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Estimation in a Mixed Proportional Hazards Model

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SUMMARY

Cox's proportional hazards model (PHM) has been widely applied in the analysis of lifetime data, and can be characterized by covariates influencing lifetime of a system, where the covariates describe operating environments (e.g. temperature, pressure, humidity). When environments are uncertain, the covariates may be often modeled as random variables. We assume that a covariate is a discrete random variable, and propose a new mixture type of PHM, called the mixed PHM. We develop the Expectation-Maximization(EM) algorithm to estimate the model parameters. Two types of observations are considered; one type of observations is obtained from experimental units, which are tested in laboratories and the other type of observations is obtained from field units which are operated by customers. An illustrative example is given.

Keywords : Proportional hazards model, Mixture model, Estimation, EM algorithm, Failure data analysis.

1. Introduction

Notation

s : a random variable of a covariate

s_k : the kth element of a covariate

g : the number of elements of s

n : the number of uncategorized field units

m_i : the number of categorized experimental units whose covariates are s_i

m_{sum} : $\sum_{i=1}^{g} m_i$

x_j : the failure time of the jth uncategorized unit.

y_{ij} : the failure time of the jth unit among the categorized units whose covariates are s_i.

θ : a vector of lifetime distribution parameters.

η : a vector of a regression parameter

ϕ : (η, θ)

ψ : $(\mathbf{p}, \eta, \theta)$

An important problem in the failure data analysis is that all parts of the data have not always been collected under similar conditions. For example, we often encounter the situation where a piece of equipment may have been used in different environments or may have a different age or modification status. Such different environments will affect the equipment's inherent reliability characteristics obviously. Therefore, it may be useful to take account of the environmental factors in equipment reliability modeling. The proportional hazards model (PHM), which was proposed by Cox, has been considered as a useful tool to deal with environmental factors in the analysis of lifetime data. Solomon[17] indicated that significant effects for covariates would be obtained even in the cases where the model was not wholly appropriate, and showed that the PHM is relatively robust to departures from the proportional hazards assumption. The application of PHM to reliability data has been considered by a number of authors, for example, Ansell & Phillips[1] and Jozwiak[8]. For a list of more recent papers, see the review paper by Kumar and Klefsjö[10].

Let T be a non-negative random variable and denote the failure time of an item under consideration.
The failure nature of this item can be modeled by the hazard rate $\lambda(t)$:

$$\lambda(t) = \lim_{h \to 0} \frac{P(t \le T < t + h | T \ge t)}{h} \tag{1}$$

The assumption in the PHM, in most cases, is that the hazard rate of a system is the product of an arbitrary and unspecified baseline hazard rate $\lambda_0(t)$ depending on only time, and a positive functional term $\omega(s;\eta)$, which is basically independent of time. The function $\omega(s;\eta)$ is introduced to incorporate the effects of a number of covariates such as temperature, pressure and changes in design. Thus, the hazard rate in the PHM is given by

$$\lambda(t;s) = \omega(s;\eta)\lambda_0(t) \tag{2}$$

where s is a row vector consisting of the covariates and η is a column vector consisting of the regression parameters. The reliability function and the density function in the generic PHM are given by

$$R(t;s) = \exp\left[-\int_0^t \lambda_0(u)\omega(s;\eta)du\right] \tag{3}$$

$$f(t;s) = \lambda(t;s)R(t;s) \tag{4}$$

There are two ways to model the baseline hazard rate $\lambda_0(t)$; parametric model and non-parametric model. In the parametric model, we assume a suitable theoretical distribution for $\lambda_0(t)$. On the other hand, in the non-parametric model, no specific distribution is assumed. Note that the non-parametric method cannot always guarantee an accurate estimation because of the lack of knowledge on the lifetime distribution. In this paper, two representative lifetime distributions; the exponential and the Weibull distributions, are assumed for $\lambda_0(t)$. It is also assumed in many cases that the functional form of $\omega(s;\eta)$ is known. Various types of functional forms of $\omega(s;\eta)$ have been proposed in the past literature. Some of these are: the exponential form, $\exp(s\eta)$; the logistic form, $\log(1+\exp(s\eta))$; the inverse linear form, $1/(1+s\eta)$; and the linear form, $1+s\eta$. In this paper, we assume the exponential form which has been most widely used in applications. Covariates are associated with the equipment's environmental and operational conditions and η is the effects of the covariates.

We consider a situation where equipment's environmental and operational conditions are various. In Martorell, Sanchez & Serradell,[12], it was reported that the equipment at nuclear power plants works under very different operating conditions. In addition, very different environmental conditions appear in a nuclear power plant. That is, some components are placed in a very hard environment, for instance, under high temperature and doses of radiation, while others remain in a comfortable environment. In such a case, the covariates can be modeled as variables. Also, we cannot figure out the condition under which a product is operated before installing it. These variability and uncertainty of the covariates make us consider the covariates as random variables.

For notational and computational convenience, suppose that the number of the covariates for one unit, is only one. Define the probability mass function of the random covariate s by

$$p(s)=\begin{cases} p_1 & when \quad s=s_1 \\ \vdots & \qquad\quad \vdots \\ p_g & when \quad s=s_g \end{cases} \tag{5}$$

It is assumed in this section that the support of the random variable, s, is known.

Under these assumptions, the probability density function of the time to failure is represented in the following finite mixture form,

$$f(t)=\sum_{i=1}^{g} f(t,s_i)=\sum_{i=1}^{g} p_i f(t;s_i)=\sum_{i=1}^{g} p_i\lambda_0(t)\omega(s_i;\eta)\exp\left[-\int_0^t \lambda_0(u)\omega(s_i;\eta)du\right] \tag{6}$$

The main purpose of this article is to estimate lifetime distributions of the products whose failures phenomena can be modeled by the mixed PHM. We assume that data are collected from two types of observations; one type of observations is obtained from experimental units, which are tested at laboratories and the other type of observations is obtained from field units which are operated by customers. It is also assumed that the covariate of an experimental unit is known before testing and so m_i's are constant ; however, for a field unit, we don't know the value of the covariate but know the support of it's discrete probability mass function.

It represents the real-world condition that products are tested in laboratories under all possible stress levels of the real fields. For an example of air-conditioners, they might be tested under various temperatures at laboratories. The assumption that the support of the covariate is known, describes that a temperature under which a sold product is operated is one among temperatures under which products are tested at laboratories. Generally, temperatures can be controlled at laboratories, and so we can know it for each air-conditioner. However, it is very difficult to investigate the temperature for every air-conditioner failed at fields, and so we may not know the temperatures for field units.

With these two types of observations, we develop maximum likelihood techniques of model parameters; distribution parameters, mixing proportions and a regression parameter, based on the EM algorithm.

The mixed PHM is a kind of mixture model. The extensive applicability of the mixed distributions has generated many research problems. The existing results for estimating model parameters in the mixture model were classified and introduced by Titterington et al[19], Everitt & Hann[2], and McLachlan & Basford[14]. The finite mixed exponential distribution and the finite mixed Weibull distribution are good candidates to represent failure times. McClean[13] considers the fitting of mixed exponential distribution to the grouped follow-up data when the number of components is known. Lau[11] estimates hazard rate in both mixture of geometrics and mixture of exponentials model. Jiang & Kececioglu[6] and Jiang & Murthy[5] use graphical approaches and Jiang & Kececioglu[7] propose maximum likelihood estimates(MLE) from censored data for estimation of the mixed Weibull distributions. Jaisingh et al[4] considere the influence of the work environment using a Weibull & inverse Gaussian mixture. Hirose[3] deals with the power-law mixture model which extends the power law in accelerated life testing. Sy & Taylor[18] and Peng & Dear[16] involve the mixture models in PHM for estimating cure rate. They assumed no specific theoretical distribution for the baseline hazard function and use two non-parametric mixture models.

2. Estimation.

2.1 Maximum Likelihood Estimation

In this section we introduce the maximum likelihood method for estimating parameters of the mixed proportional hazards model. Not only is it appealing on intuition grounds, but it also possesses desirable statistical properties. For example, under very general conditions the estimators obtained by the method are consistent and they are asymptotically normally distributed

As mentioned before, we consider both the observations in laboratories and the observations in field. Both of them are incomplete, because the values of the covariates are missed in field units and it is impossible to estimate the mixing proportions using observations from only experimental units. Consider a sample consisting of both n independent field units and m_{sum} independent experimental units.

The observed full likelihood function for this sample is defined by

$$L(\psi)=\prod_{i=1}^{n}\sum_{j=1}^{g}p_j f(x_i;s_j,\phi)\times\prod_{i=1}^{g}\prod_{j=1}^{m_i}f(y_{ij};s_i,\phi) \tag{7}$$

The problem is to obtain the estimates $\hat{\psi}$ which maximize $L(\psi)$. However, it is not easy to find the MLEs in the traditional way of differentiating L with respect to ψ and setting it equal to zero, because the likelihood function often becomes a complex multi-modal function. An alternative way is to apply an iterative algorithm such as the EM algorithm.

The estimate of p_k can be calculated by the similar method to the generic mixture distributions. To maximize this likelihood subject to the constraint, $\sum p_k = 1$, we introduce a Lagrange multiplier and maximize

$$\log L(\psi)=\sum_{i=1}^{n}\log\left(\sum_{j=1}^{g}p_j f(x_i;\phi,s_j)\right)+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\log f(y_{ij};\phi,s_i)-\gamma\left(\sum_{i=1}^{g}p_i-1\right) \tag{8}$$

This yields

$$\frac{\partial \log L(\psi)}{\partial p_k}=\sum_{i=1}^{n}\left(f(x_i;\phi,s_k)\Big/\sum_{j=1}^{g}p_j f(x_i;\phi,s_j)\right)-\gamma=\sum_{i=1}^{n}(f(x_i;\phi,s_k)/f(x_i))-\gamma=0 \tag{9}$$

The Lagrange multiplier, γ can be founded by multiplying (9) by p_k and summing over k to give $n-\gamma=0$.
The posterior probability that the covariate for the jth field unit become s_i, is given by

$$\hat{\tau}_{ij}=\tau_i(x_j;\psi)=p_i f(x_j|s_i;\theta)\Big/\sum_{k=1}^{g}p_k f(x_j|s_k;\theta) \tag{10}$$

If we multiply (9) by p_k, we can express the MLE, $\hat{p}_k$ in the following form:

$$\hat{p}_k=\sum_{j=1}^{n}\hat{\tau}_{kj}/n \qquad for \quad k=1,\ldots,g \tag{11}$$

The above relation is used in the following EM algorithm.

2.2 EM Algorithm

The EM algorithm is a broadly applicable approach to the iterative computation of maximum likelihood estimates, useful in a variety of incomplete-data problems. The EM algorithm finds estimate by iteratively performing two steps : the expectation step (E-step) and the maximization step(M-step). In the E-step we calculate the conditional expectation of the log likelihood function for complete data. In the M-step, we search parameter values maximizing the conditional expectation. Similar to the classical mixture models, the EM algorithm can be applied to the mixed PHM by augmenting the observed data with the unobserved indicator variables which are the values of the covariates of field units. That is, in order to pose this problem as an incomplete-data one, we now introduce as the unobservable or missing data, the vector

$$z=(z_1^T,\ldots,z_n^T)^T \tag{12}$$

where z_j is a g-dimensional vector of zero-one indicator variables and where z_{ij} is one or zero according as whether the covariate for x_j is s_i or not and z_j^T is the transpose of z_j.
Then the log likelihood for the complete data is given by

$$\log L_C(\psi)=\sum_{i=1}^{g}\sum_{j=1}^{n}z_{ij}(\log p_i+\log f(x_j;\phi,s_i))+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\log f(y_{ij};\phi,s_i) \tag{13}$$

The w-th E-step requires the calculation of the expectation of the complete data log likelihood, $\log L_C(\psi)$, conditional on the observed data and the current fit $\psi^{(w-1)}$ for ψ.

$$\begin{aligned}Q(\psi,\psi^{(w-1)})&=E\{\log L_C(\psi)|X,Y;\psi^{(w-1)}\}\\&=\sum_{i=1}^{g}\sum_{j=1}^{n}E(z_{ij}|x_j;\psi^{(w-1)})(\log p_i+\log f(x_j;\phi,s_i))+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\log f(y_{ij};\phi,s_i)\end{aligned} \tag{14}$$

This step is affected here simply by replacing each indicator variable z_{ij} by its expectation conditional on x_j which is given by

$$E\left(z_{ij}\middle|x_j;\psi^{(w-1)}\right)=\tau_i\left(x_j;\psi^{(w-1)}\right) \tag{15}$$

On the w-th M-step, the intent is to choose the new value of ψ, say $\psi^{(w)}$, that maximize $Q\left(\psi,\psi^{(w-1)}\right)$ which, from the E step, is equal here to $\log L_C(\psi)$ with each z_{ij} replaced by $\tau_i\left(x_j;\psi^{(w-1)}\right)$.

2.3 An application to known functions

In this section, we apply an exponential functional form of $\omega(s;\eta)$ and the Weibull functions for the baseline hazard function because they are most general. The lifetime density function for a field unit is given by

$$f(t)=\sum_{i=1}^{g} p_i\lambda\beta t^{\beta-1}e^{s_i\eta}\exp\left(-\lambda t^{\beta}e^{s_i\eta}\right) \tag{16}$$

The likelihood function and the log likelihood function are

$$L(\psi)=\prod_{i=1}^{n}\sum_{j=1}^{g}p_je^{s_j\eta}\lambda\beta x_i^{\beta-1}\exp\left(-\lambda x_i^{\beta}e^{s_j\eta}\right)\times\prod_{i=1}^{g}\prod_{j=1}^{m}e^{s_i\eta}\lambda\beta y_{ij}^{\beta-1}\exp\left(-\lambda y_{ij}^{\beta}e^{s_i\eta}\right) \tag{17}$$

$$\begin{aligned}\log L(\psi)&=\sum_{i=1}^{n}\log\left\{\sum_{j=1}^{g}p_je^{s_j\eta}\lambda\beta x_i^{\beta-1}\exp\left(-\lambda x_i^{\beta}e^{s_j\eta}\right)\right\}\\&+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\left\{\log\beta+\log\lambda+(\beta-1)\log y_{ij}+s_i\eta-\lambda y_{ij}^{\beta}e^{s_i\eta}\right\}\end{aligned} \tag{18}$$

respectively. As mentioned in Section 2.1 and 2.2, the EM algorithm is applied for estimating the parameters. On the w-th E-step and M-step, the expectation of the complete data log likelihood conditional on the observed data and the current fit is given by

$$\begin{aligned}&Q\left(\psi,\psi^{(w-1)}\right)=E\left\{\log L_C(\psi)\middle|X,Y;\psi^{(w-1)}\right\}\\&=\sum_{i=1}^{g}\sum_{j=1}^{n}E\left(z_{ij}\middle|x_j;\psi^{(w-1)}\right)\left(\log p_i+\log\beta+\log\lambda+(\beta-1)\log x_j+s_i\eta-\lambda x_j^{\beta}e^{s_i\eta}\right)\\&+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\left(\log\beta+\log\lambda+(\beta-1)\log y_{ij}+s_i\eta-\lambda y_{ij}^{\beta}e^{s_i\eta}\right)\end{aligned} \tag{19}$$

In the E-step, we calculate $E\left(z_{ij}\middle|x_j;\psi^{(w-1)}\right)$ as

$$\hat{z}_{ij}=\frac{p_i^{(w-1)}e^{s_i\eta^{(w-1)}}\lambda^{(w-1)}\beta^{(w-1)}x_j^{\beta^{(w-1)}-1}\exp\left(-\lambda^{(w-1)}x_j^{\beta^{(w-1)}}e^{s_i\eta^{(w-1)}}\right)}{\sum_{k=1}^{g}p_k^{(w-1)}e^{s_k\eta^{(w-1)}}\lambda^{(w-1)}\beta^{(w-1)}x_j^{\beta^{(w-1)}-1}\exp\left(-\lambda^{(w-1)}x_j^{\beta^{(w-1)}}e^{s_k\eta^{(w-1)}}\right)} \tag{20}$$

In the M-step, we find the new values of ψ, say $\psi^{(w)}$, that maximize $Q\left(\psi,\psi^{(w-1)}\right)$. One nice feature of the EM algorithm is that the solution to the M-step often exists in a closed form. However, we can't obtain the closed form of ψ, in our case.

Differentiating the function Q of Equation (19) with respect to λ, β and η, and setting them equal to zero yields

$$\frac{\partial Q}{\partial \lambda}=\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{\tau}_{ij}\left(\frac{1}{\lambda}-e^{s_i\eta}x_j{}^{\beta}\right)+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\left(\frac{1}{\lambda}-e^{s_i\eta}y_{ij}{}^{\beta}\right)=0 \tag{21}$$

$$\frac{\partial Q}{\partial \beta}=\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{\tau}_{ij}\left(\frac{1}{\beta}+\log x_j-\lambda x_j{}^{\beta}e^{s_i\eta}\log x_j\right)+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\left(\frac{1}{\beta}+\log y_{ij}-\lambda y_{ij}{}^{\beta}e^{s_i\eta}\log y_{ij}\right)=0 \tag{22}$$

$$\frac{\partial Q}{\partial \eta}=\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{\tau}_{ij}\left(s_i-\lambda s_i x_j{}^{\beta}e^{s_i\eta}\right)+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\left(s_i-\lambda s_i y_{ij}{}^{\beta}e^{s_i\eta}\right)=0 \tag{23}$$

Equation (21), (22), and (23) do not give the closed forms for the values maximizing Equation (19); instead we use the following simple procedure to find them.

Step 1: Set initial values of $\lambda_{old}=\lambda^{(w-1)}$, $\beta_{old}=\beta^{(w-1)}$ and $\eta_{old}=\eta^{(w-1)}$.

Step 2 : Calculate λ_{new} from Equation (21) and replace λ_{old} with λ_{new}

$$\lambda_{new}=\left(n+m_{sum}\right)\Big/\left(\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{\tau}_{ij}x_j{}^{\beta_{old}}e^{s_i\eta_{old}}+\sum_{i=1}^{g}\sum_{j=1}^{m_i}y_{ij}{}^{\beta_{old}}e^{s_i\eta_{old}}\right) \tag{24}$$

Step 3 : Find β_{new} from Equation (22) using a line search and set $\beta_{old}=\beta_{new}$.

Step 4: Find η_{new} from Equation (23) using a line search and set $\eta_{old}=\eta_{new}$.

Step 5: If $|\phi_{new}-\phi_{old}|<\varepsilon$, terminate the procedure, otherwise go to Step 2.

Theorem 1.

For fixed $(\mathbf{p},\beta,\eta)$, the function Q of Equation (19) is concave with respect to λ and for fixed $(\mathbf{p},\lambda,\eta)$ or $(\mathbf{p},\lambda,\beta)$, the function Q is concave with respect to β or η.

Proof: The second order conditions for the parameters λ, β and η are derived as

$$\frac{\partial^2 Q}{\partial \lambda^2}=-\frac{1}{\lambda^2}(n+m)<0$$

$$\frac{\partial^2 Q}{\partial \beta^2}=\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{\tau}_{ij}\left(-\frac{1}{\beta^2}-\lambda x_j{}^{\beta}e^{s_i\eta}\left(\log x_j\right)^2\right)+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\left(-\frac{1}{\beta^2}-\lambda y_{ij}{}^{\beta}e^{s_i\eta}\left(\log y_{ij}\right)^2\right)<0$$

$$\frac{\partial^2 Q}{\partial \eta^2}=\sum_{i=1}^{g}\sum_{j=1}^{n}\hat{\tau}_{ij}\left(-\lambda s_i{}^2x_j{}^{\beta}e^{s_i\eta}\right)+\sum_{i=1}^{g}\sum_{j=1}^{m_i}\left(-\lambda s_i{}^2y_{ij}{}^{\beta}e^{s_i\eta}\right)<0$$

They are negative in λ, β and η, respectively. ■

Theorem 1 guarantees the accuracy and effectiveness of the line search techniques in step 2 and 3. It is well known that even if the M-step is numerically performed, the accuracy for the solution of the EM is not crucial.

$p_k^{(w)}$ is obtained from the relation of Equation (11), that is :

$$p_k^{(w)}=\sum_{j=1}^{n}\hat{\tau}_{kj}\big/n \quad for\ \ k=1,\ldots,g \tag{25}$$

We can also have the same result by differentiating the $Q(\psi, \psi^{(w-1)})$ with respect to $\mathbf{p}$.

Note that the maximization procedure and Equation (25) do not give the estimators explicitly; instead they must be solved using the general EM iterative procedure.

3. An Example

An example is made with the sample given by Nelson[15] to illustrate the practical application of results obtained here. The data in Nelson[15](pp.129) are the times to oil breakdown under high test voltage and they are used as an example for accelerated testing. The data under 30, 34, 38kv have been selected from the data under 26, 28, 30, 32, 34, 36, and 38kv, that is the number of groups is three. The data are considered as the experimental data in this example. Since the field data are indispensable for this model but there are no uncategorized data in the data in Nelson[15], the artificial field data are randomly selected from the experimental data with $p_1 = 0.3$, $p_2 = 0.5$ and $p_3 = 0.2$. Data for this example are given below.

Experimental data

Group	Covariate	Data										
1	30KV	7.74	17.05	20.46	21.02	22.66	43.40	47.30	139.07	144.12	175.88	194.90
2	34KV	0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	6.50	7.35
		8.01	8.27	12.06	31.75	32.52	33.91	36.71	72.89			
3	38KV	0.09	0.39	0.47	0.73	0.74	1.13	1.40	2.38			

Field data

0.47	0.96	1.13	4.15	8.01	12.06	20.46	31.75	43.4	139.07

It is intended in the example to estimate the lifetime distribution of the field units which is modeled by the mixed proportional hazards model. Before estimating the parameters, the probability plot can be roughly used to test the fitness of the model to a given set of data. The experimental data are plotted in Figure 1. The conditional probability of the lifetime given a covariate follows the Weibull distribution with a different scale parameter and a same shape parameter from it's baseline distribution because the covariate just changes the scale parameter in the case of the Weibull baseline hazard rate in this model. Therefore, data should be nearby three straight lines and the straight lines should be parallel. Figure 1 shows that these conditions are nearly satisfied in this example. Using the proposed method, we have $\hat{\lambda} = 5.89 \times 10^{-9}$, $\hat{\beta} = 0.9208$, $\hat{\eta} = 0.492$, $(\hat{p}_1, \hat{p}_2, \hat{p}_3) = (0.27, 0.53, 0.2)$. The probability density function for the example is graphed in Figure 2.

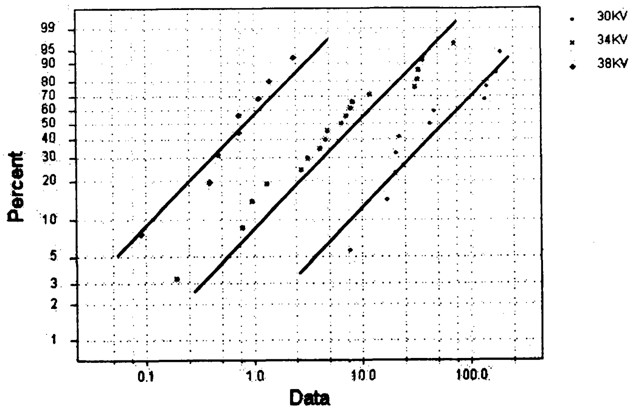

Figure 1. Weibull probability plot

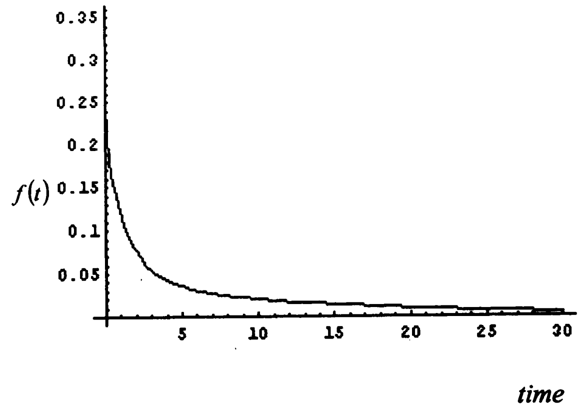

Figure 2. Probability density function

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