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行列群の連続有限型因子環上への非コサイクル同値な連続個の作用

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1. 前置き

ここでは、行列のなす群に対して、有限連続型因子環 M の自己同型写像群としての、一連の表現 α_s, $(s\in[0,1/2])$ を、与える。

ここでの因子環 M としては、次のようなものが現れる：

i) 近似的有限次元型環 R

ii) 自由群　$F_n,(n=2,3,\cdots,\infty)$ の左正則表現により生成されるノイマン環 $L(F_n)$.

i) の場合には、行列群 G として $SL(n,\mathbb{Z}),n\geq 3$, $Sp(n,\mathbb{Z}),n\geq 2$ を取れば、接合積 $R\times_{\alpha_s}G$ は Kazhdan の性質 T を持つ因子環となる。ところが、ii) の場合, 特に $n=\infty$ の時には、同じ群をもってきても、この接合積は、因子環となるが。Kazhdan の性質 T を持つことは、出来ない。

全ての場合において、これらの連続個の表現は 互いに異なる s と t に対して α_s と α_t とは互いに非共役である。又、とくに、行列群が特殊群 $SL(n,\mathbb{Z}),n\geq 3$ のときには、更にコサイクル同値に、なることもできない連続個の表現である。

更に、これらの表現に現れる個々の自己同型写像 $\alpha_s(T)$ に対する Connes-Stormer のエントロピー $H(\alpha_T^s)$ は次のような関係式をみたす。

$$\log(\max\{\prod_{i=1}^n\mu_i,\prod_{i=1}^n\mu_i'\})\ \leq\ H(\alpha_T^s)\leq\log(\prod_{i=1}^n\mu_i\mu_i')$$

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となる。ただし、T は可逆な n 次正方行列で、その固有値 $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ に対して、

$$\mu_i = \max(|\lambda_i|, 1), \quad \mu_i' = \max(\frac{1}{|\lambda_i|}, 1)$$

である。

II. 行列群の連続有限型因子環上への作用

この章では、行列群 $GL(m, \mathbb{Z}), m \geq 2$ の超有限連続有限型因子環 R と n $(2 \leq n \leq \infty)$, 個の生成元 をもつ自由群　F_n　の左正則表現によって与えられる因子環 $L(F_n)$ の自己同型写像群としての作用を、与える。

2.1. 自由積に対するシンプレティック　形式

先ず通常のシンプレティック　形式　の概念を拡張して、自由積に対するシンプレティック　形式を定義する。

2.1.1. Let n be a positive integer. Let $B(a, b)$ be the symplectic form of a and b in the vector space $\mathbb{R}^{2n}$ ($\mathbb{R}$ is the set of all real numbers) :

$$B(a, b) = \sum_{i=1}^{n} a_i b_{n+i} - \sum_{i=1}^{n} a_{n+i} b_i,$$

where $a = (a_1, \cdots, a_n, a_{n+1}, \cdots, a_{2n}) \in \mathbb{R}^{2n}$. Let E_n be the identity matrix of the order n and let $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$. Then $B(a, b) = (a, Jb)$, the natural inner product of $\mathbb{R}^{2n}$. The symplectic group $Sp(n, \mathbb{R})$ is the set of a matrix T such that $B(a, b) = B(Ta, Tb)$ for all $a, b \in \mathbb{R}^{2n}$.

2.1.2. We denote by A_i a copy of the additive group $\mathbb{Z}^{2n}$ for all $i \in \mathbb{Z}$. Let $I \subset \mathbb{Z}$ be a subset, and let $A(I) = \underset{i \in I}{*} A_i$ be the free product group. Each $a \in A(I)$ is expressed by the unique form that

(*) $\qquad a = a_1 a_2 \cdots a_p, \quad a_i \in A_{\iota_i}, a_i \neq 0 \quad \iota_i \in I, \quad \iota_1 \neq \iota_2 \neq \cdots \neq \iota_p.$

The p is called the length of a and the set $\{\iota_1, \iota_2, \cdots, \iota_p\}$ is called the alphabet for the word a, which we denote by $J(a)$. Given $i \in J(a)$, let $I(a,i) = \{j \in I : \iota_j = i\}$. Let $\Phi_i : A(I) \to A_i$ be the homomorphism defined by

$$\Phi_i(a) = \sum_{j \in I(a,i)} a_j,$$

for $a \in A(I)$ with the reduced form (*). If $I(a,i) = \emptyset$, we let $\Phi_i(a) = 0$.

Define a homomorphism $\alpha : sp(n, \mathbb{Z}) \to \mathrm{Aut}(A(I))$ by

$$\alpha_T(a) = T(a_1)T(a_2)\cdots T(a_p), \quad \text{for} \quad T \in Sp(n, \mathbb{Z}).$$

Here $Sp(n, \mathbb{Z})$ is the symplectic group whose components are integers, and $a \in A(I)$ has the form (*).

2.1.3. Let

$$B_I(a,b) = \sum_{i \in J(a) \cap J(b)} B(\Phi_i(a), \Phi_i(b)), \quad a, b \in A(I).$$

In the case where $J(a) \cap J(b) = \emptyset$, we let $B_I(a,b) = 0$. Then we have the following proposition by calculations.

2.1.4. Proposition. *The following basic properties of the symplectic form $B(\cdot,\cdot)$ in 2.1.1 are satisfied ;*

$$\begin{cases} B_I(a,1) = B_I(1,a) = B_I(a,a^{-1}) = 0, \\ B_I(a,b) = -B_I(b,a), \\ B_I(ab,c) = B_I(a,c) + B_I(b,c), \\ B_I(a,b) = B_I(\alpha_T(a), \alpha_T(b)), \end{cases}$$

where $a, b, c \in A(I)$, $T \in sp(n, \mathbb{Z})$ and 1 is the identity of the group $A(I)$, and

$$B_I(a,b) = 0, \quad \text{if} \quad a \in A_i, \ b \in A_j, \ i \neq j.$$

2.2. 自由積群に対する 2-コサイクル.

2.2.1. Let $A(I) = \underset{i\in I}{*} A_i$ be the same group as in the section 2.1. We consider the semidirect product

$$G(I) = sp(n, \mathbb{Z}) \times_s A(I),$$

where the product of whose elements are defined by

$$(S,a)(T,b) = (ST, \alpha_T^{-1}(a)b), \quad (S,T \in sp(n,\mathbb{Z}),\ a,b \in A(I)).$$

Let $s \in [0, \pi/2]$ be irrational $(mod. 2\pi)$. Put

$$\mu_s((S,a),(T,b)) = e^{\sqrt{-1}\ s\pi B_I(\alpha_T^{-1}(a),b)}, \quad (S,T \in sp(n,\mathbb{Z}), a,b \in A(I)).$$

Then we have the following Proposition using the symplectic property of $B_I(\cdot,\cdot)$.

2.2.2. Proposition. *The μ_s is a normalized 2-cocycle of $G(I) \times G(I)$ to the torus $\mathbb{T}$:*

$$\begin{cases} \mu_s(1,g) = \mu_s(g,1) = \mu_s(g,g^{-1}) = 1 \\ \mu_s(f,g)\mu_s(fg,h) = \mu_s(g,h)\mu_s(f,gh) \end{cases}$$

2.3. 自由群因子環 $L(F_n)$ への $GL(n,\mathbb{Z})$ の作用.

上で、準備したことを用いて、自由群因子環 $L(F_n)$ への $GL(n,\mathbb{Z})$ の作用を、与える。この与え方は、下の命題 2.3.2 で、示すように、特殊な場合には、超有限連続有限型因子環 R への作用を、同時に与えることになることを、注意する。

2.3.1. Let μ_s be the normalized 2-cocycle defined in 2.2. The left μ_s-representation λ^s of the group $G(I) = sp(n,\mathbb{Z}) \times_s A(I)$ is defined by

$$(\lambda^s(g)\xi)(h) = \mu_s(h^{-1},g)\xi(g^{-1}h), \quad g,h \in G(I),\ \xi \in l^2(G(I)).$$

Then λ^s is a μ_s-cocycle unitary representation of $G(I)$ on $l^2(G(I))$:

$$\lambda^s(g)\lambda^s(h) = \mu_s(g,h)\lambda^s(gh).$$

We denote by $N_s(I)$ the von Neumann algebra generated by $\lambda^s(A(I))$.

Let $L(F_k)$ be the von Neumann algebra generated by the left regular repersentation of the free group F_k with k generators, $k \geq 2$.

2.3.2. Proposition. *Let $|I|$ be the cardinality of the set I.*

(1) If $|I| = 1$, then $N_s(I)$ is isomorphic to the hyperfinite II_1 factor R.

(2) If $|I| \geq 2$, then the von Neumann algebra $N_s(I)$ is isomorphic to the free group factor $L(F_{|I|})$.

Define the homomorphism $\alpha^s : sp(n, \mathbb{Z}) \to \mathrm{Aut}(N_s(I))$ by

$$\alpha^s_T(x) = \lambda^s(T,1)x\lambda^s(T,1)^*, \quad (T \in sp(n,\mathbb{Z}),\ x \in N_s(I)).$$

Then we have

$$\alpha^s_T(\lambda^s(1,a)) = \lambda^s(1, \alpha_T(a)), \quad (T \in sp(n,\mathbb{Z}),\ a \in A(I)).$$

Let η be the imbedding of $GL(n,\mathbb{R})$ into $sp(n,\mathbb{R})$ given by

$$\eta(T) = \begin{pmatrix} T & 0 \\ 0 & (T^t)^{-1} \end{pmatrix}.$$

Here T^t means the transposed matrix of T.

2.3.3. Definition. We define the action α^s of $GL(n,\mathbb{Z})$ on $N_s(I)$ by the homomorphism $\alpha^s \circ \eta : GL(n,\mathbb{Z}) \to \mathrm{Aut}(N_s(I))$.

Let M be a II_1 factor with the unique trace τ. and let G be a discrete group. An action β of G on M is "outer" if β_g is an outer automorphism for all $g \in G$, and β is " mixing" if given $x, y \in M$ and $\epsilon > 0$ there exists a finite subset K of G such that

$$|\tau(y\beta_g(x)) - \tau(x)\tau(y)| < \epsilon, \quad \forall g \notin K.$$

Remark that if β is mixing, then β is ergodic.

2.3.4. Proposition. *Let $G \subset GL(n, \mathbb{Z}), (n \geq 2)$ be a non-trivial subgroup.*

(1) The action $\alpha^s : G \to Aut(N_s(I))$ is outer.

(2) If given finite subsets $S_1,\ S_2 \subset \mathbb{Z}^{2n}$, G contains a finite subset K such that

$$S_1 \cap \{\eta(T)a; a \in S_2\} = \emptyset, \quad \forall\, T \notin K,$$

then α^s is mixing for all s.

Let $\{e_j; 1 \leq j \leq 2n\}$ be the standard basis in $\mathbb{Z}^{2n}$, whose copy in A_i we denote by $\{e(i;j); 1 \leq j \leq 2n\}$. Let S be a subset of $\{1, 2, \cdots, n\}$ and let

$$\mathcal{A}_S = \{a \in A(I); a = a_1 a_2 \cdots a_p,\ a_k \in \cup_{l \in S} \mathbb{Z} e(\iota_k, l), \iota_1 \neq \cdots \neq \iota_p\}.$$

2.3.5. Lemma. *Let $G \subset GL(n, \mathbb{Z})$ be a subgroup, and let θ be an isomorphism of $N_s(I)$ onto $N_t(I)$ such that $\theta \alpha^s_T = \alpha^t_T \theta$ for all $T \in G$.*

(1) Let us fix an integer j with $1 \leq j \leq n$.

(1-1) Assume that $n \geq 3$: If G contains matrices $\{T_{1,l}, T_{2,k}; 1 \leq l, k \leq n, l \neq j,\ k \neq n\}$ whose p, q component $T(p,q)$ satisfies that $T_{1,l}(p,p) = 1$ for all p, $T_{1,l}(j,l) \neq 0$ and $T_{1,l}(p,q) = 0$ otherwise, and $T_{2,k}(p,p) = 1$ for all p, $T_{2,k}(k,n) \neq 0$ and $T_{2,k}(p,q) = 0$ otherwise, then for all $i \in I$ we have the Fourier expansion :

$$\theta(\lambda^s(e(i;j))) = \sum_{a \in \mathcal{A}_j} c(a) \lambda^t(a), \ (c(a) \in \mathbb{C}), \ \textit{in the } \|\cdot\|_2 \textit{ convergence topology.}$$

Moreover if G contains a matrix T_3 suct that $T_3(j,j) = 1, T_3(p,j) \neq 0$ for some $p \neq j$, then there exist a permutation σ of I and an $m_j \in \mathbb{Z}$ so that

$$\theta(\lambda^s(e(i;j))) = \lambda^t(m_j\ e(\sigma(i), j)), \quad \forall i \in I.$$

(1-2) Case that $n = 2$: If G contains the above matrix $\{T_{1,l}; l \neq j\}$, then for all $i \in I$ we have the Fourier expansion for $S = \{j, 5 - j\}$:

$$\theta(\lambda^s(e(i;j))) = \sum_{a \in \mathcal{A}_S} c(a) \lambda^t(a), \ (c(a) \in \mathbb{C}), \ \textit{in the } \|\cdot\|_2 \textit{ convergence topology.}$$

Moreover if G contains the above matrix T_3, then there exist a permutation σ of I, $c \in \mathbb{T}$ and an $m_k \in \mathbb{Z}, (k = 1, 2)$ so that

$$\theta(\lambda^s(e(i;j)) = c_j \lambda^t(m_1\ e(\sigma(i), j))\lambda^t(m_2\ e(\sigma(i), 5-j)).$$

(2) Let us fix an integer $j, (n+1 \le j \le 2n)$. Same formulas hold if conditions are satisfied by replacing the matrices T to $(T^t)^{-1}$.

2.3.6. Proposition. *Let $G \subset GL(n, \mathbb{Z}), (n \ge 2)$ be a subgroup which contains the matrices in Lemma 2.3.4 for all j. Then $\alpha^s : G \to Aut(N_s(I))$ is not conjugate to $\alpha^t : G \to Aut(N_t(I))$ if $s \ne t$.*

2.3.7. Corollary. *The groups $SL(n, \mathbb{Z}), GL(n, \mathbb{Z}), (n \ge 2)$ and the free group F_2 have a continuous family of non-conjugate mixing outer actions on the free grooup factor $L(F_m)$ for all $m = 2, 3, \cdots, \infty$.*

2.4. コホモロジー類　とカズダンの性質 T

この節では、特に、行列群のうちでも、カズダンの性質 T　を持つ群を中心に取り扱う。

Let α be an ergodic action of a discrete group G on a II_1 factor M, and let $U(M)$ be the unitary operators of M. We denote by $Z^1_{\alpha,erg}$ the set of 1-cocycle unitary representation u of G on M for α such that $\mathrm{Ad}(u_g) \cdot \alpha_g$ is also ergodic :

$$Z^1_{\alpha,erg} = \{u : G \to U(M) |\ \mathrm{Ad}(u_g) \circ \alpha_g \text{ is ergodic, } u_g\alpha_g(u_h) = u_{gh},\ \ \forall\, g, h \in G, \}.$$

Two cocycles u_g, v_g are said to be cohomologous and denoted by $u_g \sim v_g$ if there exists a unitary $u \in M$ such that $u_g = uv_g\alpha_g(u^*)$. Let

$$H^1_{\alpha,erg} = \mathbb{Z}^1_{\alpha,erg} \,/ \sim .$$

2.4.1. Theorem. *Let $G \subset GL(n,\mathbb{Z}), (n \geq 2)$ be a subgroup which contains the matrices in Lemma 2.3.5 for all $j, (1 \leq j \leq n)$. If G has the property T of Kazhdan, then the actions $\alpha^s : G \to Aut(L(F_m))$ gives a continuous family, any two of which are not cocycle conjugate for all $m = 2, 3, \cdots, \infty$.*

2.4.2. Corollary. *Each of the group $SL(n,\mathbb{Z}), n \geq 3$ and $Sp(n,\mathbb{Z}), n \geq 2$ has a continuous family of ergodic outer actions on the free group factor $L(F_m), m \geq 2$ such that each two of them are not cocycle conjugate.*

Proof. The group $SL(n,\mathbb{Z}), n \geq 3$ and $Sp(n,\mathbb{Z}), n \geq 2$ have the property T of Kazhdan by [K, DK], and satisfy the conditions in Theorem 2.4.1. □

2.5. 上記の作用 α^s に関する接合積環の性質

Let us consider the crossed product $M_s(n, I) = N_s(I) \rtimes_{\alpha^s} SL(n,\mathbb{Z})$. Then the von Neumann algebra $M_s(n, I)$ is generated by $\lambda_s(\eta(SL(n,\mathbb{Z})) \times_s A(I))$. Moreover, $M_s(n, I) \cong R \rtimes_{\alpha^s} SL(n,\mathbb{Z})$ if $|I| = 1$, where R is the hyperfinite II_1 factor, and $M_s(n, I) \cong L(F_m) \rtimes_{\alpha^s} SL(n,\mathbb{Z})$ if $2 \leq |I| = m \leq \infty$. Since the action α^s is outer by Proposition 2.3.4, the $M_s(n, I)$ is a type II_1 factor.

2.5.1. In this section, we remark that the crossed products $M_s(2, I) = N_s(I) \rtimes_{\alpha^s} SL(2,\mathbb{Z})$ is a factor which have "HT free group subfactor $L(F_m)$" if $|I| = m \geq 2$. The notion of "HT free group subfactor $L(F_n)$" is a modification of HT Cartan subalgebra in the sense of Popa ([Po2]) as follows :

Let M be a finite von Neumann algebra, and let $B \subset M$ be a von Neumann subalgebra. The embedding $B \subset M$ has *the Property T*, if it has the property which is a notion in von Neumann algebra text of Margulis' property T ([M], cf. [dHV]) for the pair of groups, that is, there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$

of M and $\epsilon > 0$ such that if H is a Hilbert M bimodule with $\xi \in H$ a unit vector which satisfies that $||x_i\xi - \xi x_i|| \leq \epsilon$ for all i, then there exists a non zero vector $\xi_0 \in H$ such that $b\xi_0 = \xi_0 b$ for all $b \in B$. When M is a type II_1 factor, the embedding $B \subset M$ is said to have *the property H* if it has a property which is a generalization of Hagerup's compact approximation property ([Po2 : Definition 2.3]), and $B \subset M = B \rtimes_\sigma G$ has the property H if G has positive definite functions ϕ_n such that

$$\phi_n(1) = 1, \quad \lim_{g\to\infty} \phi_n(g) = 0, (\forall n), \quad \lim_{n\to\infty} \phi_n(g) = 1, (\forall g \in G).$$

Furthermore, an abelian C^*-subalgebra B of a type II_1 factor M is called a HT *Cartan subalgebra* of M if it satisfies the following conditions :

1) $B' \cap M = B$ and $N_M(B) = \{unitary\ u \in M : uBu^* = B\}$ generates M.

2) $B \subset M$ has the property H.

3) B has a von Neumann subalgebra $B_0 \subset B$ such that $B_0' \cap M = B$ and such that $B_0 \subset M$ has the property T.

Popa remarked about "HT hyperfinite subfactor" R of the factor $R \rtimes_\sigma G_0$ in [Po2 : Remark 6.6].

Here, we consider a notion corresponding HT Cartan subalgebra for subfactors which is isomorphic to the free group factor $L(F_n), n \geq 2$. We say that a subfactor Q of a type II_1 factor M is a HT *free group subfactor* of M if it satisfies the following conditions :

1) $Q' \cap M = \mathbb{C}$ and $N_M(Q)$ generates M.

2) $Q \subset M$ has the property H.

3) Q has a von Neumann subalgebra $Q_0 \subset Q$ such that $Q_0' \cap M = Q_0' \cap Q$ and such that $Q_0 \subset M$ has the property T.

2.5.2. Proposition. *Assume that $|I| \geq 2$. The type II_1 factor $M_s(2, I)$ has a HT free group subfactor isomorphic to $L(F_{|I|})$ for all $s \in [0, \pi/2]$ mod. 2π.*

Remark by the same proof that $R = N_s(I)$ is a HT hyperfinite subfactor of M when $|I| = 1$.

2.5.3. Remark. Assume that $n \geq 3$. As we showed in [Ch2], if $|I| = 1$, then the factor $M_s(n, I)$ has property T of Connes-Jones ([CJ]) because the group $\eta(SL(n,\mathbb{Z})) \times_s \mathbb{Z}^{2n}$ has property T of Kadhdan. If $|I| = \infty$, then the group $\eta(SL(n,\mathbb{Z})) \times_s A(I)$ does not have property T because it has infinite generators and can not have property T by [K : Theorem 2], so that the factor $M_s(n, I)$ does not have property T by [CJ : Theorem 2]. We don't know whether $M_s(n, I)$ has property T or not, in the case where $1 \neq |I| < \infty$.

III. 作用の中に現れる個々の自己同型写像のエントロピーの値

The each automorphism of $L(F_m), m \geq 2$ in the actions that we discussed in the section 2 is given essentially as the free products of those in [Ch2]. Using this fact, in this section, we give an estimation of the Connes-Stϕrmer entropy $H(\alpha_T^s)$ for each automorphism α_T^s, $(T \in GL(n,\mathbb{Z}))$ of the type II_1 factor $N_s(I)$. The cocycle actions of Popa are given as the reduced actions of the free permutation $\sigma \in \mathrm{Aut}(L_\infty)$, and it is known that $H(\sigma) = 0$ ([S1], cf. [S2, BC, D4]).

3.1. Entropy $ht_\phi(\alpha)$

To obtain an estimation of the values for the entropies, we need an entropy defined in [Ch5], which ia a slight modification of Voiculescu's topological entropy ([V2], cf.[Br]). First, we review the definition and basic properties of the entropy

By a C^*-dynamical system (A, α, ϕ), we mean that A is a separable unital C^*-algebra, α is a $*$-automorphism of A and ϕ is an α-invariant state of A.

3.1.1. Given a C^*-dynamical system (A, α, ϕ), let π be a faithful $*$-representation of A on a Hilbert space H, and let $\xi \in H$ be a unit vector such that $\phi = \omega_\xi \circ \pi$. Here ω_ξ is the vector state $< \cdot\, \xi, \xi >$. Let $CPA(A, B(H))$ be the set of all triplets (ϱ, η, C) of a finite dimensional C^*-algebra C and unital completely positive maps $A \underset{\varrho}{\rightarrow} C \underset{\eta}{\rightarrow} B(H)$. The von Neumann entropy of a state ψ on a finite dimensional C^*-algebra is denoted by $S(\psi)$. For a finite subset $\omega \subset A$, and a $\delta > 0$, put

$$scp_\phi(\pi; \omega, \delta) = \inf\{S(\omega_\xi \circ \eta) : (\varrho, \eta, C) \in CPA(A, B(H))$$
$$\text{and } ||\eta \circ \varrho(a) - \pi(a)|| < \delta||a||, \quad \text{for all } a \in \omega\}.$$

The $scp_\phi(\pi; \omega, \delta)$ is defined to be ∞ if no such approximation exists. Let

$$ht_\phi(\pi; \alpha, \omega, \delta) = \overline{\lim_{N \to \infty}} \frac{1}{N} scp_\phi(\pi; \bigcup_{i=0}^{N-1} \alpha^i(\omega), \delta),$$

$$ht_\phi(\pi; \alpha, \omega) = \sup_{\delta > 0} ht_\phi(\pi; \alpha, \omega, \delta).$$

$$ht_\phi(\pi; \alpha) = \sup_{\omega} ht_\phi(\pi; \alpha, \omega).$$

3.1.2. Remark. In the case where a C^*-dynamical system (A, α, ϕ) has a faithful $*$-representation $\pi : A \to B(H)$ and a cyclic unit vector $\xi \in H$ such that $\phi = \omega_\xi \circ \pi$, we can prove that the value $scp_\phi(\pi; \omega, \delta)$ does not depend of the choice of π.

3.1.3. Remark. A unital C^*-algebra A is exact if and only if for some C^*-algebra B there exists an embedding $\iota : A \to B$ which is nuclear, that is, for arbitrary $\epsilon > 0$ and for every finite set $\omega \subset A$ there exist a finite dimensional C^*-algebra C and unital completely positive maps $A \underset{\rho}{\rightarrow} C \underset{\eta}{\rightarrow} B$ such that $||\iota(a) - \eta \circ \rho(a)|| < \epsilon||a||$

for all $a \in \omega$. ([Kir : Theorem 4.1], [Was]). Let $(H_\phi, \pi_\phi, \xi_\phi)$ be the GNS(Gelfand-Naimark-Segal representation)-triplet associated with a state ϕ of an exact C^*-algebra A. Consider the completely positive extension ϱ of the map $\pi_\phi \circ \iota^{-1}$: $\iota(A) \to B(H_\phi)$ to B ([Ar]), then $A \underset{\rho}{\to} C \underset{\varrho\circ\eta}{\to} B(H_\phi)$ implies the nuclearity of π_ϕ (so that the approximation approach for $scp_\phi(\omega;\delta)$ is reasonable).

3.1.4. Definition. Let (A, α, ϕ) be a C^*-dynamical system, where A is exact and the GNS-representation π_ϕ is faithful. We define the entropy $ht_\phi(\alpha)$ by $ht_\phi(\pi_\phi, \alpha)$.

The faithfulness of π_ϕ is not always necessary in the definition of $ht_\phi(\alpha)$, but it is essential when we discuss the entropy of the free product $\alpha * \beta$ of two automorphisms α and β.

3.1.5. Remark. In the case where A is nuclear, this entropy coincides with that defined in in [Ch4], and in the form of $scp_\phi(\omega;\delta)$, we only need triplets (ϱ, η, C), where C is a finite dimensional C^*-algebra, and $A \underset{\varrho}{\to} C \underset{\eta}{\to} A$ are unital completely positive maps .

3.1.6. Proposition. *Let (A, α, ϕ) be a C^*-dynamical system such that A is exact and ϕ has the faithful GNS-representation. Then the $ht_\phi(\alpha)$ takes the value between the Connes-Narnhofer-Thirring entropy $h_\phi(\alpha)$ and the Brown- Voiculescu's topological entropy $ht(\alpha)$:*

$$h_\phi(\alpha) \leq ht_\phi(\alpha) \leq ht(\alpha).$$

If A ia abelian, then we have

$$h_\phi(\alpha) = ht_\phi(\alpha).$$

3.1.7. Theorem. *For each $i \in I$, let A_i be a unital exact C^*-algebra, and let ϕ_i be a state of A_i whose GNS-representation π_i is faithful. Let A and ϕ be the C^*-algebra and the state given by the reduced free product construction :*

$$(A, \phi) = \underset{i \in I}{*} (A_i, \phi_i).$$

If $\alpha_i \in \mathrm{Aut}(A_i)$ satisfies $\phi_i \circ \alpha_i = \phi_i$ for all $i \in I$, then free product automorphism $\alpha = \underset{i \in I}{*} \alpha_i \in \mathrm{Aut}(A)$ preserves the state ϕ and

$$ht_\phi(\alpha) = \sup_{i \in I} \; ht_{\phi_i}(\alpha_i).$$

3.1.8. Remark. About the topological entropy, Brown-Dykema-Shlyakhtenko proved in [BDS : Theorem 5.7] that

$$ht_\phi(\alpha) = \sup_{i \in I} \; ht_{\phi_i}(\alpha_i)$$

under the same conditions as in Theorem 3.1.7.

3.2. Now we discuss on the Connes-Stϕrmer entropy $H(\cdot)$ for each automorphism α_T^s of $N_s(I)$ $(T \in GL(n, \mathbb{Z}))$. Here α^s is the action of $GL(n, \mathbb{Z})$ on $N_s(I)$ defined in 2.3.3, and $N_s(I)$ is the hyperfinite II_1 factor R if $|I| = 1$ and is the free group factor $L(F_m)$ if $2 \leq |I| = m \leq \infty$.

3.2.1. We denote by $C_r^*(A(I), \mu_s)$ the C^*-algebra generated by $\lambda^s(A(I))$ with respect to the μ_s-representation λ^s of $G(I)$ in 2.3.1, and by τ the tracial state of $C_r^*(A(I), \mu_s)$ given by $\tau(\lambda^s(g)) = 0, (e \neq g \in G(I))$. For a $T \in GL(n, \mathbb{Z})$, we denote by β_T^s the automorphism of $C_r^*(A(I), \mu_s)$ induced from the automorphism $\alpha_{\eta(T)}$ of $A(I)$ defined in 2.1.2. The C^*-algebra $C_r^*(A(I), \mu_s)$ is weakly dense in the factor $N_s(I)$, and the automorphism α_T^s of $N_s(I)$ is the extension of β_T^s.

3.2.2. Proposition. *Let $n \geq 2$, and let $T \in GL(n,\mathbb{Z})$. For all $s \in [0,\pi/2]$ irrational mod. 2π, the Connes-Størmer entropy $H(\cdot)$, Connes-Narnhofer-Thirring entropy $h_\tau(\cdot)$, Brown-Voiculescu's topological entropy $ht(\cdot)$ and $ht_\tau(\cdot)$ satisfy that*

$$H(\alpha_T^s) = h_\tau(\beta_T^s) \ \leq \ ht_\tau(\beta_T^s) \leq ht(\beta_T^s) \leq \log(\prod_{i=1}^n \mu_i\mu_i').$$

Here $\mu_i = \max(|\lambda_i|, 1)$ and $\mu_i' = \max(\frac{1}{|\lambda_i|}, 1)$, for the eigenvalue list $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ of the matrix $T \in GL(n,\mathbb{Z})$.

In particular if $T \in SL(n,\mathbb{Z})$, then

$$\log(\max\{\prod_{i=1}^n \mu_i, \prod_{i=1}^n \mu_i'\}) \ \leq \ H(\alpha_T^s) = h_\tau(\beta_T^s) \ \leq \ ht_\tau(\beta_T^s) \leq ht(\beta_T^s) \leq \log(\prod_{i=1}^n \mu_i\mu_i').$$

3.2.3. Remark. We gave actions $\{\alpha^s\}_s$ in Section 2, in order to obtain a non cocycle conjugate continuous family of actions on the free group factors. However, from a point of view of entropy theory, it would be interesting to treat the action such that we can get the exact value of the entropy for each automorphism appearing the action. As an example of such an action on the free group factors, we have the followings :

Let $I \subset \mathbb{Z}$, and let A_i be the copy of the the group C^*-algebra $C^*(\mathbb{Z}^n)$. Denote by τ_i the tracial state of C_i taking 0 for $g \in \mathbb{Z}^n, g \neq 1$. Consider the reduced free product $(C,\tau) = \underset{i\in I}{*}(C_i,\tau_i)$. Let M_i be the von Neumann algebra generated by $\pi_i(C_i)$, where π_i is the GNS-representation by τ_i, and let M be the von Neumann algebra generated by $\pi_\tau(C)$. Then M is isomorphic to $L(F_m)$ by Dykema [D2 : Corollary 5.3], where $m = |I|$. Let $\gamma_{i,T}$ be the automorphism of C_i induced by $T \in SL(n,\mathbb{Z})$, then $\tau_i \cdot \gamma_{i,T} = \tau_i$, and we have the automorphism $\gamma_T = \underset{i\in I}{*}\gamma_{i,T}$ of C such that $\tau \cdot \gamma = \tau$ (cf. [Ch3], [BD]). By the proof of Proposition 3.2.2,

$$h_\tau(\gamma_T) \leq ht_\tau(\gamma_T) = ht_{\tau_i}(\gamma_{i,T}) = h_{\tau_i}(\gamma_{i,T}) = \log(\prod_{i=1}^n \mu_i),$$

where $\{\mu_i; 1 \leq i \leq n\}$ are the same as in Proposition 3.2.2. We denote by $\hat{\gamma}_T$ (resp. $\hat{\gamma}_{i,T}$) the extension of γ_T (resp. $\hat{\gamma}_{i,T}$) to M (resp. M_i). Then

$$\log(\prod_{i=1}^{n} \mu_i) = H(\hat{\gamma}_{i,T}) \leq H(\hat{\gamma}_T) = h_\tau(\gamma_T).$$

Thus we have the action $\hat{\gamma}$ of $SL(n, \mathbb{Z})$ on $L(F_m)$ such that

$$H(\hat{\gamma}_T) = \log(\prod_{i=1}^{n} \mu_i), \quad \forall T \in SL(n, \mathbb{Z}).$$

REFERENCES

[Ar] W. Arveson : Subalgebras of C^*-algebras, *Acta Math.* **123** (1969), 141-224.

[Br] N. Brown : Topological entropy in exact C^*-algebras, *Math. Ann,* **314** (1999), 347-367.

[BC] N. Brown and M. Choda : Approximation entropies in crossed products with an application to free shifts, *Pacific J. Math.,* (to appear).

[BD] E. Blanchard and K. J. Dykema : Embeddings of reduced amalgamated free product C^*-algebras, Preprint.

[BDS] N. Brown, K. J. Dykema and D. Shlyakhtenko : Topological entropy of free product automorphisms, Preprint.

[BG] N. Brown and E. Germain : Dual entropy in discrete groups with amenable actions, Preprint.

[C] A. Connes : A type II_1 factor with countable fundamental group, J. Operator Theory 4(1980), 151-153.

[Ch1] M. Choda : A continuum of non-conjugate property T actions of SL(n,Z) on the hyperfinite II_1 -factor, Math. Japonica, 30(1985), 133-150.

[Ch2] M. Choda : Outer actions of groups with property T on the hyperfinite II_1-factor, Math. Japonica, 31(1986), 533-551.

[Ch3] M. Choda : Reduced free products of completely positive maps and entropy for free products of automorphisms, *Publ. RIMS, Kyoto Univ.* **32** (1996), 179-190.

[Ch4] M. Choda : A C^*-dynamical entropy and applications to canonical endomorphisms, *J. Funct. Anal.,* **173** (2000), 453-480.

[Ch5] M. Choda : Dynamical entropy for automorphisms on exact C^*-algebras, Preprint.

[CJ] A. Connes and V. Jones : Property T for von Neumann algebras, Bull. London Math. Soc., 17(1985), 57-62.

[CNT] A. Connes H. Narnhofer and W. Thirring : Dynamical entropy of C^* algebras and von Neumann algebras, *Commun. Math. Phys.*, **112** (1987), 691-719.

[CS] A. Connes and E. Stϕrmer : Entropy of II_1 von Neumann algebras, *Acta Math.,* **134** (1975), 289-306.

[DK] C. Delaroche and A. Kirilov : Sur les relations entre l'espace dual d'un groupe et la structure de ses sous-groupes fermes, Seminaire Bourbaki, no. 343, 1967-1968.

[D1] K. J. Dykema : On certain free product factors via an extended matrix model, J. Funct. Anal., 112(1993), 31-60.

[D2] K. J. Dykema : Free products of hyperfinite von Neumann algebras and free dimension, Duke Math. J., 69(1993), 97-119.

[D3] K. J. Dykema : Interpolated free group factors, Pacific J. Math., 163(1994), 123-135.

[D4] K. J. Dykema : Topological entropy of some automorphisms of reduced amalgamated free product C^*-algebras, Preprint, 1999.

[H] U. Haagerup : An example of non-nuclear C^*-algebra which has the metric approximation property, Invent. Math. 50(1979), 279-293.

dHV] P. de la Harpe and A. Valette : "La propiete T de Kazhdan pour les groupes localement compacts, Asterisque" 175 (1989).

[K] D. Kazhdan : Connection of the dual space of a group with the structure of its closed subgroups, Funct. Anal. and its Appl., 1(1967), 63-65.

[Ka] Y. Kawahigashi : Cohomology of actions of discrete groups on factors of type II_1, Pacific J. Math., 149(1991), 303-317.

[Kir] E. Kirchberg : On subalgebras of the CAR-algebra, *Jour. Funct. Anal.* **129** (1995), 35-63.

[M] G. A. Margulis : Finitely-additive invariant measures on Euclidean spaces, *Ergod. Th. & Dynam. Sys.* **2** (1982), 383-396.

[Pe] K. Petersen : Ergodic Theory, C.U.P., Cambridge, 1983.

[Po1] S. Popa : Some rigidity result for actions non-commutative Bernoulli shifts, Preprint, (2001).

[Po2] S. Popa : On a class of type II_1 factors with Betti numbers invariants, Preprint, (2001).

[S1] E. Stϕrmer : Entropy of some automorphisms of the II_1-factor of the free group in infinite number of generators, *Invent. Math.* **110** (1992), 63-73.

[S2] E. Stϕrmer : States and shifts on infinite free products of C^* - algebras, *Fields Inst. Commun.* **12** (1997), 281-291.

[V1] D. Voiculescu : Symmetries of somereduced free product C*–algebras, in Operator Algebras and Their Connections with Topology and Ergodic Theory, Lecture Notes in Math., Springer–Verlag **1132** (1985), 556–588.

VDN] D. Voiculescu , K. Dykema and A. Nica : Free randam variables CRM Monograph series, Amer. Math. Soc. (1992).

[V2] D. Voiculescu : Dynamical approximation entropies and topological entropy in operator algebras, *Commun. Math. Phys.*, **170** (1995), 249-281.

[Was] S. Wassermann : Exact C^*-algebras and related topics, *Seoul National University Lecture Notes Series*, **19** (1994).

[Wan] P.S. Wang : On isolated points in the dual space of locally compact groups, Math. Ann., 218(1975), 19-34.