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Jørgensen groups of parabolic type I (Finite type)

静岡大学理工学研究科 李 長軍 (Changjun Li) Shizuoka University
静岡大学理工学研究科 大市 牧人 (Makito Oichi) Shizuoka University
静岡大学理学部 佐藤 宏樹 (Hiroki Sato) Shizuoka University *

ABSTRACT. In this paper we will state extreme discrete groups (Jørgensen groups) of parabolic type - finite type - for Jørgensen's inequality. There are exactly 16 Jørgensen groups of such type.

1. Introduction.

1.1. It is one of the most important problem in the theory of Kleinian groups to decide whether or not a subgroup G of the Möbius transformation group is discrete. For this problem there are two important and useful theorems:

One is Poincaré's polyhedron theorem, which is a sufficient condition for G to be discrete. The other is Jørgensen's inequality, which is a necessary condition for a two-generator Möbius transformation group $G = \langle A, B\rangle$ to be discrete.

1.2. Let Möb denote the set of all linear fractional transformations (Möbius transformations)

$$A(z) = \frac{az+b}{cz+d}$$

of the extended complex plane $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, where a, b, c, d are complex numbers and the determinant $ad - bc = 1$. There is an isomorphism between Möb and $PSL(2, \mathbf{C})$. Throughout this paper we will always write elements of Möb as matrices with determinant 1. We recall that Möb $(= PSL(2, \mathbf{C}))$ acts on the upper half space H^3 of $\mathbf{R}^3$ as the group of conformal isometries of hyperbolic 3−space.

In this paper we use a Kleinian group in the same meaning as a discrete group. Namely, a Kleinian group is a discrete subgroup of Möb. A subgroup G of Möb is said to be *elementary* if there exists a finite G−orbit in $\mathbf{R}^3$.

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1.3. The *trace* $\mathrm{tr}(A)$ of the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (ad - bc = 1)$$

in $SL(2, \mathbf{C})$ is defined by $\mathrm{tr}(A) = a + d$. We remark that the trace of an element of Möb ($= PSL(2, \mathbf{C})$) is not well-defined, but Jørgensen number (defined later) is still well-defined after choosing matrix representatives.

1.4. In 1976 Jørgensen obtained the following important theorem called *Jørgensen's inequality*, which gives a necessary condition for a non-elementary Möbius transformtion group $G = \langle A, B\rangle$ to be discrete.

THEOREM A (Jørgensen [**1**]). *Suppose that the Möbius transformations A and B generate a non-elementary discrete group. Then*

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 *is best possible.*

1.5.

DEFINITION 1. Let A and B be Möbius transformations. The *Jørgensen number* $J(A, B)$ of the ordered pair (A, B) is defined as

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

DEFINITION 2. Let G be a non-elementary two-generator subgroup of Möb. The *Jørgensen number* $J(G)$ for G is defined as

$J(G) := \inf\{J(A, B) \mid A \text{ and } B \text{ generate } G\}$.

DEFINITION 3. A subgroup G of Möb is called a *Jørgensen group* if G satisfies the following four conditions:
(1) G is a two-generator group.
(2) G is a discrete group.
(3) G is a non-elementary group.
(4) There exist generators A and B of G such that $J(A, B) = 1$.

REMARK The fourth condition in Definition 3 is equivalent to the following condition: There exist generators of A and B of G such that $J(G) = J(A, B) = 1$.

That is, G is a Jørgensen group if and only if
(1) G is a two-generator group.
(2) G is a discrete group.
(3) G is a non-elementary group.
(4) $J(G) = 1$.

1.6. Jørgensen and Kiikka showed the following.

THEOREM B (Jørgensen-Kiikka [2]). *Let $\langle A, B\rangle$ be a non-elementary discrete group with $J(A,B) = 1$. Then A is elliptic of order at least seven or A is parabolic.*

If $\langle A, B\rangle$ is a Jørgensen group such that A is parabolic, then we call it a *Jørgensen group of parabolic type.* There are infinite number of Jørgensen groups (see Jørgensen-Lascurain-Pignataro [3], Sato [8]).

The following familiar groups are all Jørgensen groups of parabolic type:
(1) The modular group.
(2) The Picared group (Jørgensen-Lascurain-Pignataro [3], Sato [9], Sato-Yamada [11]).
(3) The figure-eight knot group (Sato [8]).
(4) "The Gehring-Maskit group" (Sato [8]), where "the Gehring-Maskit group" is the group studied in Maskit [6].

Now it gives rise to the following problem.

PROBLEM . Find all Jørgensen groups of parabolic type.

1.7. Let $\langle A, B\rangle$ be a marked two-generator group such that A is parabolic. Then we can normalize A and B as follows:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := B_{\sigma,\mu} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix},$$

where $\sigma \in \mathbf{C} \setminus \{0\}$ and $\mu \in \mathbf{C}$.

We denote by $G_{\sigma,\mu}$ the marked group generated by A and $B_{\sigma,\mu}$: $G_{\sigma,\mu} = \langle A, B_{\sigma,\mu}\rangle$. We say that $(\sigma,\mu) \in (\mathbf{C}\setminus\{0\}) \times \mathbf{C}$ *is the point representing a marked group $G_{\sigma,\mu}$ and that $G_{\sigma,\mu}$ is the marked group corresponding to a point (σ,μ).*

1.8. In the previous paper [8], we considered the case of $\mu = ik$ $(k \in \mathbf{R})$. Namely, we considered marked two-generator group $G_{\sigma,ik} = \langle A, B_{\sigma,ik}\rangle$ generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\sigma,ik} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix},$$

where $\sigma \in \mathbf{C} \setminus \{0\}$ and $k \in \mathbf{R}$.

Now we have the following conjecture.

CONJECTURE. For any Jørgensen group G of parabolic type there exists a marked group $G_{\sigma,ik}$ $(\sigma \in \mathbf{C}\setminus\{0\}, k \in \mathbf{R})$ such that $G_{\sigma,ik}$ is conjugate to G.

If this conjecture is true, then we only consider the case of $\mu = ik$ in order to find all Jørgensen groups of parabolic type.

1.9. Let C be the following cylinder:

$C = \{(\sigma, ik) \mid |\sigma| = 1,\ k \in \mathbf{R}\}$.

THEOREM C (Sato [8]). (i) *If a marked two-generator group* $G_{\sigma,\mu}$ $(\sigma \in \mathbf{C} \setminus \{0\}, \mu \in \mathbf{C})$ *is a Jørgensen group, then* $|\sigma| = 1$.

(ii) *If a marked two-generator group* $G_{\sigma,ik}$ $(\sigma \in \mathbf{C} \setminus \{0\}, k \in \mathbf{R})$ is a Jørgensen group, then the point (σ, ik) *representing* $G_{\sigma,ik}$ *lies on the cylinder* C.

If we set $\sigma = -ire^{i\theta}$, which is used in the previous paper [8], then we can represent the familiar Jørgensen groups stated before by using the $(-ire^{i\theta}, ik)-$ coordinate as follows.

THEOREM D (Jørgensen-Lascurain-Pignataro [3], Sato [8, 9], Sato-Yamada [11]).
(1) *The modular group corresponds to* $(-ie^{\pi i/2}, 0)$.
(2) *The Picard group corresponds to* $(-ie^{\pi i/2}, i/2)$.
(3) *The figure-eight knot group corresponds to* $(-ie^{\pi i/6}, i\sqrt{3}/2)$.
(4) *The "Gehring - Maskit group" corresponds to* $(-i, i\sqrt{3}/2)$.

REMARK (Sato [10] The Whitehead link corresponds to $(\sqrt{2}e^{3\pi i/4}, -i/2)$. Therefore the Whitehead link is not a Jørgensen group.

Now it gives rise to the following problem.

PROBLEM 1. Find all Jørgensen groups of parabolic type.

PROBLEM 2. Find all Jørgensen groups of parabolic type (σ, ik).

We devide Jørgensen groups of this type into two parts as follows:

Part 1. $|k| \leq \sqrt{3}/2 \quad 0 \leq \theta \leq 2\pi$.

Part 2. $\sqrt{3}/2 < |k| \quad 0 \leq \theta \leq 2\pi$.

We call Jørgensen groups in Part 1 of *finite type*. In this paper we will state that we found all Jørgensen groups of finite type.

§2. Theorems

In this section we will state theorems. We can prove the theorems by using Poincarë,s polyhedron theorem (cf. Maskit [5]) and Jøregensen's inequality. The proofs will appear elsewhere.

MAIN THEOREM (Li - Oichi - Sato [4]). (1) T*here are 16 Jørgensen groups on the region* $\{(\theta, k) \mid 0 \leq \theta \leq \pi/2,\ 0 \leq k \leq \sqrt{3}/2\}$.

(2) *9 groups of them are Kleinian groups of the first kind and 7 groups are of the second kind. Where* G : *is a Kleinian group of the first kind if the hyperbolic volume* $V(\mathbf{H}^3/G)$ *of the* $\mathbf{H}^3/G$ *is finite, and otherwise* G *is of the second kind.*

Let A and $B_{ik,\theta}$ ($k \in \mathbf{R}$, $0 \leq \theta < \pi/2$) be the following matrices :

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\theta,ik} = \begin{pmatrix} ke^{i\theta} & ik^2e^{i\theta} - ie^{-i\theta} \\ -ie^{i\theta} & ke^{i\theta} \end{pmatrix}.$$

THEOREM 1. *Let $G_{\theta,ik} = \langle A, B_{\theta,ik}\rangle$ be the group generated by A and $B_{\theta,ik}$. If $0 < \theta < \pi/6$, $\pi/6 < \theta < \pi/4$, $\pi/4 < \theta < \pi/3$ or $\pi/3 < \theta < \pi/2$, then $G_{\theta,ik} = \langle A, B_{\theta,ik}\rangle$ is not a Kleinian group for every $k \in \mathbf{R}$ and so not a Jørgensen group for every $k \in \mathbf{R}$.*

THEOREM 2 (The case of $\theta = 0$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_k := B_{0,ik} = \begin{pmatrix} k & i(k^2-1) \\ -i & k \end{pmatrix}$$

and let $G_k = \langle A, B_k\rangle$ be the group generated by A and B_k ($k \in \mathbf{R}$). Then the following hold.

(i) *In the case of $k = 0$, G_k is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(ii) *In the case of $0 < |k| < 1/2$, G_k is not a Kleinian group and not a Jørgensen group for every k.*

(iii) *In the case of $k = 1/2$, G_k is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(iv) *In the case of $1/2 < |k| < \sqrt{2}/2$, G_k is not a Kleinian group and not a Jørgensen group for every k.*

(v) *In the case of $k = \sqrt{2}/2$, G_k is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(vi) *In the case of $\sqrt{2}/2 < |k| < (1+\sqrt{5})/4$, G_k is not a Kleinian group and not a Jørgensen group for every k.*

(vii) *In the case of $k = (1+\sqrt{5})/4$, G_k is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(viii) *In the case of $(1+\sqrt{5})/4 < |k| < \sqrt{3}/2$, G_k is not a Kleinian group and not a Jørgensen group for every k.*

(ix) *In the case of $k = \sqrt{3}/2$, G_k is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_k)/G_k$ is a Riemann surface with signature* $(0; 2, 3, \infty)$.

THEOREM 3 (The case of $\theta = \pi/6$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$B_k := B_{\pi/6,ik} = \begin{pmatrix} ke^{\pi i/6} & i(k^2e^{\pi i/6} - e^{-\pi i/6}) \\ -ie^{\pi i/6} & ke^{\pi i/6} \end{pmatrix}$$

and let $G_k = \langle A, B_k \rangle$ *be the group generated by* A *and* B_k $(k \in \mathbf{R})$. *Then the following hold.*

(i) *In the case of* $k = 0$, G_k *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_0)$ *of* 3*-orbifold for* G_0 *is as follows*:

$$V(G_0) = 3L(\pi/3),$$

where $L(\theta)$ *is the Lobachevskiĭ function*:

$$L(\theta) = -\int_0^\theta \log|2\sin u|du.$$

(ii) *In the case of* $0 < |k| < \sqrt{3}/2$, G_k *is not a Kleinian group and not a Jørgensen group for every* k.

(iii) *In the case of* $k = \sqrt{3}/2$, *then* G_k *is a Kleinian group of the first kind and a Jørgensen group (the figure-eight knot group). The volume* $V(G_{\sqrt{3}/2})$ *of* 3*-orbifold for* $G_{\sqrt{3}/2}$ *is as follows*:

$$V(G_{\sqrt{3}/2}) = 6L(\pi/3).$$

REMARK. Oichi [7] gave an alternative presentation for the figure-eight knot group.

THEOREM 4 (The case of $\theta = \pi/4$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$B_k := B_{\pi/4,ik} = \begin{pmatrix} ke^{\pi i/4} & i(k^2e^{\pi i/4} - e^{-\pi i/4}) \\ -ie^{\pi i/4} & ke^{\pi i/4} \end{pmatrix}$$

and let $G_k = \langle A, B_k \rangle$ *be the group generated by* A *and* B_k $(k \in \mathbf{R})$. *Then the following hold.*

(i) *In the case of* $k = 0$, G_k *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(ii) *In the case of* $0 < |k| < 1/2$, G_k *is not a Kleinian group and not a Jørgensen group for every* k.

(iii) *In the case of* $k = 1/2$, G_k *is a Kleinian group of the first kind and a Jørgensen group.*

(iv) *In the case of* $1/2 < |k| \leq \sqrt{3}/2$, G_k *is not a Kleinian group and not a Jørgensen group for every* k. *The volume* $V(G_{1/2})$ *of* 3*-orbifold for* $G_{1/2}$ *is as follows:*

$$V(G_{\sqrt{2}/2}) = 1/2\{2L(\pi/4) - L(5\pi/12) - L(\pi/12)\}.$$

THEOREM 5 (The case of $\theta = \pi/3$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$B_k := B_{\pi/3,ik} = \begin{pmatrix} ke^{\pi i/3} & i(k^2e^{\pi i/3} - e^{-\pi i/3}) \\ -ie^{\pi i/3} & ke^{\pi i/3} \end{pmatrix}$$

and let $G_k = \langle A, B_k \rangle$ *be the group generated by* A *and* B_k $(k \in \mathbf{R})$. *Then the following hold.*

(i) *In the case of* $k = 0$, G_k *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_0)$ *of* 3*-orbifold for* G_0 *is as follows:*

$$V(G_{\sqrt{2}/2}) = 3L(\pi/3).$$

(ii) *In the case of* $0 < |k| < \sqrt{3}/2$, G_k *is not a Kleinian group and not a Jørgensen group for every* k.

(iii) *In the case of* $k = \sqrt{3}/2$, G_k *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_{\sqrt{2}/2})$ *of* 3*-orbifold for* $G_{\sqrt{3}/2}$ *is as follows:*

$$V(G_{\sqrt{3}/2}) = 3L(\pi/3).$$

THEOREM 6 (The case of $\theta = \pi/2$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$B_k := B_{\pi/2,ik} = \begin{pmatrix} ik & -(k^2+1) \\ 1 & ik \end{pmatrix}$$

and let $G_k = \langle A, B_k \rangle$ *be the group generated by* A *and* B_k $(k \in \mathbf{R})$. *Then the following hold.*

(i) *In the case of* $k=0$, G_k *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_k)/G_k$ *is a union of two Riemann surfaces with signature* $(0;2,3,\infty)$ *(The modular group).*

(ii) *In the case of* $0<k<1/2$, G_k *is not a Kleinian group and not a Jørgensen group for every* k.

(iii) *In the case of* $k=1/2$, G_k *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_{1/2})$ *of* 3*-orbifold for* $G_{1/2}$ *is as follows*:

$$V(G_{1/2}) = 7L(\pi/3)/2 - L(\varphi_0+\pi/6) + L(\varphi_0-\pi/6),$$

where φ_0 is φ satisfying $\tan\theta = 2\sin\varphi$.

(iv) *In the case of* $1/2<k<\sqrt{2}/2$, G_k *is not a Kleinian group and not a Jørgensen group for every* k.

(v) *In the case of* $k=\sqrt{2}/2$, G_k *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_{\sqrt{2}/2})$ *of* 3*-orbifold for* $G_{\sqrt{2}/2}$ *is as follows*:

$$V(G_{\sqrt{2}/2}) = 2\{2L(\pi/4) - L(5\pi/12) - L(\pi/12)\}.$$

(vi) *In the case of* $\sqrt{2}/2<k<(1+\sqrt{5})/4$, G_k *is not a Kleinian group and not a Jørgensen group for every* k.

(vii) *In the case of* $k=(1+\sqrt{5})/4$, G_k *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_{(1+\sqrt{5})/4})$ *of* 3*-orbifold for* $G_{(1+\sqrt{5})/4}$ *is as follows*:

$$V(G_{(1+\sqrt{5})/4}) = 2L(\pi/10)+2L(2\pi/5)-L(4\pi/15)-L(\varphi_0+2\pi/5)+L(\pi/15)+L(\varphi_0-2\pi/5),$$

where φ_0 is φ satisfying $\tan\theta = 2\sin\varphi$.

(viii) *In the case of* $(1+\sqrt{5})/4<k<\sqrt{3}/2$, G_k *is not a Kleinian group and not a Jørgensen group for every* k.

(ix) *In the case of* $k=\sqrt{3}/2$, G_k *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_{\sqrt{3}/2})$ *of* 3*-orbifold for* $G_{\sqrt{3}/2}$ *is as follows*:

$$V(G_{\sqrt{3}/2}) = 5L(\pi/3).$$

Next we consider the case where k is fixed and θ moves, namely we consider discreteness of $G_{ik,\theta}$ on horizontal lines.

Let A and $B_{ik,\theta}$ ($k\in\mathbf{R}$, $0\le\theta\le\pi/2$) be the following matrices :

$$A=\begin{pmatrix}1&1\\0&1\end{pmatrix} \quad\text{and}\quad B_{\theta,ik}=\begin{pmatrix}ke^{i\theta} & ik^2e^{i\theta}-ie^{-i\theta}\\ -ie^{i\theta} & ke^{i\theta}\end{pmatrix}.$$

THEOREM 7. *Let* $G_{\theta,ik} = \langle A, B_{\theta,ik}\rangle$ *be the group generated by* A *and* $B_{\theta,ik}$. *If* $0 < k < 1/2$, $1/2 < k < \sqrt{2}/2$, $\sqrt{2}/2 < k < (1+\sqrt{5})/4$ or $(1+\sqrt{5})/4 < k < \sqrt{3}/2$, *then* $G_{\theta,ik} = \langle A, B_{\theta,ik}\rangle$ *is not a Kleinian group and so not a Jørgensen group for every* θ $(0 \le \theta \le \pi/2)$.

THEOREM 8 (The case of $k = 0$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$B_\theta := B_{\theta,0} = \begin{pmatrix} 0 & -ie^{-i\theta} \\ -ie^{i\theta} & 0 \end{pmatrix}$$

and let $G_\theta = \langle A, B_\theta\rangle$ *be the group generated by* A *and* B_θ $(0 \le \theta \le \pi/2)$. *Then the following hold.*

(i) *In the case of* $\theta = 0$, G_θ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_\theta)/G_\theta$ *is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(ii) *In the case of* $0 < \theta < \pi/6$, G_θ *is not a Kleinian group and not a Jørgensen group for every* θ.

(iii) *In the case of* $\theta = \pi/6$, G_θ *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_{\pi/6})$ *of* 3*-orbifold for* $G_{\pi/6}$ *is as follows*:

$$V(G_{\pi/6}) = 3L(\pi/3).$$

(iv) *In the case of* $\pi/6 < \theta < \pi/4$, G_θ *is not a Kleinian group and not a Jørgensen group for every* θ.

(v) *In the case of* $\theta = \pi/4$, G_θ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_\theta)/G_\theta$ *is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(vi) *In the case of* $\pi/4 < \theta < \pi/3$, G_θ *is not a Kleinian group and not a Jørgensen group for every* θ.

(vii) *In the case of* $\theta = \pi/3$, G_θ *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_{\pi/3})$ *of* 3*-orbifold for* $G_{\pi/3}$ *is as follows*:

$$V(G_{\pi/3}) = 3L(\pi/3).$$

(viii) *In the case of* $\pi/3 < \theta < \pi/2$, G_θ *is not a Kleinian group and not a Jørgensen group for every* θ.

(ix) *In the case of* $\theta = \pi/2$, G_θ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_\theta)/G_\theta$ *is a union of two Riemann surfaces with signature* $(0; 2, 3, \infty)$. (*The modular group*).

THEOREM 9 (The case of $k = 1/2$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$B_\theta := B_{i/2,\theta} = \begin{pmatrix} e^{i\theta}/2 & i(e^{i\theta}/4 - e^{-i\theta}) \\ -ie^{\theta} & e^{i\theta}/2 \end{pmatrix}$$

and let $G_\theta = \langle A, B_\theta \rangle$ *be the group generated by* A *and* B_θ $(0 \le \theta \le \pi/2)$. *Then the following hold.*

(i) *In the case of* $\theta = 0$, G_θ *is a Kleinian group of the second kind, a Jørgensen group and* $\Omega(G_\theta)/G_\theta$ *is a Riemann surfaces with signature* $(0; 2, 3, \infty)$.

(ii) *In the case of* $0 < \theta < \pi/4$, G_θ *is not a Kleinian group and not a Jørgensen group for every* θ.

(iii) *In the case of* $\theta = \pi/4$, G_θ *is a Kleinian group of the first kind and a Jørgensen group. The volume* $V(G_{\pi/4})$ *of* 3*-orbifold for* $G_{\pi/4}$ *is as follows*:

$$V(G_{\pi/4}) = 1/2\{2L(\pi/4) - L(5\pi/12) - L(\pi/12)\}.$$

(iv) *In the case of* $\pi/4 < \theta < \pi/2$, G_θ *is not a Kleinian group and not a Jørgensen group for every* θ.

(v) *In the case of* $\theta = \pi/2$, G_θ *is a Kleinian group of the first kind and a Jørgensen group (the Picard group). The volume* $V(G_{\pi/2})$ *of* 3*-orbifold for* $G_{\pi/2}$ *is as follows*:

$$V(G_{\pi/2}) = 7L(\pi/3)/2 - L(\varphi_0 + \pi/6) + L(\varphi_0 - \pi/6),$$

where φ_0 is φ satisfying $\tan\theta = 2\sin\varphi$.

THEOREM 10 (The case of $k = \sqrt{2}/2$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$B_\theta := B_{\theta,i\sqrt{2}/2} = \begin{pmatrix} \sqrt{2}e^{i\theta}/2 & i(e^{i\theta}/2 - e^{-i\theta}) \\ -ie^{\theta} & \sqrt{2}e^{i\theta}/2 \end{pmatrix}$$

and let $G_\theta = \langle A, B_\theta \rangle$ be the group generated by A and B_θ $(0 \leq \theta \leq \pi/2)$. Then the following hold.

(i) *In the case of $\theta = 0$, G_θ is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_\theta)/G_\theta$ is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(ii) *In the case of $0 < \theta < \pi/2$, G_θ is not a Kleinian group and not a Jørgensen group for every θ.*

(iii) *In the case of $\theta = \pi/2$, G_θ is a Kleinian group of the first kind and a Jørgensen group. The volume $V(G_{\pi/2})$ of 3-orbifold for $G_{\pi/2}$ is as follows:*

$$V(G_{\pi/2}) = 2\{2L(\pi/4) - L(5\pi/12) - L(\pi/12)\}.$$

THEOREM 11 (The case of $k = (1+\sqrt{5})/4$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$B_\theta := B_{\theta,i(1+\sqrt{5})/4} = \begin{pmatrix} (1+\sqrt{5})e^{i\theta}/4 & i((3+\sqrt{5})e^{i\theta}/8 - e^{-i\theta}) \\ -ie^{\theta} & (1+\sqrt{5})e^{i\theta}/4 \end{pmatrix}$$

and let $G_\theta = \langle A, B_\theta \rangle$ be the group generated by A and B_θ $(0 \leq \theta \leq \pi/2)$. Then the following hold.

(i) *In the case of $\theta = 0$, G_θ is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_\theta)/G_\theta$ is a Riemann surface with signature* $(0; 2, 3, \infty)$.

(ii) *In the case of $0 < \theta < \pi/2$, G_θ is not a Kleinian group and not a Jørgensen group for every θ.*

(iii) *In the case of $\theta = \pi/2$, G_θ is a Kleinian group of the first kind and a Jørgensen group. The volume $V(G_{\pi/2}$ of 3-orbifold for $G_{\pi/2}$ is as follows:*

$$V(G_{\pi/2}) = 2L(\pi/10) + 2L(2\pi/5) - L(4\pi/15) - L(\varphi_0 + 2\pi/5) + L(\pi/15) + L(\varphi_0 - 2\pi/5).$$

where φ_0 is φ satisfying $\tan\theta = 2\sin\varphi$.

THEOREM 12 (The case of $k = \sqrt{3}/2$). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$B_\theta := B_{\theta, i\sqrt{3}/2} = \begin{pmatrix} \sqrt{3}e^{i\theta}/2 & i(3e^{i\theta}/4 - e^{-i\theta}) \\ -ie^{i\theta} & \sqrt{3}e^{i\theta}/2 \end{pmatrix}$$

and let $G_\theta = \langle A, B_\theta \rangle$ be the group generated by A and B_θ $(0 \le \theta \le \pi/2)$. Then the following hold.

(i) *In the case of $\theta = 0$, G_θ is a Kleinian group of the second kind, a Jørgensen group and $\Omega(G_\theta)/G_\theta$ is a Riemann surface with signature $(0; 2, 3, \infty)$.*

(ii) *In the case of $0 < \theta < \pi/6$, G_θ is not a Kleinian group and not a Jørgensen group for every θ.*

(iii) *In the case of $\theta = \pi/6$, G_θ is a Kleinian group of the first kind and a Jørgensen group (the figure-eight knot group). The volume $V(G_{\pi/6})$ of 3-orbifold for $G_{\pi/6}$ is as follows:*

$$V(G_{\pi/6}) = 6L(\pi/3).$$

(iv) *In the case of $\pi/6 < \theta < \pi/3$, G_θ is not a Kleinian group and not a Jørgensen group for every θ.*

(v) *In the case of $\theta = \pi/3$, G_θ is a Kleinian group of the first kind and a Jørgensen group. The volume $V(G_{\pi/3})$ of 3-orbifold for $G_{\pi/6}$ is as follows:*

$$V(G_{\pi/3}) = 3L(\pi/3).$$

(vi) *In the case of $\pi/3 < \theta < \pi/2$, G_θ is not a Kleinian group and not a Jørgensen group for every θ.*

(vii) *In the case of $\theta = \pi/2$, G_θ is a Kleinian group of the first kind and a Jørgensen group. The volume $V(G_{\pi/2})$ of 3-orbifold for $G_{\pi/2}$ is as follows:*

$$V(G_{\pi/2}) = 5L(\pi/3).$$

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Department of Mathematics
Faculty of Science
Shizuoka University
Ohya Shizuoka 422-8529
Japan
e-mail:r5144012@ipc.shizuoka.ac.jp
e-mail:r5544012@ipc.shizuoka.ac.jp
e-mail: smhsato@ipc.shizuoka.ac.jp