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Beurling's Minimum Principle in a Cone

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Abstract

This paper shows that some characterizations of the harmonic majorization of the Martin function connected with a domain having smooth boundary without a corner e.g a ball and a half-space also hold for a special domain with corners, i.e. a cone.

1. *Introduction.*

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and all positive real numbers, respectively. We denote by $\mathbf{R}^n$ $(n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P=(X,y), X=(x_1,x_2,\ldots,x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P-Q|$. Also $|P-O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set S in $\mathbf{R}^n$ are denoted by ∂S and $\bar{S}$, respectively.

We introduce a system of spherical coordinates $(r,\Theta), \Theta=(\theta_1,\theta_2,\ldots,\theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1,x_2,\ldots,x_{n-1},y)$ by

$$x_1 = r(\Pi_{j=1}^{n-1}\sin\theta_j) \quad (n \geq 2), \qquad y = r\cos\theta_1,$$

and if $n \geq 3$, then

$$x_{n+1-k} = r(\Pi_{j=1}^{k-1}\sin\theta_j)\cos\theta_k \quad (2 \leq k \leq n-1),$$

where $0 \leq r < +\infty$, $-\frac{1}{2}\pi \leq \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi$ $(1 \leq j \leq n-2)$.

The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1,\Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1,\Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set

$$\{(r,\Theta) \in \mathbf{R}^n;\ r \in \Lambda,\ (1,\Theta) \in \Omega\}$$

in $\mathbf{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half-space

$$\mathbf{R}_+ \times \mathbf{S}_+^{n-1} = \{(X, y) \in \mathbf{R}^n;\ y > 0\}$$

will be denoted by $\mathbf{T}_n$.

To extend a result of Beurling [7] for n=2, Armitage and Kuran [4] said that a sequence $\{P_m\}$ of points $P_m = (X_m, y_m) \in \mathbf{T}_n$, $|P_m| \to +\infty$ $(m \to +\infty)$ "characterizes the positive harmonic majorization of y", if every positive harmonic function h in $\mathbf{T}_n$ which majorizes the function y on the set $\{P_m;\ m = 1, 2, ..\}$ majorizes y everywhere in $\mathbf{T}_n$, i.e.

$$\inf_{P \in \mathbf{T}_n} \frac{h(P)}{y} = \inf_m \frac{h(P_m)}{y_m} \qquad (P = (X, y) \in \mathbf{T}_n).$$

They proved

THEOREM A (Beurling [7] for $n = 2$, Armitage and Kuran [4, Theorem 1] for $n \geq 2$). *Let $\{P_m\}$ be a sequence of points $\{P_m\}$,*

$$P_m = (r_m, \Theta_m) \in \mathbf{T}_n, \Theta_m = (\theta_{1.m}, \theta_{2,m}, ..., \theta_{(n-1),m})$$

in $\mathbf{T}_n$ satisfying

$$r_{m+1} \geq a\ r_m \qquad (m = 1, 2, ...) \tag{1.1}$$

for a certain $a > 1$. Then the sequence $\{P_m\}$ characterizes the positive harmonic majorization of y if and only if

$$\sum_{m=1}^{\infty} (\cos \theta_{1,m})^n = +\infty. \tag{1.2}$$

We remark that y is the Martin function at the infinite Martin boundary point ∞ of $\mathbf{T}_n$, i.e. y is (up to a positive multiplicative constant) the only positive harmonic function in $\mathbf{T}_n$ which vanishes on $\partial \mathbf{T}_n$. The "if" part of Teorem A is a minimm principle, since if h is a positive harmonic function $h(P)$ of $P = (X, y) \in \mathbf{T}_n$, then

$$\liminf_{P \in \mathbf{T}_n,\ P \to P'} \{h(P) - y\} \geq 0$$

for every P' on $\partial \mathbf{T}_n$ and the majorization of the function y by h on the set of points P_m satisfying (1.1) and (1.2) replaces

$$\liminf_{P \in \mathbf{T}_n,\ |P| \to +\infty} \{h(P) - y\} \geq 0.$$

Hence this sort of sequence was said to be "equivalent to ∞" in Beurling [**7**] and this type of Theorem A was called "Beurling minimum principle" in Ancona [**3**, p.18] and Maz'ya [**15**]. This Theorem A was also extended by Maz'ya [**15**] to positive solutions of a second-order elliptic differential equation in an n-dimensional bounded domain with smooth boundary of class $C^{1,\alpha}$ $(0 < \alpha < 1)$.

Let D be a domain in $\mathbf{R}^n$ and $\Delta(D)$ be the Martin boundary of D. The Martin function at $Q \in \Delta(D)$ is denoted by $K_Q(P)$ $(P \in D)$. Following Armitage and Kuran [**4**], we say that a subset E of D *characterizes the positve harmonic majorization of* $K_Q(P)$, if every positive harmonic function h in D which majorizes $K_Q(P)$ on E majorizes $K_Q(P)$ everywhere in D, i.e.

$$\inf_{P\in D} \frac{h(P)}{K_Q(P)} = \inf_{P\in E} \frac{h(P)}{K_Q(P)}. \tag{1.3}$$

We set

$$B(P,r) = \{P' \in \mathbf{R}^n;\ |P' - P| < r\} \quad (r > 0)$$

and

$$d(P) = \inf_{Q \notin D} |P - Q|$$

for any $P \in D$. For a subset E of D and a number ρ $(0 < \rho < 1)$ we put

$$E_\rho = \cup_{P\in E} B(P, \rho d(P)). \tag{1.4}$$

Dahlberg proved

THEOREM B (Dahlberg [**10**, Theorem 1]). *Let D be a Liapunov-Dini domain in $\mathbf{R}^n$ and $Q \in \partial D$. If $E \subset D$, then the following conditions on E are equivalent;*

(*i*) *E characterizes the positve harmonic majorization of $K_Q(P)$;*

(*ii*) *for every $\rho, 0 < \rho < 1$*

$$\int_{E_\rho} \frac{dP}{|P - Q|^n} = +\infty,$$

(*iii*) *for some $\rho, 0 < \rho < 1$*

$$\int_{E_\rho} \frac{dP}{|P - Q|^n} = +\infty.$$

Since (1.3) is closely related to the notion of minimally thinness of E_ρ in (1.4) (see Sjögren [**17**], Ancona [**3**] and Zhang [**19**]), which is also seen in Theorem 1 of this paper, Aikawa and Essén [**2**, Corollary 7.4.7] also proved Theorem B in a different way from Dahlberg's.

By using a suitable Kelvin transformation which maps $\mathbf{T}_n$ onto a ball, the following Theorem C follows from Theorem B.

THEOREM C (Dahlberg [**10**, Theorem 3]). *If $E \subset \mathbf{T}_n$, then the following conditions on E are equivalent;*

(*i*) *E characterizes the positive harmonic majorization of y;*

(*ii*) *for every $\rho, 0 < \rho < 1$*

$$\int_{E_\rho} \frac{dP}{(1+|P|)^n} = +\infty,$$

where

$$E_\rho = \cup_{P=(X,y)\in E} B(P, \rho y);$$

(*iii*) *for some $\rho, 0 < \rho < 1$*

$$\int_{E_\rho} \frac{dP}{(1+|P|)^n} = +\infty.$$

The methods of proving these Theorems A and B were based on the smoothness of the boundary having no wedges e.g. a ball. For a domain having more rough boundary e.g. a Lipschiz domain, Ancona [**3**, Theorem 7.4] and Zhang [**19**, Theorem 3] gave more complicated results which generalize Theorem A.

For a Lipshitz domain and an NTA domain D, Zhang [**19**, Corollary 1] and Aikawa [**1**, Remark and Theorem 1] gave a necessary and sufficient qualitative condition for a subset E of D characterizing the positive harmonic majorization of $K_Q(P)$ by connecting with minimally thinness of E_ρ in (1.4), respectively. In his paper Aikawa says that since a general NTA domain may have wedges, Theorem B does not hold for an NTA domain. But when we see our results in this paper, we may ask whether quantitative Theorem B can just be extended to a Lipshitz domain and an NTA domain.

In this paper we shall prove Theorems A and C can be extended to a result at a corner point of a wedge i.e. a result at ∞ of a cone. We remark that a half-space is one of cones.

2. *Statements of results.*

Let Ω be a domain on $\mathbf{S}^{n-1} (n \geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$\begin{aligned} (\Lambda_n + \tau) f &= 0 \quad \text{on } \Omega \\ f &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Λ_n is the spherical part of the Laplace operator Δ_n

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n$$

We denote the least positive eigenvalue of this boundary value problem by τ_Ω and the normalized positive eigenfunction corresponding to τ_Ω by $f_\Omega(\Theta)$;

$$\int_\Omega f_\Omega^2(\Theta)d\sigma_\Theta = 1,$$

where $d\sigma_\Theta$ is the surface element on $\mathbf{S}^{n-1}$. We denote the solutions of the equation

$$t^2 + (n-2)t - \tau_\Omega = 0$$

by $\alpha_\Omega, -\beta_\Omega$ $(\alpha_\Omega, \beta_\Omega > 0)$. If $\Omega = \mathbf{S}_+^{n-1}$, then $\alpha_\Omega = 1$, $\beta_\Omega = n-1$ and

$$f_\Omega(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1,$$

where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

To make simplify our consideration in the following, we shall assume that if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ (e.g. see Gilbarg and Trudinger [**12**, pp.88-89] for the definition of $C^{2,\alpha}$-domain). By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}(n \geq 2)$. We call it a cone. Then $\mathbf{T}_n$ is a cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$. It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, each of which is a minimal Martin boundary point. The Martin kernel at ∞ with respect to a reference point chosen suitably is $K_\infty(P) = r^{\alpha_\Omega} f_\Omega(\Theta)$ $(P = (r, \Theta) \in C_n(\Omega))$.

A subset E of a domain D in $\mathbf{R}^n$ is said to be *minimally thin* at $Q \in \Delta(D)$ (Brelot [**8**, p.122], Doob [**11**, p.208]), if there exists a point $P \in D$ such that

$$\hat{R}^E_{K_Q(\cdot)}(P) = K_Q(P),$$

where $\hat{R}^E_{K_Q(\cdot)}(P)$ is the regularized reduced function of $K_Q(P)$ relative to E (Helms [**14**, p.134]).

The following Theorem 1 which is used to prove Theorem 2 is a specialized version of Aikawa [**1**, Theorem 1]. Since his proof is so complicated because of an NTA domain we shall give a simple proof based on a function which is a conical version of Dahlberg's [**10**, pp.240-241].

THEOREM 1. *Let E be a subset of $C_n(\Omega)$. The following conditions on E are equivalent:*

(i) E characterizes the positive harmonic majorization of $K_\infty(P)$;
(ii) for any ρ, $0 < \rho < 1$, E_ρ is not minimally thin at ∞,
(iii) for some ρ, $0 < \rho < 1$, E_ρ is not minimally thin at ∞ .

The following Theorem 2 extends Theorem C.

THEOREM 2. *Suppose that $E \subset C_n(\Omega)$. Then the following conditions on E are equivalent:*

(i) E characterizes the positive harmonic majorization of $K_\infty(P)$;
(ii) for every ρ $(0 < \rho < 1)$,

$$\int_{E_\rho} \frac{dP}{(1+|P|)^n} = +\infty,$$

(iii) for some ρ $(0 < \rho < 1)$,

$$\int_{E_\rho} \frac{dP}{(1+|P|)^n} = +\infty.$$

A sequence $\{P_m\}$ of points $P_m \in D$ is said to be *separated*, if there exists a positive constant c such that

$$|P_i - P_j| \geq cd(P_i) \quad (i, j = 1, 2, ..., i = j)$$

(e.g. see Ancona [**3**, p.18], Aikawa and Essén [**2**, p.156]).

From Theorem 2 we immediately have the following Corollary which extends Theorem A.

COROLLARY. *Let $\{P_m\}$, $P_m \in C_n(\Omega)$ be a separated sequence satisfying*

$$\inf_m |P_m| > 0.$$

The sequence $\{P_m\}$ characterizes the positive harmonic majorization of $K_\infty(P)$ if and only if

$$\sum_{m=1}^{\infty} \left(\frac{d(P_m)}{|P_m|}\right)^n = +\infty.$$

3. *Lemmas and proof of Theorem 1.*

Let f and g be two positive real valued functions defined on a set Z. Then we shall write $f \approx g$, if there exists two constants $A_1, A_2, 0 < A_1 \leq A_2$ such that $A_1 g \leq f \leq A_2 g$ everywhere on Z. For a subset S in $\mathbf{R}^n$, the interior of S and the diameter of S are denoted by int S and diam S, respectively. For two subsets S_1 and S_2 in $\mathbf{R}^n$, the distance between S_1 and S_2 is denoted by dist(S_1, S_2). A cube of $\mathcal{M}_k$ is of the form

$$[l_1 2^{-k}, (l_1+1)2^{-k}] \times \cdots \times [l_n 2^{-k}, (l_n+1)2^{-k}] \quad (k = 0, \pm 1, \pm 2, \ldots)$$

where $l_1, \ldots, l_n$ are integers. Let ρ be a number satisfying $0 < \rho \leq \frac{1}{2}$. A family of the Whitney cubes of $C_n(\Omega)$ with ρ is the set of cubes having the following properties;
(*i*) $\bigcup_i W_i = C_n(\Omega)$,

(*ii*) $\operatorname{int} W_i \cap \operatorname{int} W_k = \emptyset \quad (i = k)$,
(*iii*) $\left[\frac{8}{3\rho}\right] \operatorname{diam} W_i \leq \operatorname{dist}(W_i, \mathbf{R}^n \setminus C_n(\Omega)) \leq 2\left(\left[\frac{8}{3\rho}\right]+1\right) \operatorname{diam} W_i$,
where [a] denotes the integer satisfying $[a] \leq a < [a]+1$ (Stein [**18**, p.167, Theorem 1]).

The following Lemma 1 is fundamental in this paper.

LEMMA 1 (I.Miyamoto, M.Yanagishita and H.Yoshida [**16**, Theorems 2 and 3]). *Let a Borel subset E of $C_n(\Omega)$ be minimally thin at ∞. Then we have*

$$\int_E \frac{dP}{(1+|P|)^n} < +\infty. \tag{3.1}$$

If E is a union of cubes from a family of the Whitney cubes of $C_n(\Omega)$ with ρ $(0 < \rho \leq \frac{1}{2})$, then (3.1) is also sufficient for E to be minimally thin at ∞.

For a set $E \subset C_n(\Omega)$ and a number ρ $(0 < \rho \leq \frac{1}{2})$, define E_ρ and $E_{\frac{\rho}{4}}$ as in (1.4).

LEMMA 2. *Let $\{W_i\}_{i\geq 1}$ be a family of the Whitney cubes of $C_n(\Omega)$ with ρ. Then there exists a subsequence $\{W_{i_j}\}_{j\geq 1}$ of $\{W_i\}_{i\geq 1}$ such that*
(*i*) $\bigcup_j W_{i_j} \subset E_\rho$,
(*ii*) $W_{i_j} \cap E_{\frac{\rho}{4}} = \emptyset \quad (j = 1, 2, \ldots)$, $\quad E_{\frac{\rho}{4}} \subset \bigcup_j W_{i_j}$.

Proof of Lemma 2. Let k be an integer. Let $c = \left(\left[\frac{8}{3\rho}\right]+1\right)$ and set

$$I_k = \left\{P \in C_n(\Omega)\ ; c\sqrt{n}2^{-k} < \operatorname{dist}(P, \partial C_n(\Omega)) \leq c\sqrt{n}2^{-k+1}\right\}.$$

Let $\{W_{i_j}\}_{j\geq 1}$ be a sequence of all Whitney cubes of $\{W_i\}_{i\geq 1}$ such that

$$W_{i_j} \cap E_{\frac{\rho}{4}} = \emptyset \quad (j = 1, 2, \ldots).$$

Then it is evident that (*ii*) holds. We shall also show that this $\{W_{i_j}\}_{j\geq 1}$ satisfies (*i*). Take any $W_{i_{j0}}$ and let $W_{i_{j0}}$ be a cube of $\mathcal{M}_{k_0}$. Since $W_{i_{j0}} \cap E_{\frac{\rho}{4}} = \emptyset$, there exists a point P_{j0} in E such that

$$B(P_{j0}, \frac{\rho}{4} d(P_{j0})) \cap W_{i_{j0}} = \emptyset. \tag{3.2}$$

Then $P_{j0} \in I_{k_0+1} \cup I_{k_0} \cup I_{k_0-1}$. Because, if $P \in I_k$ and W_{i_j} is a Whitney cube satisfying $W_{i_j} \cap B(P, \frac{\rho}{4} d(P)) = \emptyset$, then $W_{i_j} \in \mathcal{M}_{k+1} \cup \mathcal{M}_k \cup \mathcal{M}_{k-1}$.

If $P_{j0} \in I_{k_0+1}$, then

$$\rho d(P_{j0}) - \frac{\rho}{4} d(P_{j0}) = \frac{3}{4}\rho d(P_{j0}) > \frac{3}{4}\rho\left(\left[\frac{8}{3\rho}\right]+1\right)\sqrt{n}2^{-(k_0+1)} > \sqrt{n}2^{-k_0}.$$

Since the diameter of $W_{i_{j0}}$ is $\sqrt{n}2^{-k_0}$, we have from (3.2) that $W_{i_{j0}} \subset B(P_{j0}, \rho d(P_{j0}))$ and hence $W_{i_{j0}} \subset E_\rho$. Even if $P_{j0} \in I_{k_0}$ or $P_{j0} \in I_{k_0-1}$, we similarly have $W_{i_{j0}} \subset E_\rho$.

Thus all cubes of $\{W_{i_j}\}_{j\geq 1}$ are contained in E_ρ, which is just (i).

Proof of Theorem 1. Proof of $(i)\Rightarrow(ii)$. First of all, we shall remark the following fact. Let c be a positive constant. Since E characterizes the positive harmonic majorization of $K_\infty(P)$, $E_1=\{P\in E; K_\infty(P)>c\}$ also characterizes the positive harmonic majorization of $K_\infty(P)$. For otherwise there exists a positive harmonic function $h(P)$ on $C_n(\Omega)$, satisfying

$$a=\inf_{P\in C_n(\Omega)}\frac{h(P)}{K_\infty(P)}<\inf_{P\in E_1}\frac{h(P)}{K_\infty(P)}=b.$$

If we put $h_1(P)=h(P)+bc\quad (P\in C_n(\Omega))$, then $h_1(P)\geq bK_\infty(P)$ for all $P\in E$ and hence

$$\inf_{P\in C_n(\Omega)}\frac{h_1(P)}{K_\infty(P)}=a<b\leq\inf_{P\in E}\frac{h_1(P)}{K_\infty(P)},$$

which contradicts the assumption that E characterizes the positive harmonic majorization of $K_\infty(P)$. If we can show that for any ρ $(0<\rho<1)$ $(E_1)_\rho$ is not minimally thin at ∞, then for any ρ $(0<\rho<1)$ E_ρ is also not minimally thin at ∞. Hence by applying the following argument to E_1 if necessary, we may assume that $K_\infty(P)>c$ for every $P\in E$, without generality.

Suppose that for some number ρ $(0<\rho<1)$ E_ρ is minimally thin at ∞. Then to obtain a contradiction to (i) we shall make a positive harmonic function $h(P)$ on $C_n(\Omega)$ satisfying

$$\inf_{P\in C_n(\Omega)}\frac{h(P)}{K_\infty(P)}<\inf_{P\in E}\frac{h(P)}{K_\infty(P)}.$$

If E is a bounded subset of $C_n(\Omega)$, then let h be a constant function. When E is unbounded, we shall follow Dahlberg [**10**, p.240] to make it.

We can assume $\rho\leq\frac{1}{2}$. Let $\{P_j\}$ be a sequence of points P_j which are the central points of cubes W_{i_j} in Lemma 2. Then $\{P_j\}$ can not accumulate to any finite boundary point of $C_n(\Omega)$ and hence $|P_j|\to+\infty$, because $P_j\in E_\rho$ from (i) of Lemma 2 and $K_\infty(P)>c$ for any $P\in E$. Since E_ρ is minimally thin at ∞ and

$$\int_{W_{i_j}}\frac{dP}{(1+|P|)^n}\approx\left(\frac{d(P_j)}{|P_j|}\right)^n\quad (j=1,2,\ldots.),$$

Lemma 1 and (i) of Lemma 2 give

$$\sum_{j=1}^{\infty}\left(\frac{d(P_j)}{|P_j|}\right)^n<+\infty. \tag{3.3}$$

Now we shall assume that $d(P_j)\leq\frac{1}{2}|P_j|\quad (j=1,2,\ldots)$. The general case will be treated at the end of this proof. Take a point $Q_j=(t_j,\Phi_j)\in\partial C_n(\Omega)\setminus\{O\}$ satisfying

$$|P_j-Q_j|=d(P_j)\quad (j=1,2,\ldots).$$

Then we also see $|Q_j| \geq \frac{1}{2}|P_j|$ and hence $|Q_j| \to +\infty$ $(j \to +\infty)$. We define a function $h(P)$ by

$$h(P) = \sum_{j=1}^{\infty} \mathbf{P}_{Q_j}(P) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_\Omega}}, \quad \mathbf{P}_{Q_j}(P) = \frac{\partial G(P, Q_j)}{\partial n_{Q_j}} \quad (P \in C_n(\Omega)),$$

where $G(P_1, P_2)$ $(P_1, P_2 \in C_n(\Omega))$ is the Green function of $C_n(\Omega)$ and $\frac{\partial}{\partial n_Q}$ denotes the differentiation at $Q \in \partial C_n(\Omega)$ along the inward normal into $C_n(\Omega)$. Then h is well-defined and hence is a positive harmonic function on $C_n(\Omega)$, because at any fixed $P = (r, \Theta) \in C_n(\Omega)$

$$\mathbf{P}_{Q_j}(P) \approx r^{\alpha_\Omega} f_\Omega(\Theta) t_j^{-\beta_\Omega - 1} \frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j)$$

for every Q_j satisfying $t_j \geq 2r$ (see Azarin [**6**, Lemma 1]).

Now we shall show

$$\inf_{P \in E} \frac{h(P)}{K_\infty(P)} > 0.$$

To see first

$$\frac{h(P_j)}{K_\infty(P_j)} \geq A \quad (j = 1, 2, \ldots) \tag{3.4}$$

for some positive constant A, denote the Poisson kernel of the ball $B_j = B(P_j, d(P_j))$ by $\mathbf{P}_j(P, Q)$ $(P \in B_j, Q \in \partial B_j)$. Then we see

$$\mathbf{P}_{Q_j}(P) \geq \mathbf{P}_j(P, Q_j) \quad (P \in B_j;\ j = 1, 2, \ldots)$$

and hence

$$\mathbf{P}_{Q_j}(P_j) \geq \mathbf{P}_j(P_j, Q_j) = s_n^{-1}\{d(P_j)\}^{1-n} \quad (j = 1, 2, \ldots).$$

Since

$$f_\Omega(\Theta) \approx d(P') \quad (P' = (1, \Theta), \Theta \in \Omega),$$

we obtain

$$h(P_j) \geq \mathbf{P}_{Q_j}(P_j) \frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_\Omega}} \geq A K_\infty(P_j) \quad (j = 1, 2, \ldots).$$

Next take any $P \in E$. Then by (ii) of Lemma 2 there exist a point P_j such that

$$|P - P_j| < \frac{1}{2} \mathrm{diam}(W_{i_j}) \leq \delta d(P_j),$$

where $\delta = \frac{1}{2}\left[\frac{8}{3\rho}\right]^{-1}$. Hence we see

$$h(P) \geq \frac{1 - \delta}{(1 + \delta)^{n-1}} h(P_j) \quad \text{and} \quad K_\infty(P) \leq \frac{1 + \delta}{(1 - \delta)^{n-1}} K_\infty(P_j)$$

from the Harnack's inequalities (see Armitage and Gardiner [5, Theorem 1.4.1]). Thus we have

$$\frac{h(P)}{K_\infty(P)} \geq \left(\frac{1-\delta}{1+\delta}\right)^n \frac{h(P_j)}{K_\infty(P_j)} \geq \left(\frac{1-\delta}{1+\delta}\right)^n A$$

from (3.4), which shows

$$\inf_{P\in E} \frac{h(P)}{K_\infty(P)} > 0.$$

To show $\inf_{P\in C_n(\Omega)} \dfrac{h(P)}{K_\infty(P)} = 0$, fix a ray L which is inside $C_n(\Omega)$ and starts from O. We shall show

$$\lim_{|P|\to+\infty,\ P\in L} \frac{h(P)}{K_\infty(P)} = 0. \tag{3.5}$$

Put

$$g_j(P) = \frac{\mathbf{P}_{Q_j}(P)}{K_\infty(P)}|P_j|^{\beta_\Omega+1} \quad (P\in C_n(\Omega), j=1,2,\ldots).$$

Then we have

$$\frac{h(P)}{K_\infty(P)} = \sum_{j=1}^{\infty} g_j(P)\left(\frac{d(P_j)}{|P_j|}\right)^n.$$

Since

$$\mathbf{P}_{Q_j}(P) \approx t_j^{\alpha_\Omega-1} r^{-\beta_\Omega} f_\Omega(\Theta)\frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j) \quad (P=(r,\Theta)\in C_n(\Omega), r\geq 2t_j) \tag{3.6}$$

(see Azarin [6, Lemma 1]), we see

$$\lim_{|P|\to+\infty,\ P\in L} g_j(P) = 0$$

for any fixed j. Hence if we can show that

$$|g_j(P)| \leq M \quad (P\in L, j=1,2,\ldots) \tag{3.7}$$

for some constant M, then we shall have (3.5) from (3.3).

Now we shall prove (3.7) by dividing into three cases. If $r\leq \frac{t_j}{2}$, then we have

$$\mathbf{P}_{Q_j}(P) \approx r^{\alpha_\Omega} t_j^{-\beta_\Omega-1} f_\Omega(\Theta)\frac{\partial}{\partial n_{\Phi_j}} f_\Omega(\Phi_j)$$

and hence we have

$$|g_j(P)| \leq M \quad (P=(r,\Theta)\in C_n(\Omega), j=1,2,\ldots).$$

If $r\geq 2t_j$, then we have

$$|g_j(P)| \leq M \quad (P=(r,\Theta)\in C_n(\Omega);\ j=1,2,\ldots)$$

from (3.6). Lastly, put $R_1 = \frac{r}{t_j}$, $u = t_j$ and $\Theta_1 = \Theta$ in

$$u^{n-2}G((uR_1,\Theta_1),(uR_2,\Theta_2)) = G((R_1,\Theta_1),(R_2,\Theta_2)) \quad ((R_1,\Theta_1),(R_2,\Theta_2) \in C_n(\Omega)).$$

When (R_2,Θ_2) approaches to $(1,\Phi_j)$ along the inward normal, we obtain

$$\frac{\partial G(P,Q_j)}{\partial n_{Q_j}} = \frac{1}{t_j^{n-1}}\frac{\partial G}{\partial n_{Q'_j}}\left(\left(\frac{r}{t_j},\Theta\right),(1,\Phi_j)\right). \tag{3.8}$$

If $\frac{1}{2}t_j \le r \le 2t_j$, then

$$t_j^{n-1}\mathbf{P}_{Q_j}(P) \le M' \quad (P=(r,\Theta)\in L;\ j=1,2,\dots)$$

for some constant M' and hence

$$|g_j(P)| \le M \quad (P \in L;\ j=1,2,\dots).$$

Finally, even if there is a j such that $d(P_j) > \frac{1}{2}|P_j|$, there also exists a J such that $d(P_j) \le \frac{1}{2}|P_j|$ for every $j \ge J$. Define h_2 by

$$h_2(P) = \sum_{j=J}^{\infty}\mathbf{P}_{Q_j}(P)\frac{\{d(P_j)\}^n}{|P_j|^{1-\alpha_\Omega}} \quad (P \in C_n(\Omega)),$$

which satisfies

$$h_2(P_j) \ge AK_\infty(P_j) \quad (j \ge J) \quad \text{and} \quad \inf_{P\in C_n(\Omega)}\frac{h_2(P)}{K_\infty(P)} = 0.$$

Put $\gamma = \max_{1\le j<J} K_\infty(P_j)$. Then the function $h(P) = h_2(P)+\gamma$ is a positive harmonic function on $C_n(\Omega)$ such that

$$\inf_{P\in C_n(\Omega)}\frac{h(P)}{K_\infty(P)} = 0$$

and

$$h(P_j) \ge \min\{A,1\}K_\infty(P_j) \quad (j=1,2,\dots)$$

from which it follows in the same way as above that

$$\inf_{P\in E}\frac{h(P)}{K_\infty(P)} > 0.$$

Proof of $(iii) \Rightarrow (i)$.

Suppose that E does not characterize the positive harmonic majorization of $K_\infty(P)$. Then there exists a positive harmonic function $h(P)$ in $C_n(\Omega)$ such that

$$a = \inf_{P\in C_n(\Omega)}\frac{h(P)}{K_\infty(P)} < \inf_{P\in E}\frac{h(P)}{K_\infty(P)} = b.$$

If we put $v(P) = h(P) - aK_\infty(P)$ $(P \in C_n(\Omega))$, then $v(P)$ is a positive harmonic function on $C_n(\Omega)$ satisfying $\inf_{P\in C_n(\Omega)} \frac{v(P)}{K_\infty(P)} = 0$. Let ρ be any positive number satisfying $0 < \rho < 1$. For any $P \in E_\rho$, there exists a point $P' \in E$ such that $|P - P'| < \rho d(P')$ and hence

$$\left(\frac{1-\rho}{1+\rho}\right)^n \frac{v(P')}{K_\infty(P')} \leq \frac{v(P)}{K_\infty(P)}$$

by Harnack's inequality. (e.g. Armitage and Gardiner [**5**, Theorem 1.4.1]). Hence we have

$$\inf_{P\in E_\rho} \frac{v(P)}{K_\infty(P)} \geq \left(\frac{1-\rho}{1+\rho}\right)^n \inf_{P\in E} \frac{v(P)}{K_\infty(P)} = \left(\frac{1-\rho}{1+\rho}\right)^n (b-a) > 0.$$

Therefore we obtain

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K_\infty(P)} < \inf_{P\in E_\rho} \frac{v(P)}{K_\infty(P)}.$$

Since $v(P)$ is also a positive superharmonic function, E_ρ is minimally thin at ∞ (e.g. Miyamoto, Yanagishita and Yoshida [**16**. Theorem 1]). This contradicts (iii).

4. *Proofs of Theorem 2 and Corollary*

Proof of Theorem 2. Proof of $(i) \Rightarrow (ii)$. Suppose that

$$\int_{E_\rho} \frac{dP}{(1+|P|)^n} < +\infty$$

for some ρ $(0 < \rho < 1)$. We can assume that this ρ satisfies $0 < \rho \leq \frac{1}{2}$. Let $\{W_{i_j}\}_{j\geq 1}$ be a subsequence of $\{W_i\}_{i\geq 1}$ in Lemma 2. Then from (i) of Lemma 2 we also have

$$\int_{\cup_j W_{i_j}} \frac{dP}{(1+|P|)^n} < +\infty.$$

Since $\cup_j W_{i_j}$ is a union of cubes from the Whitney cubes of $C_n(\Omega)$ with ρ, we see from the second part of Lemma 1 that $\cup_j W_{i_j}$ is minimally thin at ∞, and hence from (ii) of Lemma 2 that $E_{\frac{\rho}{4}}$ is minimally thin at ∞. Since E characterizes the positive harmonic majorization of $K_\infty(P)$, it follows from Theorem 1 that $E_{\frac{\rho}{4}}$ is not minimally thin at ∞, which contradicts the conclusion obtained above.

Proof of $(iii) \Rightarrow (i)$. Suppose that E does not characterize the positive harmonic majorization of $K_\infty(P)$. Then we see from Theorem 1 that for any ρ $(0 < \rho < 1)$ E_ρ is minimally thin at ∞. Lemma 1 gives that for any ρ $(0 < \rho < 1)$

$$\int_{E_\rho} \frac{dP}{(1+|P|)^n} < +\infty.$$

This contradicts (iii).

Proof of Corollary. It is easy to see that if $\{P_m\}$ is a separated sequence, then

$$B(P_i, \rho d(P_i)) \cap B(P_j, \rho d(P_j)) = \emptyset \quad (i, j = 1, 2, ...; \; i = j)$$

for a sufficiently small ρ $(0 < \rho < 1)$ and hence

$$\int_{E_\rho} \frac{dP}{(1+|P|)^n} \approx \sum_{m=1}^{\infty} \left(\frac{d(P_m)}{|P_m|}\right)^n.$$

This corollary immediately follows from (iii) of Theorem 2.

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