Lindelöf type theorems for monotone Sobolev functions on half spaces

Abstract
This paper deals with Lindelöf type theorems for monotone functions in weighted Sobolev spaces.

1 Introduction

Let $\mathbb{R}^n$ ($n \geq 2$) denote the $n$-dimensional Euclidean space. We use the notation $D$ to denote the upper half space of $\mathbb{R}^n$, that is,

$$D = \{ x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0 \}.$$

We denote by $\rho_D(x)$ the distance of $x$ from the boundary $\partial D$, that is, $\rho_D(x) = |x_n|$ for $x = (x', x_n)$. Denote by $B(x, r)$ the open ball centered at $x$ with radius $r$, and set $\sigma B(x, r) = B(x, \sigma r)$ for $\sigma > 0$ and $S(x, r) = \partial B(x, r)$.

A continuous function $u$ on $D$ is called monotone in the sense of Lebesgue (see [6]) if the equalities

$$\max_{\partial G} u = \max u \quad \text{and} \quad \min_{\partial G} u = \min u$$

hold whenever $G$ is a domain with compact closure $\overline{G} \subset D$. If $u$ is a monotone Sobolev function in $D$ and $p > n - 1$, then

$$|u(x) - u(y)| \leq Mr \left( \frac{1}{r^n} \int_{2B} |\nabla u(z)|^p dz \right)^{1/p}$$

(1)

for all $x, y \in B$, where $B$ is an arbitrary ball of radius $r$ with $2B \subset D$ (see [7, Theorem 1] and [5, Theorem 2.8]). For further results of monotone functions, we refer to [3], [14] and [16].

2000 Mathematics Subject Classification : Primary 31B25 (46E35)
Key words and phrases : monotone functions, Sobolev functions, Lindelöf theorem, Hausdorff measures, weighted $p$-capacity
Our aim in the present note is to extend the second author's result [13, Theorem 2] to weighted case.

Let $\mu$ be a Borel measure on $\mathbb{R}^n$ satisfying the doubling condition:

$$\mu(2B) \leq c_1 \mu(B)$$

for every ball $B \subset \mathbb{R}^n$. We further assume that

$$\frac{\mu(B')}{\mu(B)} \geq c_2 \left(\frac{r'}{r}\right)^s$$

for all $B' = B(\xi', r')$ and $B = B(\xi, r)$ with $\xi', \xi \in \partial D$ and $B' \subset B$, where $s > 1$.

**Theorem 1.** Let $u$ be a Sobolev function on $D$ satisfying

$$|u(x) - u(y)| \leq M \rho_D(z) \left(\int_{\sigma B} |\nabla u(z)|^p d\mu\right)^{1/p}$$

for every $x, y \in B = B(z, \rho_D(z)/(2\sigma))$ with $z \in D$ and

$$\int_D |\nabla u(z)|^p d\mu(z) < \infty.$$

Define

$$E_1 = \left\{ \xi \in \partial D : \int_{B(\xi, 1) \cap D} |\xi - y|^{1-n} |\nabla u(y)| dy = \infty \right\}$$

and

$$E_2 = \left\{ \xi \in \partial D : \limsup_{r \to 0} (r^{-p} \mu(B(\xi, r)))^{-1} \int_{B(\xi, r) \cap D} |\nabla u(y)|^p d\mu(y) > 0 \right\}.$$

Then $u$ has a nontangential limit at every $\xi \in \partial D \setminus (E_1 \cup E_2)$.

Remark 1. Note here that $E_1 \cup E_2$ is of $C_{1,p,\mu}$-capacity zero. In Manfredi-Villamor [9], the exceptional sets are characterized by Hausdorff dimension, so that their result follows from this nontangential limit result.

**Theorem 2.** Let $u$ be a function on $D$ for which there exist a nonnegative function $g \in L^p_{loc}(D; \mu)$, $M > 0$ and $\sigma \geq 1$ such that

$$|u(x) - u(y)| \leq M \rho_D(z) \left(\int_{\sigma B} g^p d\mu\right)^{1/p}$$

for every $x, y \in B = B(z, \rho_D(z)/(2\sigma))$ with $z \in D$ and

$$\int_D g(z)^p d\mu(z) < \infty.$$
Suppose $p > s - 1$ and set

$$E = \left\{ \xi \in \partial D : \limsup_{r \to 0} \left( r^{-p} \mu(B(\xi, r)) \right)^{-1} \int_{B(\xi, r) \cap D} g(z)^{p} d\mu(z) > 0 \right\}.$$  

If $\xi \in \partial D \setminus E$ and there exists a curve $\gamma$ in $D$ tending to $\xi$ along which $u$ has a finite limit $\beta$, then $u$ has a nontangential limit $\beta$ at $\xi$.

For $\alpha > -1$, we consider

$$d\nu(x) = |x_{n}|^{\alpha} dx$$

as a measure, which satisfies

$$\nu(B(\xi, r)) = \nu(B(0, 1)) r^{n+\alpha}$$

for all $\xi \in \partial D$ and $r > 0$.

Then we obtain the following result.

**Corollary 1.** Let $u$ be a monotone Sobolev function on $D$ satisfying

$$\int_{D} |\nabla u(z)|^{p} z_{n}^{\alpha} dz < \infty$$

for $p > n - 1$ and $-1 < \alpha < p - n + 1$. Consider the set

$$E_{n+\alpha-p} = \left\{ \xi \in \partial D : \limsup_{r \to 0} r^{p-\alpha-n} \int_{B(\xi, r) \cap D} |\nabla u(z)|^{p} z_{n}^{\alpha} dz > 0 \right\}.$$  

If $\xi \in \partial D \setminus E_{n+\alpha-p}$ and there exists a curve $\gamma$ in $D$ tending to $\xi$ along which $u$ has a finite limit $\beta$, then $u$ has a nontangential limit $\beta$ at $\xi$.

**Remark 2.** We know that $\mathcal{H}^{n+\alpha-p}(E_{n+\alpha-p}) = 0$, where $\mathcal{H}^{d}$ denotes the $d$-dimensional Hausdorff measure, and hence it is of $C_{1-\alpha/p,p}$-capacity zero; for these results, see Meyers [10, 11] and the second author’s book [14].

**2 Proof of Theorem 2**

A sequence $\{x_{j}\}$ is called regular at $\xi \in \partial D$ if $x_{j} \to \xi$ and

$$|x_{j+1} - \xi| < |x_{j} - \xi| < c|x_{j+1} - \xi|$$

for some constant $c > 1$.

First we give the following result, which can be proved by (4).

**Lemma 1.** Let $u$ and $g$ be as in Theorem 2. If $\xi \in \partial D \setminus E$ and there exists a regular sequence $\{x_{j}\} \subset D$ with $x_{j} = \xi + (0, \ldots, 0, r_{j})$ such that $u(x_{j})$ has a finite limit $\beta$, then $u$ has a nontangential limit $\beta$ at $\xi$. 
PROOF OF THEOREM 2: For $r > 0$ sufficiently small, take $C(r) \in \gamma \cap S(\xi, r)$. Letting $C_1(r) = \xi + (0,\ldots,0, r)$, take an end point $C_2(r) \in \partial \mathbb{D}$ of a quarter of circle containing $C_1(r)$ and $C(r)$.

We take a finite chain of balls $B_1, B_2,\ldots, B_N$ ($N$ may depend on $r$) with the following properties:

(i) $B_j = B(z_j, \rho_D(z_j)/(2\sigma))$ with $z_j \in C(r) C_1(r)$, $z_1 = C(r)$ and $z_N = C_1(r)$;

(ii) $\rho_D(z_j) \leq \rho_D(z_{j+1})$ and $z_{j+1} \notin B_j$;

(iii) $B_j \cap B_{j+1} \neq \emptyset$ for each $j$;

(iv) $|C_2(r) - z| \leq 3\rho_D(z)$ for $z \in A(\xi, r) = \bigcup_{j=1}^{N} \sigma B_j \subset B(\xi, 2r) \cap \mathbb{D}$;

(v) $\sum_j \chi_{\sigma B_j} \leq c_3$, where $\chi_A$ denotes the characteristic function of $A$ and $c_3$ is a constant depending only on $c_1$ and $\sigma$;

see Heinonen [2] and Hajłasz-Koskela [1].

Pick $x_j \in B_{j+1} \cap B_j$ for $1 \leq j \leq N - 1$. By (4), we see that

$$|u(x_j) - u(x_{j-1})| \leq M \rho_D(z_j) \left( \int_{\sigma B_j} g(z)^p d\mu(z) \right)^{1/p}$$

for $1 \leq j \leq N$, where $x_0 = C(r)$ and $x_N = C_1(r)$.

Since $p > s - 1$ by our assumption, there is $\delta > 0$ such that $s - p < \delta < 1$. We have by Hölder's inequality

$$|u(C_1(r)) - u(C(r))| \leq |u(x_1) - u(x_0)| + |u(x_2) - u(x_1)| + \cdots + |u(x_N) - u(x_{N-1})|$$

$$\leq M \sum_{j=1}^{N} \rho_D(z_j)^{1+\delta/p} \mu(\sigma B_j)^{-1/p'} \left( \int_{\sigma B_j} g(z)^p \rho_D(z)^{-\delta} d\mu(z) \right)^{1/p'}$$

where $1/p + 1/p' = 1$. Here note that $\mu(B(C_2(r), \rho_D(z_j))) \leq c_4 \mu(\sigma B_j)$, where $c_4$ is a positive constant depending only on the doubling constant $c_1$. Since $\delta > s - p$, we see from (2) that

$$\sum_{j=1}^{N} \rho_D(z_j)^{p'(1+\delta)/p} \mu(\sigma B_j)^{-p'/p'} \leq M \sum_{j=1}^{N} \rho_D(z_j)^{p'(1+\delta)/p} \mu(B(C_2(r), \rho_D(z_j)))^{-p'/p'}$$
\[
M \sum_{j=1}^{N} \rho_D(z_j)^{p'(p+\delta)/p} \left( \frac{\rho_D(z_j)}{2r} \right)^{-sp'/p} \mu(B(\xi, 2r))^{-p'/p} \leq M r^{sp/p'} \mu(B(\xi, r))^{-p'/p} \sum_{j=1}^{N} \rho_D(z_j)^{p'(p+\delta-s)/p}.
\]

Moreover, since \(0 < \delta < 1\), we note that
\[
\int_{2^{-j}}^{2^{-j+1}} |C_2(r) - z|^{-\delta} dr \leq \int_{2^{-j}}^{2^{-j+1}} |r - |z||^{-\delta} dr \leq M 2^{-j(1-\delta)}.
\]

Hence it follows from (6) that
\[
\int_{2^{-j}}^{2^{-j+1}} \frac{|u(C_1(r)) - u(C(r))|^p dr}{r} \leq M \int_{2^{-j}}^{2^{-j+1}} r^\delta (r^{-p} \mu(B(\xi, r)))^{-1} \left( \int_{B(\xi,2r) \cap D} g(z)^p |C_2(r) - z|^{-\delta} d\mu(z) \right) \frac{dr}{r}.
\]

Since \(\xi \in \partial D \setminus E\), we can find a sequence \(\{r_j\}\) such that \(2^{-j} < r_j < 2^{-j+1}\) and
\[
\lim_{j \to \infty} |u(C_1(r_j)) - u(C(r_j))| = 0.
\]
By our assumption we see that \(u(C_1(r_j))\) has a finite limit \(\beta\) as \(j \to \infty\). If we note that \(\{C_1(r_j)\}\) is regular at \(\xi\), then Lemma 1 proves the required conclusion of the theorem.

\section{A_q weights}

Let \(w\) be a Muckenhoupt \(A_q\) weight, and define
\[
d\nu(y) = w(y) dy.
\]
Let \(u\) be a monotone Sobolev function on \(D\) such that
\[
\int_D |\nabla u(x)|^p d\nu(x) < \infty.
\]
Suppose that $1 < q < p/(n-1)$. Since $p_1 = p/q > n-1$, applying inequality (1) we obtain
\[
|u(x) - u(y)| \leq M r \left( \frac{1}{r^n} \int_{2B} |\nabla u(z)|^{p_1} dz \right)^{1/p_1}
\]
for all $x, y \in B$, where $B$ is an arbitrary ball of radius $r$ with $2B \subset D$. As in the proof of Theorem 2, we insist that
\[
\int_{2^{-j}}^{2^{-j+1}} |u(C_1(r)) - u(C(r))|^{p_1} \frac{dr}{r} \leq M 2^{-jp_1} |B(\xi, 2^{-j})|^{-1} \int_{B(\xi, 2^{-j+2}) \cap D} |\nabla u(z)|^{p_1} dz.
\]
Using Hölder inequality and $A_q$-condition of $w$, we have
\[
\int_{2^{-j}}^{2^{-j+1}} |u(C_1(r)) - u(C(r))|^{p_1} \frac{dr}{r} \leq M 2^{-jp_1} |B(\xi, 2^{-j})|^{-1} \left( \int_{B(\xi, 2^{-j+2}) \cap D} |\nabla u(z)|^{p_1} w(z) dz \right)^{1/q} \left( \int_{B(\xi, 2^{-j+2}) \cap D} w(z)^{-q'/q} dz \right)^{1/q'}
\]
\[
\leq M \left( 2^{jp} \nu(B(\xi, 2^{-j+2})) \right)^{-1} \int_{B(\xi, 2^{-j+2}) \cap D} |\nabla u(z)|^{p} d\nu(z),
\]
where $1/q + 1/q' = 1$. Thus we obtain the following result (cf. Manfredi-Villamor [9]), as in the proof of Theorem 2.

**Corollary 2.** Let $1 \leq q < p/(n-1)$. Let $w \in A_q$ and set $d\nu(y) = w(y) dy$. Suppose that $u$ is a monotone Sobolev function on $D$ satisfying
\[
\int_{D} |\nabla u(z)|^{p} d\nu(z) < \infty.
\]
Set
\[
E = \left\{ \xi \in \partial D : \limsup_{r \to 0} \left( r^{-p} \nu(B(\xi, r)) \right)^{-1} \int_{B(\xi, r) \cap D} |\nabla u(z)|^{p} d\nu(z) > 0 \right\}.
\]
If $\xi \in \partial D \setminus E$ and there exists a curve $\gamma$ in $D$ tending to $\xi$ along which $u$ has a finite limit $\beta$, then $u$ has a nontangential limit $\beta$ at $\xi$.

**Remark 3.** Let $1 \leq q < p/(n-1)$. Let $w$ be a Muckenhoupt $A_q$ weight, and define
\[
d\nu(y) = w(y) dy.
\]
Suppose that $u$ is a monotone Sobolev function on $D$ satisfying (7). Applying Hölder’s inequality to (1) with $p$ replaced by $p/q$, we see that
\[
|u(x) - u(y)| \leq M \left[ \int_{2B} |\nabla u(z)|^{p} d\nu(z) \right]^{1/p}
\]
for all $x, y \in B$, where $B$ is an arbitrary ball of radius $r$ with $2B \subset D$ (see also Manfredi-Villamor [9]).

**Remark 4.** Consider $w(y) = |y_n|^\alpha$. Then $w \in A_q$ if and only if $-1 < \alpha < q - 1$. In this case, Corollary 2 does not imply Corollary 1 when $n \geq 3$. 


4 Generalizations of Lindelöf theorems

For an integer $d$, $1 \leq d < n$, let $P_d : \mathbb{R}^n \rightarrow \mathbb{R}^d$ be the projection, that is,

$$P_d(x) = (x_1, \ldots, x_d, 0, \ldots, 0) \quad \text{for} \quad x = (x_1, x_2, \ldots, x_n).$$

We say that $\Gamma \subset D$ is a $(\lambda_1, \lambda_2, d)$-approach set at $\xi$, where $\lambda_1 \geq 1$ and $\lambda_2 > 0$, if there exists a sequence of positive numbers $\{r_j\}$ tending to zero such that $r_{j+1} < r_j < \lambda_1 r_{j+1}$ and

$$\mathcal{H}^d(P_d(\Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1})))) \geq \lambda_2 r_j^d.$$

Theorem 3. Let $u$ be a function on $D$ with $g$ satisfying (4) and

$$\int_D g(z)^p d\mu(z) < \infty.$$

Suppose $p > s - d$, and define

$$E = \left\{ \xi \in \partial D : \limsup_{r \to 0} (r^{-p} \mu(B(\xi, r)))^{-1} \int_{B(\xi, r) \cap D} g(z)^p d\mu(z) > 0 \right\}.$$

If $\xi \in \partial D \setminus E$ and there exists a $(\lambda_1, \lambda_2, d)$-approach set $\Gamma \subset D$ at $\xi$ along which $u$ has a finite limit $\beta$ at $\xi$, then $u$ has a nontangential limit $\beta$ at $\xi$.

PROOF. By our assumption, we can take $\delta > 0$ such that $s - p < \delta < d$. Set

$$G_j = P_d(\Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1}))).$$

For $X \in G_j$, take $C(X) \in \Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1}))$, and set $r(X) = r = |\xi - C(X)|$.

Let $C_1(X) = \xi + (0, \ldots, 0, r)$ and $D(X) = P_{n-1}(C(X))$.

We take a finite chain of balls $B_1, B_2, \ldots, B_N$ with the following properties:

(i) $B_j = B(z_j, \rho_D(z_j)/(2\sigma))$ with $z_j \in C(X)C_1(X)$, $z_1 = C(X)$ and $z_N = C_1(X)$;
(ii) $\rho_D(z_j) \leq \rho_D(z_{j+1})$ and $z_{j+1} \notin B_j$;
(iii) $B_j \cap B_{j+1} \neq \emptyset$ for each $j$;
(iv) $|D(X) - z| \leq 3\rho_D(z)$ for $z \in A(\xi, r) = \bigcup_{j=1}^N \sigma B_j \subset B(\xi, 2r) \cap D$;
(v) $\sum_j \chi_{\sigma B_j} \leq c_3$.

Since $\delta > s - p$, we have as in the proof of Theorem 2

$$|u(C_1(X)) - u(C(X))|^p \leq M r^\delta (r^{-p} \mu(B(\xi, r)))^{-1} \int_{B(\xi, 2r) \cap D} g(z)^p |D(X) - z|^{-\delta} d\mu(z).$$
Further, since $P_d$ is 1-Lipschitz and $0 < \delta < d$, we see that

\[
\int_{G_j} |D(X) - z|^{-\delta} d\mathcal{H}^d(X) \leq \int_{G_j} |X - P_d(z)|^{-\delta} d\mathcal{H}^d(X) \\
\leq \int_{P_d(B(\xi, r_j))} |X - P_d(z)|^{-\delta} d\mathcal{H}^d(X) \\
\leq Mr_j^{d-\delta}.
\]

Hence we have

\[
\int_{G_j} |u(C_1(X)) - u(C(X))|^p d\mathcal{H}^d(X) \leq M(r_j^{-p} \mu(B(\xi, r_j))^{-1} \int_{B(\xi, 2r_j) \cap D} g(z)^p d\mu(z).
\]

Thus we can find a sequence $\{X_j\}$ such that $X_j \in G_j$ and

\[
\lim_{j \to \infty} |u(C_1(X_j)) - u(C(X_j))| = 0.
\]

Thus we see that $u(C_1(X_j))$ has a finite limit $\beta$ as $j \to \infty$. Since $\{C_1(X_j)\}$ is regular at $\xi$, we can show that $u$ has a nontangential limit $\beta$ at $\xi$ by Lemma 1.

**Corollary 3.** Let $u$ be a harmonic function on $\Omega$ satisfying

\[
\int_{\Omega \cap B(0, N)} |\nabla u(z)|^p z_n^\alpha dz < \infty
\]

for every $N > 0$, and $-1 < \alpha < p - n + d$. If $\xi \in \partial \Omega \setminus E_{n+\alpha-p}$ and there exists a $(\lambda_1, \lambda_2, d)$-approach set $\Gamma \subset \Omega$ at $\xi$ along which $u$ has a finite limit $\beta$ at $\xi$, then $u$ has a nontangential limit $\beta$ at $\xi$.

**Remark 5.** The conclusion of Corollary 3 is still valid for $\mathcal{A}$-harmonic functions and polyharmonic functions.

**References**


Department of Mathematics  
Faculty of Science  
Hiroshima University  
Higashi-Hiroshima 739-8526, Japan  
E-mail address: toshi@mis.hiroshima-u.ac.jp  
and  
The Division of Mathematical and Information Sciences  
Faculty of Integrated Arts and Sciences  
Hiroshima University  
Higashi-Hiroshima 739-8521, Japan  
E-mail address: mizuta@mis.hiroshima-u.ac.jp  
and  
Department of Mathematics  
Faculty of Education  
Hiroshima University  
Higashi-Hiroshima 739-8524, Japan  
E-mail address: tshimo@hiroshima-u.ac.jp