Enbeddings of derived functor modules into degenerate principal series

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§ 1. Formulation of the problem

Let $G$ be a real linear reductive Lie group and let $G_\mathbb{C}$ its complexification. We denote by $\mathfrak{g}_0$ (resp. $\mathfrak{g}$) the Lie algebra of $G$ (resp. $G_\mathbb{C}$) and denote by $\sigma$ the complex conjugation on $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. We fix a maximal compact subgroup $K$ of $G$ and denote by $\theta$ the corresponding Cartan involution. We denote by $\xi$ the complexified Lie algebra of $K$.

We fix a parabolic subgroup $P$ of $G$ with $\theta$-stable Levi part $M$. We denote by $N$ the nilradical of $P$. We denote by $\mathfrak{p}$, $\mathfrak{m}$, and $\mathfrak{n}$ the complexified Lie algebras of $P$, $M$, and $N$, respectively. We denote by $P_\mathbb{C}$, $M_\mathbb{C}$, and $N_\mathbb{C}$ the analytic subgroups in $G_\mathbb{C}$ with respect to $\mathfrak{p}$, $\mathfrak{m}$, and $\mathfrak{n}$, respectively.

For $X \in \mathfrak{m}$, we define

$$\delta(X) = \frac{1}{2} \text{tr} (\text{ad}_\mathfrak{g}(X)|_\mathfrak{n}).$$

Then, $\delta$ is a one-dimensional representation of $\mathfrak{m}$. We see that $2\delta$ lifts to a holomorphic group homomorphism $\xi_{2\delta} : M_\mathbb{C} \to \mathbb{C}^\times$. Defining $\xi_{2\delta}|_{N_\mathbb{C}}$ trivial, we may extend $\xi_{2\delta}$ to $P_\mathbb{C}$. We put $X = G_\mathbb{C}/P_\mathbb{C}$. Let $L$ be the holomorphic line bundle on $X$ corresponding to the canonical divisor. Namely, $L$ is the $G_\mathbb{C}$-homogeneous line bundle on $X$ associated to the character $\xi_{2\delta}$ on $P_\mathbb{C}$. We denote the restriction of $\xi_{2\delta}$ to $P$ by the same letter.

For a character $\eta : P \to \mathbb{C}^\times$, we consider the unnormalized parabolic induction $\text{Ind}^P_G(\eta)$. Namely, $\text{Ind}^P_G(\eta)$ is the $K$-finite part of the space of the $C^\infty$-sections of the $G$-homogeneous line bundle on $G/P$ associated to $\eta$. $\text{Ind}^P_G(\eta)$ is a Harish-Chandra $(\mathfrak{g}, K)$-module.

If $G/P$ is orientable, then the trivial $G$-representation is the unique irreducible quotient of $\text{Ind}^P_G(\xi_{2\delta})$. If $G/P$ is not orientable, there is a character $\omega$ on $P$ such that $\omega$ is trivial on the identical component of $P$ and the trivial $G$-representation is the unique irreducible quotient of $\text{Ind}^P_G(\xi_{2\delta} \otimes \omega)$.

Let $\mathcal{O}$ be an open $G$-orbit on $X$. We put the following assumption:
Assumption 1.1 There is a $\theta$-stable parabolic subalgebra $q$ of $g$ such that $q \in O$.

Under the above assumption, $q$ has a Levi decomposition $q = l + u$ such that $l$ is a $\theta$ and $\sigma$-stable Levi part. In fact $l$ is unique, since we have $l = \sigma(q) \cap q$.

For each open $G$-orbit $O$ on $X$, we put

$$A_O = H^\dim\unr(O, \mathcal{L})_{K\text{-finite}}.$$ 

Namely, in the terminology in [Vogan-Zuckerman 1984], we have $A_O = A_q = A_q(0)$.

We consider the following problem:

Problem 1.2 Is there an embedding: $A_O \hookrightarrow \mathcal{I}nd_P^G(\xi_{2\delta})$ or $A_O \hookrightarrow \mathcal{I}nd_P^G(\xi_{2\delta} \otimes \omega)$?

§ 2. Complex groups

Let $G$ be a connected real split reductive linear Lie group. Here, we consider Problem 1.2 for the complexification $G_C$ rather than $G$ itself. Embedding $G_C$ into $G_C \times G_C$ via $g \mapsto (g, \sigma(g))$, we may regard $G_C \times G_C$ as a complexification of $G_C$. Each parabolic subgroup of $G_C$ is the complexification of a parabolic subgroup of $G$. Let $P$ be a parabolic subgroup of $G$.

Then, the complexification of $P_C$ can be identified with $P_C \times P_C$ via the above embedding $G_C \hookrightarrow G_C \times G_C$. Hence, the complex generalized flag variety for $G_C$ is $X \times X$. We fix a $\theta$ and $\sigma$-stable Cartan subalgebra $\mathfrak{h}$ of $g$ such that $\mathfrak{h} \subseteq \mathfrak{p}$. We denote by $w_0$ (resp. $w_p$) the longest element of the Weyl group with respect to $(g, \mathfrak{h})$ (resp. $(m, \mathfrak{h})$).

We easily have:

Proposition 2.1. $X \times X$ has a unique $G_C$-orbit (say $O_C$). $O_C$ satisfies the Assumption 1.1 if and only if $w_0 w_p = w_p w_0$.

We consider "$\xi_{2\delta}$" for $G$. Then the character $\xi_{2\delta} \boxtimes \xi_{2\delta}$ on $P_C \times P_C$ is the "$\xi_{2\delta}$" for $G_C$. For characters $\mu$ and $\nu$ of $P_C$, we denote the restriction of $\mu \boxtimes \nu$ to $P_C$ realized as a real form of $P_C \times P_C$ as above by the same letter.

For the complex case, we have:

Theorem 2.2. ([Vogan-Zuckerman 1984])

$$A_O \cong \mathcal{I}nd_P^{G_C}(\xi_{2\delta} \boxtimes 1) \cong \mathcal{I}nd_P^{G_C}(1 \boxtimes \xi_{2\delta}).$$

Therefore, Problem 1.2 reduced to the problem of the existence of intertwining operators.

For $t \in \mathbb{C}$, we define the following generalized Verma module:

$$M_p(t\delta) = U(g) \otimes_{U(p)} \xi_{t\delta}.$$ 

The following result is well-known.

Proposition 2.3. For $t_1, t_2 \in 2\mathbb{Z}$,

$$\mathcal{I}nd_P^{G_C}(\xi_{t_1\delta} \boxtimes \xi_{t_2\delta}) \cong (M_p(-t_1\delta) \boxtimes M_p(-t_2\delta))_{K_{G_C}}^{\text{finite}}.$$

So, our Problem 1.2 is seriously related to the existence of homomorphisms between generalized Verma modules. In fact, the following result is known.
Theorem 2.4. ([Matumoto 1993])

Let \( t \) be a non-negative even integer. Then we have

\[
M_{\mathfrak{p}}(-(t + 2)\delta) \hookrightarrow M_{\mathfrak{p}}(t\delta)
\]

if and only if \( w_{0}\mathfrak{p} \) is a Duflo involution in the Weyl group for \((\mathfrak{g}, \mathfrak{h})\).

If \( w_{0}\mathfrak{p} \) is a Duflo involution, using Proposition 2.2 we have:

\[
\begin{align*}
\text{"Ind}_{\mathfrak{p}G}^{G}(1 \boxtimes 1) & \rightarrow \text{"Ind}_{\mathfrak{p}G}^{G}(1 \boxtimes \xi_{2\delta}) \\
\text{\text{"Ind}_{\mathfrak{p}G}^{G}(\xi_{2\delta} \boxtimes 1) & \rightarrow \text{"Ind}_{\mathfrak{p}G}^{G}(\xi_{2\delta} \boxtimes \xi_{2\delta}).}
\end{align*}
\]

In fact, we have:

Theorem 2.5. \( \mathcal{A}_{\mathfrak{O}_{0}} \leftrightarrow \text{"Ind}_{\mathfrak{p}G}^{G}(\xi_{2\delta} \boxtimes \xi_{2\delta}) \) if and only if \( w_{0}\mathfrak{p} \) is a Duflo involution in the Weyl group for \((\mathfrak{g}, \mathfrak{h})\).

§ 3. Type A case

As we seen in the case of complex groups, the statement in Problem 1.2 is not correct in general. However, for type A groups, we have affirmative answers.

3.1 \( \text{GL}(n, \mathbb{C}) \)

We retain the notation in §2. We fix a Borel subalgebra \( \mathfrak{b} \) such that \( \mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{p} \). We denote by \( \Pi \) the basis of the root system with respect to \((\mathfrak{g}, \mathfrak{h})\) corresponding to \( \mathfrak{b} \). We denote by \( S \) the subset of \( \Pi \) corresponding to \( \mathfrak{p} \). Assumption 1.1 holds if and only if \( S \) is compatible with the symmetry of the Dynkin diagram. For a Weyl group of the type A, each involution is a Duflo involution. Hence, we have:

Theorem 3.6. Under Assumption 1.1, we have \( \mathcal{A}_{\mathfrak{O}_{0}} \leftrightarrow \text{"Ind}_{\mathfrak{p}G}^{G}(\xi_{2\delta} \boxtimes \xi_{2\delta}) \).

3.2 \( \text{GL}(n, \mathbb{R}) \)

Speh proved any derived functor module of \( \text{GL}(n, \mathbb{R}) \) is parabolically induced from the external tensor product of some so-called Speh representations and possibly a one-dimensional representation. Using this fact, we can reduce Problem 1.2 to embedding Speh representations into degenerate principal series. More precisely, we consider \( G = \text{GL}(2n, \mathbb{R}) \) and let \( P \) be a maximal parabolic subgroup whose Levi part is isomorphic to \( \text{GL}(n, \mathbb{R}) \times \text{GL}(n, \mathbb{R}) \). Then, \( X = G_{C}/P_{C} \) contains a unique open \( G \)-orbit (say \( \mathcal{O} \)). In this setting, Assumption 1.1 holds. The fine structure of degenerate principal series for \( P \) has already been studied precisely. ([Sahi 1995], [Zhang 1995], [Howe-Lee 1999],[Barbasch-Sahi-Speh 1988] ) From their results, we have:

\[
\mathcal{A}_{\mathcal{O}} \leftrightarrow \text{"Ind}_{\mathfrak{p}G}^{G}(\xi_{2\delta}) \quad \text{if } n \text{ is odd,}
\]

\[
\mathcal{A}_{\mathcal{O}} \leftrightarrow \text{"Ind}_{\mathfrak{p}G}^{G}(\xi_{2\delta} \otimes \omega) \quad \text{if } n \text{ is even.}
\]

We can deduce an affirmative answer to Problem 1.2 from this.
3.3 \( \text{GL}(n, \mathbb{H}) \)

In this case, we also have an affirmative answer to Problem 1.2. The argument is similar to (and easier than) the case of \( \text{GL}(n, \mathbb{R}) \).

3.4 \( \text{U}(m, n) \)

Let \( G = \text{U}(m, n) \) and let \( P \) be an arbitrary parabolic subgroup of \( G \). In this case, Assumption 1.1 automatically holds. We denote by \( \mathcal{V} \) the set of open \( G \)-orbits on \( X = G/\mathcal{P} \). In fact, we have:

\[
\text{Socle} \left( \text{Ind}_P^G(\xi_{2k}) \right) = \bigoplus_{\mathcal{O} \in \mathcal{V}} \mathcal{A}_{\mathcal{O}}.
\]

References


