数理解析研究所講究録 1296 巻 2002 年 107-109

On applications of Katz' middle convolution functor

by S. Reiter (joint work with M. Dettweiler)

A unifying treatment for irreducible rigid local systems on the punctured affine line is given in Katz' book "Rigid local systems". The main tool is a middle convolution functor of the category of perverse sheaves into itself. This functor is denoted MC_χ, for χ a one-dimensional representation of $\pi_1(\mathbb{G}_m)$. It preserves important properties of local systems like rigidity (more general the index of rigidity), irreducibility, etc., but in general, MC_χ changes the rank and the monodromy group.

The construction of MC_χ depends on the l-adic Fourier transform in characteristic $p > 0$. Using Laumon's work on l-adic Fourier transform, the effect of MC_χ on the local monodromy can be explicitly determined.

As an application, Katz shows that any irreducible rigid local system on the punctured affine line can be obtained from a one-dimensional local system by applying iteratively a suitable sequence of middle convolutions MC_{χ_i} and scalar multiplications. This leads to an existence algorithm for such local systems, using $MC_\chi \circ MC_{\chi^{-1}} \approx \mathrm{Id}$: Test, whether there exists an irreducible rigid local system on the r-punctured affine line, for which the local monodromy has given Jordan canonical forms.

In a paper of Dettweiler-Reiter ([DR1]) a purely algebraic analogon of Katz' functor is given (using linear algebra and some module theory). This is a functor of the category of representations of the free group F_r on r generators into itself. It depends on a scalar $\lambda \in K^\times$ and is denoted MC_λ. After choosing a set of generators of F_r, MC_λ is nothing else then a transformation of tuples of invertible matrices

$$(A_1,\ldots,A_r) \in \mathrm{GL}_n(K)^r \mapsto MC_\lambda(A_1,\ldots,A_r) = (B_1,\ldots,B_r) \in \mathrm{GL}_m(K)^r,$$

where m and $B_1,\ldots,B_r$ can be explicitly determined. It is shown that MC_λ has analogous properties as MC_χ. This leads to a new proof of Katz' existence algorithm for rigid local systems for any field.

Since this approach gives an explicit matrix representation of the corresponding tuples of matrices (not only the the Jordan forms as in Katz' book) one also sees that the convolution functor commutes with the braid group action on tuples of matrices. As an application (using braid group criteria) one can realize series of classical groups regularly as Galois groups over $\mathbb{Q}$, e.g, the groups $SO_{2m+1}(q), PGO^+_{2m}(q), PGO^-_{4m}(q), Sp_{2m}(q)$, $m > q, q$ odd, appear regularly as Galois groups over $\mathbb{Q}$. One also reobtains most of the known results for classical groups.

Also an additive version of Katz' functor is defined. This is a functor of the category of representations of the free algebra f_r on r generators into itself. It depends on a scalar $\mu \in \mathbb{C}$ and is denoted mc_μ. Again, mc_μ is nothing else then a transformation of tuples of matrices

$$(a_1, \dots, a_r) \in (K^{n\times n})^r \mapsto mc_\mu(a_1, \dots, a_r) \in (K^{m\times m})^r.$$

Thus we have the following relation with differential equations. Any choice of elements $t_1, \dots, t_r \in \mathbb{C}$, together with a tuple of matrices $\mathbf{a} := (a_1, \dots, a_r) \in (\mathbb{C}^{n\times n})^r$, yields a Fuchsian system

$$D_{\mathbf{a}} : Y' = \sum_{i=1}^{r} \frac{a_i}{x - t_i} Y.$$

Then, mc_μ translates into a transformation of Fuchsian systems, sending $D_{\mathbf{a}}$ to $D_{mc_\mu(\mathbf{a})}$. This will be called the *convolution of Fuchsian systems.* The tuple of monodromy generators of $D_{\mathbf{a}}$ will be denoted $\mathrm{Mon}(D_{\mathbf{a}})$.

Among other things, the additive version of the Katz functor makes it (in principle) possible, to write down explicit Fuchsian equations whose local system of solutions (given by $\mathrm{Mon}(D_{\mathbf{a}})$) is a given irreducible rigid local system. These systems are exactly the irreducible Fuchsian systems which are free from accessory parameters. A different approach to this (in the case of semisimple monodromy) can be found in a paper by Yokoyama ([Yo]).

What is lacking in [DR1], is the exact relation between MC_λ and mc_μ, as well as an interpretation of MC_λ in cohomological terms. These questions are answered by the following theorems (s. [DR2]).

Theorem 1 *Let* $\mu \in \mathbb{C} \setminus \mathbb{Z}$, $\lambda = e^{2\pi i \mu}$ *and* $\mathbf{a} := (a_1, \dots, a_r)$, $a_i \in \mathbb{C}^{n\times n}$, *such that* $\mathrm{Mon}(D_{\mathbf{a}}) = (A_1, \dots, A_r)$ *and* $rk(a_i) = rk(A_i - 1), i = 1, \dots, r$ *and* $rk((\sum_i a_i) + \mu) = rk(\prod_i A_i)\lambda - 1))$. *Assume that* $(A_1, \dots, A_r)$ *generate a irreducible subgroup of* $GL_n(\mathbb{C})$ *and at least two elements* A_i *are* $\neq 1$, *then*

$$\mathrm{Mon}(D_{mc_\mu(\mathbf{a})}) = MC_\lambda(A_1, \dots, A_r).$$

The main ingredient of the proof is the construction of integral expressions for the solutions of $D_{mc_\mu(\mathbf{a})}$ in terms of the solutions of $D_{\mathbf{a}}$, via Euler type integrals, i.e. integrals of the form $\int_C f(x)(y-x)^\mu dx$, where C denotes a Pochhammer contour around two singularities, $\mu \in \mathbb{C}$ and $f(x)$ a vector valued holomorphic function on C. Note that Haraoka and Yokoyama ([HY]) have also obtained integral solutions for all semisimple rigid local systems.

This theorem can be used to to recognize and the construct differential equations which arise from geometry: These are differential equations which arise from Gauß-Manin connections on relative cohomology groups.

These differential equations have many remarkable properties. E.g., it is known that if the coefficients of a differential equation which arises from geometry lies in $\overline{\mathbb{Q}}$ then Grothendieck's p-curvature conjecture holds and that the solutions of such equations are so called G-functions.

Theorem 2 *The convolution of Fuchsian systems preserves the property "arising from geometry" if the used scalar* $\mu \in \mathbb{C}$ *lies in* $\mathbb{Q}$.

Using the convolution it is possible to construct explicitly a large number of differential geometric equations. E.g., one can start from any differential equation with finite monodromy, which is automatically geometric. Often one can use the convolution in order to compute the Gauss-Manin connection of curve families explicitly. This can also be applied to nonrigid examples.

On the other hand, again using the convolution, it is sometimes possible to reduce an explicitly given Fuchsian system D to a system with finite monodromy. In this case one knows that D is geometric. For systems which are free from accessory parameters, this is carried out in Katz' book, leading to the result that Fuchsian systems, which are free from accessory parameters satisfy Grothendieck's p-curvature conjecture. But in general, the existence of accessory parameters forces one to use Theorem 1, since there can be several tuples (of monodromy generators) with same local monodromy but different monodromy group.

One can also show that the convolution functor preserves global nilpotence. If the p-curvature is nilpotent for almost all primes p one calls a differential equation globally nilpotent. E.g. it is known that all differential equations coming from geometry are globally nilpotent. It is conjectured (André-Bombieri-Dwork conjecture) that the converse is true.

Literature

[DR1] Dettweiler, M. and Reiter, S., *An Algorithm of Katz and its application to the inverse Galois problem*, J. Symb. Comput. 30, No.6, 761-798 (2000).
[DR2] Dettweiler, M. and Reiter, S. *On the monodromy of Fuchsian systems*, preprint (2002).
[HY] Haraoka,Y. and Yokoyama, T., *Construction of rigid local systems and integral representations of their sections*, preprint (2002).
[Yo] Yokoyama, T., *Construction of systems of differential equations of Okubo normal form with rigid monodromy*, preprint (2002).