu [15], but the algebraic structure enters the picture here in a very different way. Product entropy can be viewed as an analytic version of entropy for discrete Abelian groups [10]. We may thus think of it as being "dual" to Voiculescu entropy in some rough sense.

We begin with some notation (see also Notation 1.1 and 2.1).

Notation 3.1. Given a set X, we denote by $Pf(X)$ the collection of finite subsets of X. If A is a C^*-algebra with subsets $X_1, X_2, \ldots, X_n$, we write $X_1 \cdot X_2 \cdot \cdots \cdot X_n$ to refer to the set

$$\{a_1 a_2 \cdots a_n : a_i \in X_i \text{ for each } i = 1, \ldots, n\}.$$

Definition 3.2. Let A be a unital C^*-algebra with Leibniz $_{\mathrm{c}}$Lip-norm L. Let $\alpha \in \mathrm{Aut}_L(A)$ (see Definition 1.4). For $\Omega \in Pf(\mathcal{L} \cap A_1)$ and $\delta > 0$ we set

$$\begin{aligned}
\mathrm{Entp}_L(\alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log D(\Omega \cdot \alpha(\Omega) \cdot \alpha^2(\Omega) \cdot \cdots \cdot \alpha^{n-1}(\Omega), \delta), \\
\mathrm{Entp}_L(\alpha, \Omega) &= \sup_{\delta>0} \mathrm{Entp}_L(\alpha, \Omega, \delta), \\
\mathrm{Entp}_L(\alpha) &= \sup_{\Omega\in Pf(\mathcal{L}\cap A_1)} \mathrm{Entp}_L(\alpha, \Omega).
\end{aligned}$$

We will refer to $\mathrm{Entp}_L(\alpha)$ as the *product entropy* of α.

It is readily seen that product entropy is invariant under conjugacies by bi-Lipschitz *-isomorphisms.

The following proposition establishes a connection between product entropy and metric dimension.

Proposition 3.3. Let $\alpha \in \mathrm{Aut}_L(A)$ and suppose that $\mathrm{Mdim}_L(A)$ is finite. Then

$$\mathrm{Entp}_L(\alpha) \le \mathrm{Mdim}_L(A) \cdot \log \max(\lambda, 1)$$

where λ is the Lipschitz number of α.

As a corollary we obtain $\mathrm{Entp}_L(\alpha) = 0$ whenever α is Lipschitz isometric. In particular, $\mathrm{Entp}_L(\mathrm{id}_a) = 0$, which suggests that the appropriate context for our notion of product entropy as a measure of dynamical growth is that of unital C^*-algebras with $_{\mathrm{c}}$Lip-norms under which the metric dimension is finite.

A standard argument yields the next proposition, using the fact that $\mathcal{L}$ is closed under multiplication by the Leibniz rule.

Proposition 3.4. If $\alpha \in \mathrm{Aut}_L(A)$ and $k \in \mathbb{Z}$ then $\mathrm{Entp}_L(\alpha^k) = |k| \,\mathrm{Entp}_L(\alpha)$.

Product entropy decreases under taking quotients in the presence of a Lipschitz cross section, and decreases when passing to an invariant ${}_{\mathrm{c}}$Lip-normed C^*-subalgebra obtained via restriction if there is a norm-contractive idempotent linear map onto the subalgebra.

Before coming to examples we introduce a version of product entropy relative to a dynamically invariant state. For notation see Notation 1.1, 2.1, 2.1, and 3.1.

Definition 3.5. Let $\alpha \in \mathrm{Aut}_L(A)$ and let σ be an α-invariant state on A. For $\Omega \in Pf(\mathcal{L} \cap A_1)$ and $\delta > 0$ we set

$$\begin{aligned} \mathrm{Entp}_{L,\sigma}(\alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log D_\sigma\big(\Omega \cdot \alpha(\Omega) \cdot \alpha^2(\Omega) \cdot \dots \cdot \alpha^{n-1}(\Omega), \delta\big), \\ \mathrm{Entp}_{L,\sigma}(\alpha, \Omega) &= \sup_{\delta>0} \mathrm{Entp}_{L,\sigma}(\alpha, \Omega, \delta), \\ \mathrm{Entp}_{L,\sigma}(\alpha) &= \sup_{\Omega\in Pf(\mathcal{L}\cap A_1)} \mathrm{Entp}_{L,\sigma}(\alpha, \Omega). \end{aligned}$$

We refer to $\mathrm{Entp}_{L,\sigma}(\alpha)$ as the *product entropy* of α with respect to σ.

Product entropy with respect to an invariant state is invariant under conjugacies by bi-Lipschitz *-automorphisms which respect the given invariant states, and it decreases under taking quotients if there exists a Lipschitz cross section. It is easy to see that

$$\mathrm{Entp}_{L,\sigma}(\alpha) \le \mathrm{Entp}_L(\alpha)$$

for any invariant state σ, and we also have the following analogue of Proposition 3.4.

Proposition 3.6. Let $\alpha \in \mathrm{Aut}_L(A)$, and let σ be an α-invariant state on A. For $k \in \mathbb{Z}$ we have $\mathrm{Entp}_L(\alpha^k) = |k| \,\mathrm{Entp}_L(\alpha)$.

We illustrate these two product entropies with two fundamental classes of examples, infinite tensor product shifts and noncommutative toral automorphisms.

Example 3.7 (tensor product shifts). Let α be the shift on the infinite tensor product $M_p^{\otimes\mathbb{Z}}$ with the ${}_{\mathrm{c}}$Lip-norm L as defined in Example 2.5 with respect to a given $\lambda \in (0,1)$. Then α is bi-Lipschitz, and it is straightforward to verify that the Lipschitz numbers of α and its inverse are bounded by λ. Let $\tau = \mathrm{tr}_p^{\otimes\mathbb{Z}}$ be the unique (and hence α-invariant) tracial state on $M_p^{\otimes\mathbb{Z}}$.

Proposition 3.8. We have

$$\mathrm{Entp}_L(\alpha) = \mathrm{Entp}_{L,\tau}(\alpha) = 2\log p.$$

This computation is made by elaborating the proof of Proposition 2.6, with the additional feature now of having to estimate the growth of products of norm-one elements in $\mathcal{L}$ with respect to subalgebras of the form $M_p^{\otimes[m,n]}$.

Example 3.9 (noncommutative toral automorphisms). Let A_ρ be a noncommutative p-torus with generators $u_1, \dots, u_p$ and canonical trace τ (see Example 2.7). We consider a $p \times p$ integral matrix $T = (s_{ij})$ with $\det T = \pm 1$, and we suppose that T defines an automorphism α_T of A by specifying

$$\alpha(u_j) = u_1^{s_{1j}} \cdots u_p^{s_{pj}}$$

on the generators. Note that τ is α_T-invariant. For an ordinary torus this yields a fundamental example of an Anosov diffeomorphism. We will also consider the *-automorphism γ_t for a given $t = (t_1, \ldots, t_p) \in \mathbb{T}^p \cong \mathbb{R}^p/(2\pi\mathbb{Z})^p$ determined by $\gamma_t(u_j) = e^{it_j}u_j$ on the generators, and the inner *-automorphism $\mathrm{Ad}\, u$ for a given unitary $u \in \mathcal{L}$. We point out that in [5] it was shown that, for an irrational rotation algebra A_θ (i.e., the case $p = 2$ here), if the angle θ satisfies a generic Diophantine property then all *-automorphisms preserving the dense *-subalgebra of C^∞ elements (i.e., all "diffeomorphisms") have the form $\mathrm{Ad}u \circ \alpha_T \circ \gamma_t$ for a C^∞ unitary u.

Using the remark after Definition 1.4 it can be verified that $\mathrm{Ad}u \circ \alpha_T \circ \gamma_t$ and its inverse are bi-Lipschitz, with Lipschitz numbers bounded by

$$2r(T)(1 + 2L(u)\mathrm{diam}(S(A)))$$

and

$$2r(T^{-1})(1 + 2L(u)\mathrm{diam}(S(A))),$$

respectively, where $r(\cdot)$ denotes the spectral radius.

Proposition 3.10. We have

$$\mathrm{Entp}_L(\mathrm{Ad}u \circ \alpha_T \circ \gamma_t) \leq \sum_{|\lambda_i| \geq 1} \log|\lambda_i|$$

and

$$\mathrm{Entp}_L(\alpha_T \circ \gamma_t) = \mathrm{Entp}_{L,\tau}(\alpha_T \circ \gamma_t) = \sum_{|\lambda_i| \geq 1} \log|\lambda_i|$$

where $\lambda_1, \cdots, \lambda_p$ are the eigenvalues of T counted with spectral multiplicity. In particular

$$\mathrm{Entp}_L(\alpha_T) = \mathrm{Entp}_{L,\tau}(\alpha_T) = \sum_{|\lambda_i| \geq 1} \log|\lambda_i|,$$

$\mathrm{Entp}_L(\gamma_t) = \mathrm{Entp}_{L,\tau}(\gamma_t) = 0$, and $\mathrm{Entp}_L(\mathrm{Ad}u) = \mathrm{Entp}_{L,\tau}(\mathrm{Ad}u) = 0$.

To establish upper bounds for the product entropies in the proposition we can adapt some classical Fourier analysis to our noncommutative setting (as in the proof of Proposition 2.8) to approximate elements in $\mathcal{L}$ by their Cesàro means, the growth of whose products we can then estimate with respect to the linear span of sets of products of the unitary generators. For the lower bounds it suffices to consider sets of products of the unitary generators, which form orthonormal bases in the Hilbert space associated to τ. Estimating the growth of products of these sets essentially reduces to the computation of the Abelian group-theoretic entropy of the action of T on $\mathbb{Z}^p$ [10].

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