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A NOTE ON DIVERGENCE OF L^p-INTEGRALS OF SUBHARMONIC FUNCTIONS AND ITS APPLICATIONS

KENSHO TAKEGOSHI

Throughout this note we always denotes (M,g) a non-compact complete Riemannian manifold of dimension m and Δ_g the Laplacian defined by $\Delta_g := \mathrm{Trace}_g\nabla\nabla$. Our interests are the divergence property of L^p-integral of a non-trivial solution u satisfying the inequality either $u\Delta_g u \geq k$ or $\Delta_g \log u \geq k$ for a locally integralble function k on M, and its several applications in differential geometry ; for instance conformal deformation of metrics, parabolicity of manifolds and Liouville theorem for harmonic maps. The non-negativity of k is the most important case. However the condition can be relaxed by the non-negativity of the interal $\int_M\ k\ dv_g$ under assuming the integrability of the negative part of k. Such an observation brings us a few interesting applications and is originated in Yau, [Y]. The proof of results stated below will appear elsewhere.

1. To formulate our result we begin with the following theorem which contains a vanishing of gradient length of certain functions whose L^p-integrals have a moderate growth.

Theorem 1.1. *Let $\mathcal{I}(r)$, and $E(r)$ (resp. $K_\pm(r)$) be non-negative absolutely continuous and non-decreasing functions (resp. non-negative continuous functions) on $\mathbf{R}_+ := [1,+\infty)$. Suppose $K_-(+\infty) := \lim_{r\to+\infty} K_-(r) < +\infty$, $\frac{d}{dr}\mathcal{I}(r) > 0$ and*

$$K_+(r) + C_1E(r) \leq K_-(r) + C_2\sqrt{\frac{d}{dr}\mathcal{I}(r)\frac{d}{dr}E(r)} \tag{1}$$

for almost all $r \geq r_0 >> 1$ and two constants $C_1 \geq 0$, $C_2 > 0$. Then the following assertions are valid : The case $C_1 > 0$: if $K_-(+\infty) \leq K_+(+\infty) := \lim_{r\to+\infty} K_+(r) \leq +\infty$, then either

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$E(r) \equiv 0$ or $1/\frac{d}{dr}\mathcal{I}(r) \in L^1(\mathbf{R}_+)$. *The case* $C_1 = 0$: *if* $K_-(+\infty) < K_+(+\infty) \le +\infty$, *then* $\liminf_{r\to+\infty} E(r)\Big/\int_1^r dt/\frac{d}{dt}\mathcal{I}(t) > 0$.

Applying Theorem 1.1 to $\mathcal{I}(r) = \mathcal{I}_x^p(u,r)$ (see below) , $E(r) = \int_{B_x(r)} |u|^{p-2}|\nabla u|^2\, dv_g$ and $K_\pm(r) \equiv 0$ respectively we can obtain the following result which implies not only Theorem 2.1 but also Theorem 2.2 in [K] simultaneously, and is a generalization of Theorem 1, b) in [S] for the Laplacian Δ_g in view of Lemma given in 5. Appendix.

Theorem 1.2. *Suppose u is a non-constant smooth function satisfying the inequality $u\Delta_g u \ge 0$ on M. Then for any $p > 1$, $r > 0$ and $x \in M$, the function $\mathcal{I}_x^p(u,r)$ defined by $\mathcal{I}_x^p(u,r) := \int_{B_x(r)} |u|^p\, dv_g$ satisfies $1/\frac{d}{dr}\mathcal{I}_x^p(u,r) \in L^1(\mathbf{R}_+)$, where $B_x(r)$ is the geodesic ball centered at $x \in M$ and of radius r.*

Remark 1. Since the distance function r_x from a point $x \in M$ is Lipschitz continuous on M and satisfies $|\nabla r_x| \equiv 1$ within the cut locus of x, letting σ_r be the $m-1$-dimensional Hausdorff measure of the geodesic sphere $S_x(r)$ induced by g, $\frac{d}{dr}\mathcal{I}_x^p(u,r)$ coincides with the integral $\int_{S_x(r)} |u|^p\, \sigma_r$ for almost all $r > 0$ by the co-area formula (cf. [F], 3.2.12, Theorem and 3.2.46).

Let $f : (M,g) \to (M,g)$ be a conformal transformation of (M,g) with $\dim_{\mathbf{R}} M \ge 3$ and u the conformal factor of f defined by $f^*g = u^{4/(m-2)}g$. It is known that u satisfies the following non-linear equality

$$c_m\Delta_g u - s_g u + K_{f^*g}u^{(m+2)/(m-2)} \equiv 0 \quad \text{on } M$$

for $c_m := 4(m-1)/(m-2)$ and K_{f^*g} is the scalar curvature of f^*g. If $s_g \le 0$ and f preserves the scalar curvature s_g, i.e., $s_g = K_{f^*g}$, then we can see

$$(u-1)\Delta_g(u-1) = \frac{-s_g u(u-1)(u^{4/(m-2)}-1)}{c_m} \ge 0 \quad \text{on } M\ .$$

Therefore Theorem 1.2 has the following interesting application which implies a uniqueness of solution for the scalar curvature equation and is known for the case $p = 2$ (cf. [BRS], Theorem 1.5).

Theorem 1.3. *Let (M, g) be a non-compact complete Riemannian manifold of dimension $m \geq 3$ and f a conformal transformation of (M, g). Suppose* (i) *the scalar curvature s_g of g is non-positive on M,* (ii) *f preserves s_g, and* (iii) *the conformal factor u satisfies $1/\max\left\{\frac{d}{dr}\mathcal{I}_x^p(u-1,r),1\right\} \notin L^1(\mathbf{R}_+)$ for a constant $p > 1$ and a point $x \in M$. Then $u \equiv 1$, i.e., f is isometric.*

Remark 2. In Theorem 1.3 the constant function 1 is a trivial solution of the above scalar curvature equation under the condition $s_g = K_{f^*g}$. However for two non-trivial solutions u_1 and u_2 of the equation, we do not know whether $u_1 \equiv u_2$ if $1/\max\left\{\frac{d}{dr}\mathcal{I}_x^p(u_1 - u_2, r), 1\right\} \notin L^1(\mathbf{R}_+)$ for a constant $p > 1$. But under this condition it is not so hard to see that Theorem 1.2 implies $|u_1 - u_2| \leq 1$ on M, and moreover $\inf_M |u_1 - u_2| = 0$ if (M, g) is not parabolic (see Corollary 2.2 below).

As a corollary, we get the following gap theorem of solution for the scalar curvature equation immediately.

Corollary 1.4. *Let (M, g) and f be as above. Suppose the conditions* (i) *and* (ii) *of Theorem 1.3 are satisfied and moreover* (iii)$'$ *(M, g) has polynomial volume growth, i.e., $\lim_{r\to+\infty} \mathrm{Vol}(B_y(r))/r^\alpha < +\infty$ for some $\alpha > 0$ and $|u(x) - 1| < C/(1 + r_y(x))^\beta$, $x \in M$, for the distance function r_y from a fixed point $y \in M$, $C > 0$ and $\beta > 0$. Then f is isometric.*

2. (M, g) is said to be *parabolic* if (M, g) admits no positive Green's function. Varopoulos showed that (M, g) is parabolic if $r/V_x(r) \notin L^1(\mathbf{R}_+)$ for some point $x \in M$, where $V_x(r) := \mathrm{Vol}(B_x(r))$ is the volume of $B_x(r)$ relative to g (cf.[V],Theorem 2 and [Gri], Corollary 2). Later Li and Tam proved the same result under the condition $1/\frac{d}{dr}V_x(r) \notin L^1(\mathbf{R}_+)$ (cf.[LT1], Corollary 2.3). Here we note that $\frac{d}{dr}V_x(r)$ is the area of the geodesic sphere $S_x(r)$ induced by g for almost all $r > 0$. In view of Lemma in 5.Appendix, Li and Tam's assertion is sharper than Varopoulos' one. The idea of proof of Theorem 1.2 allows us to give an alternative and elementary proof of their result by showing the following.

Theorem 2.1. *Suppose (M, g) admits a non-constant continuous subharmonic function bounded from above. Then $1/\frac{d}{dr}V_x(r) \in L^1(\mathbf{R}_+)$ for any point $x \in M$.*

Remark 3. Here a continuous function u is said to be subharmonic if D is any relatively compact open subset in M, $\Delta_g v \equiv 0$ on D, and $u \leq v$ on $\overline{D} \setminus D$, then $u \leq v$ on D. If u is of class C^2, then u satisfies $\Delta_g u \geq 0$. It is known that a positive Green's function produces smooth non-constant bounded subharmonic functions (cf. [CTW], Theorem 1.4).

By Theorem 2.1, we get the following in view of Remark 3.

Corollary 2.2. *If (M, g) admits a point $x \in M$ such that $1/\frac{d}{dr}V_x(r) \notin L^1(\mathbf{R}_+)$, then (M, g) is parabolic.*

Remark 4. If (M, g) is rotationally symmetric at a point $x_* \in M$, letting r_* be the distance function from $x_* \in M$, then we can see that (i) $w(x) := \int_1^{r_*(x)} dr/\frac{d}{dr}V_{x_*}(r)$ is harmonic on $M \setminus \{x_*\}$ and $\lim_{r_*(x) \to 0} w(x) = -\infty$, (ii) $\int_1^{+\infty} rdr/V_{x_*}(r) \geq \int_1^{+\infty} dr/\frac{d}{dr}V_{x_*}(r)$ if the radial curvature is non-positive (cf.[GW1] and Lemma in 5.Appendix). In particular $u := \exp w \geq 0$ defines a continuous *bounded* subharmonic function on M if and only if $1/\frac{d}{dr}V_{x_*}(r) \in L^1(\mathbf{R}_+)$. If the Ricci curvature of (M, g) is non-negative, then (M, g) is parabolic if and only if $r/V_x(r) \notin L^1(\mathbf{R}_+)$ for some point $x \in M$ (cf.[V], Theorem 2 and [LT2], Theorem 1.9). However the integrability of $r/V_x(r)$ does not always imply the existence of Green's function on (M, g) generally as observed in [V].

3. The parabolicity of manifold is related to Liouville theorem for harmonic maps. First we state the following.

Theorem 3.1. *Let $f : (M, g) \to (N, h)$ be a smooth map from (M, g) to a Riemannian manifold N provided with a smooth function φ and a continuous function $\chi > 0$ such that* $\mathrm{Hess}(\varphi) \geq \chi h$ *and $|\nabla\varphi| \leq C$ for a constant $C > 0$ on N. Suppose f is harmonic, and the energy density $\mathbf{e}(f) := (1/2)|df|^2$ of f satisfies the following condition $(*)$: $\int_{B_x(r)} \mathbf{e}(f)\, dv_g = o\left(\int_1^r dt/\frac{d}{dt}V_x(t)\right)$ for some point $x \in M$. Then f is a constant map.*

This can be induced by applying the case $C_1 = 0$ in Theorem 1.1 to $E(r) = \int_{B_x(r)} |\nabla u|^2\, dv_g$ for $u := f^*\varphi$ and $\mathcal{I}(r) = V_x(r)$. As a corollary we obtain the following Liouville theorem for harmonic maps to a manifold of asymptotically non-positive curvature.

Corollary 3.2. *Let $f : (M, g) \to (N, h)$ be a harmonic map to a complete Riemannian* *manifold (N, h) with a pole $y \in N$ whose radial curvature R_N satisfies $R_N \leq 1/4(1+r_y)^2$ on N. Suppose the energy density $\mathbf{e}(f)$ of f satisfies the condition $(*)$ in Theorem 3.1. Then f is a constant map.*

Remark 5. It is known that if (M, g) is parabolic, then any harmonic map of finite energy from (M, g) to any Hadamard manifold (N, h), i.e., N is simply connected and $R_N \leq 0$ on N, is constant (cf.[CTW],Proposition 2.1 and Theorem 3.2). On the other hand there exists a non-degenerate harmonic map from a two dimensional Euclidean space $\mathbf{R}^2$ with flat metric to a hyperbolic plane of constant curvature -1 (cf.[CT]). The energy of such a map on $B_x(r) \subset \mathbf{R}^2$ with $x = (0,0) \in \mathbf{R}^2$ diverges not slower than $\log r$ by Theorem 3.1.

However the analyticity of map yields the following Liouville theorem of holomorphic ones to a manifold of negative curvature bounded away from zero.

Theorem 3.3. *Let $f : (M, \omega_M) \to (N, \omega_N)$ be a holomorphic map from a complete Kähler manifold (M, ω_M) of dimension $m = \dim_{\mathbf{C}} M$ to a Kähler manifold (N, ω_N). If (M, ω_M) admits a point $x \in M$ such that $1\big/\frac{d}{dr}V_x(r) \notin L^1(\mathbf{R}_+)$, and (N, ω_N) admits a smooth $1-$form θ such that $\omega_N = d\theta$ and $C := \sup_N |\theta|_g < +\infty$, then f is a constant map.*

Remark 6. The target manifold (N, ω_N) in Theorem 3.3 should be non-compact. Many kinds of hyperbolic Kähler manifold admit such a Kähler metric (cf. [Gro]).

4. In Theorem 1.2 we can relax the non-negativity condition of k for the case $p = 2$.

Theorem 4.1. *Let u be a smooth non-constant solution satisfying the inequality $u\Delta_g u \geq k$ for a locally integrable function k on M. If $k_- := \max\{-k, 0\} \in L^1(M)$ and $\int_M k \, dv_g \geq 0$, then $1\big/\frac{d}{dr}\mathcal{I}_x^2(u, r) \in L^1(\mathbf{R}_+)$ for any $x \in M$.*

According to the same spirit as Theorem 4.1 we can show the following which is a generalization of Theorem 2.1 in [LY] and Theorem 1 in [Y].

Theorem 4.3. *Let u be a smooth non-constant solution satisfying the inequality $\Delta_g \log u \geq k$ on $M_+ := \{u > 0\}$. If either $k \equiv 0$ or $k_- \in L^1(M)$ and $\int_M k \, dv_g > 0$, then $1\big/\frac{d}{dr}\mathcal{I}_x^p(u, r) \in L^1(\mathbf{R}_+)$ for any $p > 0$ and $x \in M$.*

Remark 7. Under the same situation Li and Yau showed $\liminf_{r\to+\infty} \mathcal{I}_x^p(u,r)/r^2 > 0$ in [LY], which follows from $1\big/\frac{d}{dr}\mathcal{I}_x^p(u,r) \in L^1(\mathbf{R}_+)$ in view of Lemma in 5. Appendix.
The condition $\int_M k\, dv_g > 0$ follows from the non-parabolicity of (M,g).

Corollary 4.3. *Let u be a smooth non-negative solution of the inequality $\Delta_g \log u \geq k$ on M_+. If $k_- \in L^1(M)$, (M,g) is not parabolic, and $1\big/\max\left\{1, \frac{d}{dr}\mathcal{I}_x^p(u,r)\right\} \notin L^1(\mathbf{R}_+)$ for some $p > 0$ and a point $x \in M$, then u should be identically zero.*

Let $f : (M,\omega_M) \to (N,\omega_N)$ be a holomorphic map from a non-compact complete Kähler manifold (M,ω_M) to a complex Hermitian manifold (N,ω_N) whose holomorphic sectional curvature is non-positive. Letting $\mathbf{e}(f)$ be the energy density $\mathbf{e}(f)$ of f and R_M the pointwise lower bound of the Ricci curvature of M, one can show the following inequality

$$\Delta_g \log \mathbf{e}(f) \geq 2R_M \ ,$$

where $\mathbf{e}(f) \neq 0$ (see [R], Proposition 4). A complex differential geometric interpretation of Theorem 4.2 and Corollary 4.3 is the following Liouville theorem for holomorphic maps.

Theorem 4.4. *Let $f : (M,\omega_M) \to (N,\omega_N)$ be a holomorphic map from a non-compact complete Kähler manifold (M,ω_M) to a complex Hermitian manifold (N,ω_N) whose holomorphic sectional curvature is non-positive, and $R_{M,-}$ the negative part of R_M. Suppose* (i) $R_{M,-} \in L^1(M)$, (ii) *either (M,ω_M) is not parabolic or $\int_M R_M\, dv_M > 0$, and* (iii) *the energy density $\mathbf{e}(f)$ of f satisfies $1\big/\max\left\{1, \frac{d}{dr}\mathcal{I}_x^p(\mathbf{e}(f),r)\right\} \notin L^1(\mathbf{R}_+)$ for some $p > 0$ and a point $x \in M$. Then f is a constant map.*

Remark 8. In case $R_{M,-} \equiv 0$ and $p = 1$ the condition (ii) can be dropped as shown in [SY], which deals with harmonic maps of finite energy (see also [LY], theorem 3.1).

Let $f : (M,\omega_M) \to (N,\omega_N)$ be a holomorphic map from a non-compact complete Kähler manifold (M,ω_M) of dimension m to a complex hermitian manifold (N,ω_N) of the same dimension whose Ricci curvature is non-positive. Letting u_f denote the ratio f^*V_N/V_M of the volume forms V_M relative to ω_M and V_N relative to ω_N respectively and S_M the scalar curvature of (M,ω_M), one can see the following inequality

$$\Delta_g \log u_f \geq S_M$$

where $u_f \neq 0$ (see [LY], the proof of Theorem 3.5). Hence we can also show the following theorem (cf.[MY], §1 and [LY], Theorem 3.5 and Corollary 3.6).

Theorem 4.5. *Let $f : (M, \omega_M) \to (N, \omega_N)$ be a holomorphic map from a non-compact complete Kähler manifold (M, ω_M) of dimension m to a complex hermitian manifold (N, ω_N) of the same dimension whose Ricci curvature is non-positive, and $S_{M,-}$ the negative part of S_M of M. Suppose* (i) *$S_{M,-} \in L^1(M)$,* (ii) *either (M, ω_M) is not parabolic or $\int_M S_M \, dv_M > 0$, and* (iii) *$1 \big/ \max\left\{1, \frac{d}{dr}\mathcal{I}_x^p(u_f, r)\right\} \notin L^1(\mathbf{R}_+)$ for some $p > 0$ and a point $x \in M$. Then f degenerates everywhere on M.*

5. Appendix

Lemma. *Let $v(r) > 0$ be an absolutely continuous function on $[0, +\infty)$ such that $\frac{d}{dr}v(r) > 0$ for almost all $r \in [0, +\infty)$. Then v satisfies the following integral inequality :*

$$\int_2^r \frac{tdt}{v(t)} \le 4 \int_1^r \frac{dt}{\frac{d}{dt}v(t)} \quad \text{for any } r > 2 \, .$$

If $v(r)/r$ is non-decreasing (in particular $v(r)$ is convex and $v(0) = 0$), then $r/v(r) \in L^1(\mathbf{R}_+)$ if and only if $1/\frac{d}{dr}v(r) \in L^1(\mathbf{R}_+)$.

Proof. By integration by parts and Schwarz's inequality we get the following :

$$\int_1^r \frac{t-1}{v(t)} \, dt \le 2 \int_1^r \frac{dt}{\frac{d}{dt}v(t)} \quad \text{for any } r > 1 \, ,$$

which implies the desired inequality. □

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Address : Department of Mathematics, Graduate School of Science, Machikaneyama-cho 1-16, Toyonaka-shi Osaka, 560-0043, Japan

E-mail address : kensho@math.wani.osaka-u.ac.jp