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A differentiable structure and a metric structure of submanifolds in Euclidean spaces*

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1. Introduction

In this survey, we shall consider the differentiable structure and the metric structure of submanifolds in Euclidean spaces. First of all, we shall study complete submanifolds in Euclidean spaces with constant mean curvature. The theorems due to Liebmann, Hopf, Klotz and Osserman, Cheng and Nonaka and Cheng are discussed. Next, we shall investigate the differentiable structure of compact submanifolds in Euclidean spaces. We shall consider several differentiable sphere theorems of submanifolds in Euclidean spaces.

2. A metric structure of complete submanifolds

In this secton, we shall consider complete submanifolds with constant mean curvature in Euclidean spaces. It is well-known that, in 1900, Liebmann proved the following:

Theorem 1. *A strictly convex, compact surface of constant mean curvature in the Euclidean space* $\mathbf{R}^3$ *is a standard round sphere.*

As a generalization above result, in 1951, Hopf proved a much stronger theorem, namely,

Theorem 2. *The only possible differentiable immersions of sphere into the Euclidean space* $\mathbf{R}^3$ *with constant mean curvature are exactly those round spheres.*

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On the other hand, Klotz and Osserman [7] studied complete surfaces in the Euclidean space $\mathbf{R}^3$ with constant mean curvature. They proved the following:

Theorem 3. *A complete orientable surface M^2 with constant mean curvatrure H is isometric to the totally umbilical sphere $S^2(c)$, the totally geodesic plane $\mathbf{R}^2$ or the cylinder $\mathbf{R}^1 \times S^1(c)$ if its Gaussian curvature is non-negative.*

Remark 1. It is well known that Gaussian curvature is non-negative if and only if $S \leq \frac{n^2H^2}{n-1}$ holds in the case of $n = 2$. Where S denotes the squared norm of the second fundamental form.

Recently, Cheng [3] (cf. Cheng and Nonaka [4]) generalized the result due to Klotz and Osserman to higher dimensions and higher codimensions under the same condition of constant mean curvature.

Main Theorem 1. *Let M^n be an n-dimensional ($n > 2$) complete connected submanifold with constant mean curvature H in the Euclidean space $\mathbf{R}^{n+p}$. If $S \leq \frac{n^2H^2}{n-1}$ is satisfied, then M is isometric to the totally umbilical sphere $S^n(c)$, the totally geodesic Euclidean space $\mathbf{R}^n$ or the generalized cylinder $S^{n-1}(c) \times \mathbf{R}^1$. Where S denotes the squared norm of the second fundamental form of M^n.*

Remark 2. In [2], by replacing the condition of constant mean curvature in Main Theorem 1 with constant scalar curvature, Cheng [2] proved that the result in Main Theorem 1 is also true.

3. A differentiable structure of submanifolds

It is well-known that the investigation of sphere theorems on Riemannian manifolds is very important in the study of differential geometry. It is our purpose to consider differentiable sphere theorems of compact submanifolds in Euclidean spaces.

Firstly, we state the following classical theorem due to Hadamard

Theorem 4. *An n-dimensional compact connected orientable hypersurface M in a Euclidean space with positive sectional curvature is diffeomorphic to a sphere.*

Let G be the Gauss map of M. We know that G is a diffeomorphism from M onto the unit sphere $S^n(1)$.

This above result due to Hadamard was generalized by Van Heijennoort [8] and Sacksteder [10], they proved that an n-dimensional complete connected orientable hypersurface M^n in a Euclidean space is a boundary of a convex body in $\mathbf{R}^{n+1}$ if every sectional curvature of M^n is non-negative and at least one is positive. In particular, they proved the following:

Theorem 5. *An n-dimensional locally convex (that is, the second fundamental form is semi-definite) compact connected orientable hypersurface M^n in a Euclidean space is diffeomorphic to a sphere.*

It is well-known that Nash proved that every finite dimensional Riemannian manifold possesses an isometric embedding into a Euclidean space of sufficiently high dimension. Therefore, we can not expect to extend these results due to Hadamard and Van Heijennoort and Sacksteder to higher codimensions because there exist many compact manifolds with positive sectional curvature, which are not diffeomorphic to a sphere. That is, in order to obtain a differentiable sphere theorem on compact submanifolds in Euclidean spaces, the condition of positive sectional curvatures is not strong enough. From Gauss equation, we know that $n^2H^2 - S = r > 0$ if the sectional curvature is positive, where r is the scalar curvature. Hence, $S < n^2H^2$ is not strong enough yet. In [3], we studied the differentiable structure of compact submanifolds under some stronger conditions. We proved the following:

Main Theorem 2. *An n-dimensional compact connected submanifold M^n with nonzero mean curvature H in the Euclidean space $\mathbf{R}^{n+p}$ is diffeomorphic to a sphere if $S \le \frac{n^2H^2}{n-1}$ is satisfied. Where S denotes the squared norm of the second fundamental form of M^n.*

4. Proofs of Main Theorems

First of all, we prove a general result.

Proposition 1. *Let M^n be an n-dimensional complete submanifolds with bounded non-zero mean curvature H in the Euclidean space $\mathbf{R}^{n+p}$. If the following inequality holds,*

$$S \le \frac{n^2H^2}{n-1},$$

then M^n lies in a totally geodesic submanifold $\mathbf{R}^{n+1}$ of $\mathbf{R}^{n+p}$.

Proof. Since the mean curvature of M^n is not zero, we know that $e_{n+1} = \frac{\mathbf{h}}{H}$ is a normal vector field defined globally on M^n. Hence, M^n is orientable. We choose an orthonormal frame field $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}\}$ in $\mathbf{R}^{n+p}$ such that $\{e_1, \cdots, e_n\}$ are tangent to M^n. We define S_1 and S_2 by

$$S_1 := \sum_{i,j=1}^{n} (h_{ij}^{n+1} - H\delta_{ij})^2, \quad S_2 := \sum_{\alpha=n+2}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2,$$

respectively. Where h_{ij}^{α} denote components of the second fundamental form of M^n. Then, S_1 and S_2 are functions defined on M^n globally, which do not depend on the choice of the orthonormal frame $\{e_1, \cdots, e_n\}$. And

$$S - nH^2 = S_1 + S_2.$$

From the definition of the mean curvature vector $\mathbf{h}$, we know $nH = \sum_{i=1}^{n} h_{ii}^{n+1}$ and $\sum_{i=1}^{n} h_{ii}^{\alpha} = 0$ for $n+2 \le \alpha \le n+p$ on M^n. Putting $H_\alpha = (h_{ij}^{\alpha})$ and defining $N(A) = \mathrm{trace}({}^tAA)$ for $n \times n$-matrix A. Let h_{ijk}^{α} denote components of the covariant differentiation of the second fundamental form of M^n. By making use of a direct computation, we have, from Gauss equation,

$$
\begin{aligned}
\frac{1}{2}\Delta S_2 &= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \\
&= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 \\
&+ nH\sum_{\alpha=n+2}^{n+p}\operatorname{trace}(H_{n+1}H_{\alpha}^2) - \sum_{\alpha=n+2}^{n+p}[\operatorname{trace}(H_{n+1}H_{\alpha})]^2 \\
&- \sum_{\alpha,\beta=n+2}^{n+p} N(H_{\alpha}H_{\beta}-H_{\beta}H_{\alpha}) - \sum_{\alpha,\beta=n+2}^{n+p}[\operatorname{trace}(H_{\alpha}H_{\beta})]^2 \\
&+ \sum_{\alpha=n+2}^{n+p}\operatorname{trace}(H_{n+1}H_{\alpha})^2 - \sum_{\alpha=n+2}^{n+p}\operatorname{trace}(H_{n+1}^2H_{\alpha}^2).
\end{aligned}
$$

Since $e_{n+1} = \frac{\mathbf{h}}{H}$, we have $\operatorname{trace}(H_{\alpha}) = 0$ for $\alpha = n+2, \cdots, n+p$ and $\operatorname{trace}(H_{n+1}) = nH$. By a long and complicated estimate, we can infer

$$
\begin{aligned}
\frac{1}{2}\Delta S_2 &\geq \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 \\
&+ (nH^2 - \sqrt{\frac{n}{n-1}}(n-2)H\sqrt{S_1} - S_1 - \frac{3}{2}S_2)S_2 \\
&\geq \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 \\
&+ (nH^2 - \frac{n(n-2)}{2(n-1)}H^2 - \frac{n-2}{2}S_1 - S_1 - \frac{3}{2}S_2)S_2 \\
&= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 \\
&+ (nH^2 - \frac{n(n-2)}{2(n-1)}H^2 + \frac{n^2H^2}{2} - \frac{n}{2}S + \frac{(n-3)}{2}S_2)S_2 \\
&= \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \{\frac{n}{2}(\frac{n^2H^2}{n-1} - S) + \frac{(n-3)}{2}S_2\}S_2 \\
&\geq \sum_{\alpha=n+2}^{n+p}\sum_{i,j,k=1}^{n}(h_{ijk}^{\alpha})^2 + \{\frac{(n-3)}{2}S_2\}S_2 \geq 0.
\end{aligned} \tag{4.1}
$$

Since the mean curvature is bounded, from the condition $S \leq \frac{n^2H^2}{n-1}$ and Gauss equation, we know that the Ricci curvature of M^n is bounded from below. By applying the Generalized Maximum Principle due to Omori [9] and Yau [12] to the function S_2, we have that there exists a sequence $\{p_k\} \subset M^n$ such that

$$\lim_{k\to\infty} S_2(p_k) = \sup S_2 \quad \text{and} \quad \lim_{k\to\infty} \sup \Delta S_2(p_k) \leq 0. \tag{4.2}$$

Since $S \leq \frac{n^2H^2}{n-1}$, we know that $\{h^\alpha_{ij}(p_k)\}$, for any $i, j = 1, 2, \cdots, n$ and any $\alpha = n+1, \cdots, n+p$, is a bounded sequence. Hence, we can assume $\lim_{k\to\infty} h^\alpha_{ij}(p_k) = \tilde{h}^\alpha_{ij}$, if necassary, we can take a subsequence. From (4.1) and (4.2), we know that all of inequalities is equalities. Hence, $\sup S_2 = 0$ for $n > 3$. When $n = 3$, if $\sup S_2 \neq 0$, we know $\lim_{k\to\infty}(\frac{n^2H^2}{n-1} - S)(p_k) = 0$ and $\lim_{k\to\infty} \sqrt{\frac{n}{n-1}}H(p_k) = \lim_{k\to\infty} \sqrt{S_1(p_k)}$. Let $\lim_{k\to\infty} H(p_k) = \tilde{H}$, $\lim_{k\to\infty} S(p_k) = \tilde{S}$ and $\lim_{k\to\infty} S_1(p_k) = \tilde{S}_1$. We have $\frac{n^2\tilde{H}^2}{n-1} = \tilde{S}$, $\frac{n}{n-1}\tilde{H}^2 = \tilde{S}_1$ and $\tilde{S} = \sup S_2 + \tilde{S}_1 + n\tilde{H}^2 = \tilde{S} + \sup S_2$. This is impossible. Hence, we obtain $\sup S_2 = 0$. That is, $S_2 = 0$ on M^n. From (4.1), we have

$$\sum_{\alpha=n+2}^{n+p} \sum_{i,j,k=1}^{n} (h^\alpha_{ijk})^2 = 0 \tag{4.3}$$

on M^n. Thus, we infer $S_2 \equiv 0$ and (4.3) holds on M^n.

On the other hand, we have, for any $\alpha \neq n+1$,

$$\sum_{i,k=1}^{n} h^\alpha_{iik}\omega_k = -nH\omega_{\alpha n+1}.$$

Hence, (4.3) yields $\omega_{\alpha n+1} = 0$ for any α. Thus, we know that e_{n+1} is parallel in the normal bundle $T^\perp(M^n)$ of M^n. Hence, if we denote by N_1 the normal subbundle spanned by $e_{n+2}, e_{n+3}, \cdots, e_{n+p}$ of the normal bundle of M^n, then M^n is totally geodesic with respect to N_1. Since the e_{n+1} is parallel in the normal bundle, we know that the normal subbundle N_1 is invariant under parallel translation with respect to the normal connection of M^n. Then from the Theorem 1 in [13], we conclude that M^n lies in a totally geodesic submanifold $\mathbf{R}^{n+1}$ of $\mathbf{R}^{n+p}$. This finished our proof. □

Now, we shall prove our Main Theorems.

Proof of Main Theorem 1. Since the mean curvature H is constant, we have $H = 0$ or $H > 0$. In the case of $H = 0$, we have $S = 0$ on M^n since $S \leq \frac{n^2H^2}{n-1}$ holds. Therefore, we know that M^n is totally geodesic. Hence, M^n is isometric the hyperplane $\mathbf{R}^n$. Next, we assume $H > 0$. Thus $e_{n+1} = \frac{\mathbf{h}}{H}$ is a normal vector field defined globally on M^n. Hence, M^n is orientable. From the Proposition 1, M^n lies in a totally geodesic submanifold $\mathbf{R}^{n+1}$ of $\mathbf{R}^{n+p}$. We denote by H' the mean curvature of M^n in $\mathbf{R}^{n+1}$. Since $\mathbf{R}^{n+1}$ is totally geodesic in $\mathbf{R}^{n+p}$, we have $H = H'$, that is, the mean curvature H' of M^n in $\mathbf{R}^{n+1}$ is the same as in $\mathbf{R}^{n+p}$. We also know that the squared norm S' of the second fundamental form of M^n in R^{n+1} is the same as in $\mathbf{R}^{n+p}$. Hence, we imply $S' \leq \frac{n^2(H')^2}{n-1}$ and $H' \neq 0$. We choose a local orthonormal frame field $\{e_1, \cdots, e_n\}$ such that $h_{ij} = \lambda_i \delta_{ij}$ for $i,\ j = 1,\ 2, \cdots,\ n$. Where h_{ij} and λ_i denote components of the second fundamental form and principal curvatures of M^n in $\mathbf{R}^{n+1}$, respectively. Thus, we obtain

$$\sum_{i=1}^{n}(\lambda_i)^2 \leq \frac{(\sum_{i=1}^{n}\lambda_i)^2}{n-1}.$$

From the Lemma 4.1 in Chen [1, p.56], we have, for any $i,\ j$,

$$\lambda_i \lambda_j \geq 0.$$

Hence, M^n is a complete hypersurface in $\mathbf{R}^{n+1}$ with non-negative sectional curvatures. From the Theorem due to Cheng and Yau [5], we know that M^n is isometric to $S^n(c)$ or $S^{n-1}(c) \times \mathbf{R}$. Thus, We finished the proof of Main Theorem 1.

Proof of Main Theorem 2. Since M^n is compact, we know that the mean curvature is bounded. From the condtion $S \leq \frac{n^2 H^2}{n-1}$, we know that the Proposition 1 is true. Therefore, M^n lies in a totally geodesic submanifold $\mathbf{R}^{n+1}$ of $\mathbf{R}^{n+p}$. By making use of the same assertion as in the proof of Main Theorem 1, we know that M^n is a compact hypersurface in $\mathbf{R}^{n+1}$ with non-negative sectional curvatures. Thus, we infer that the principal curvatures are non-negative on M^n because the mean curvature is not zero on M^n. Namely, M^n is locally convex. Therefore, M^n is diffeomorphic to a sphere by the Theorem 5 due to Van Heijennoort [8] and Sacksteder [10]. We completed the proof of Main Theorem 2.

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