Minimax Theorems of Convexlike Functions

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1 Introduction

In this study we consider fuzzy numbers with bounded supports due to [3] and we treat some type of fuzzy optimization problems, which arise from linear optimization problems and are analyzed under assumptions of the fuzzy goal and fuzzy constraints of decision makers. [5] gives an existence criterion for optimal solutions of the fuzzy optimization problems. In Section 2 the existence of optimal solutions means that there exists at least one solution for systems of inequalities concerning concave functions by applying Ky Fan's theorem. In Section 3 we show an extension of Ky Fan's theorem, in which functions are not convex but quasiconvex. In proving the extension we apply fixed point theorems for set-valued mappings. In Section 4 we deal with definitions of convexlike or concavelike functions in the similar way to Chapter 6 in [7] as well as we get minimax theorems under conditions that functions of two variables are lower semi-continuous and quasiconvex in one variable and concavelike in the other.

2 Existence Criterion

Let us denote by \( \mathbb{R} \) the set of real numbers and \( I = [0, 1] \). In [3] the set of fuzzy numbers is characterized by membership functions as follows:

Definition 1 Let \( \mathcal{F}(\mathbb{R}) \) be the set of fuzzy numbers \( u : \mathbb{R} \to I \) satisfying the following conditions (i)-(iii) (see [3]):

(i) \( u(\cdot) \) is upper semi-continuous on \( \mathbb{R} \);
(ii) the \( \alpha \)-cut set \( L_{\alpha}(u) = \{y \in \mathbb{R} : u(y) \geq \alpha\} \) is bounded for \( \alpha > 0 \) and \( L_{0}(u) = \bigcup_{0<\alpha\leq1}L_{\alpha}(u) \) is bounded;
(iii) \( u(\cdot) \) is fuzzy convex, i.e.,
\[
u(\lambda y_{1}+(1-\lambda)y_{2}) \geq \min[u(y_{1}), u(y_{2})]
\]
for \( y_{i} \in \mathbb{R}, i = 1, 2 \) and \( \lambda \in \mathbb{R} \) with \( 0 \leq \lambda \leq 1 \);
(iv) there exists one and only one \( m \in \mathbb{R} \) such that \( u(m) = 1 \).

The \( \alpha \)-cut set \( L_{\alpha}(u) \) is compact in \( \mathbb{R} \) for each \( \alpha \in I \) from the above Conditions (i) and (ii), since (i) means that \( L_{\alpha}(u) \) is closed for \( \alpha \in I \).
Remark 1 Under the above Conditions (ii) and (iv) the following statements (a)-(d) concerning the function $u : \mathbb{R} \rightarrow I$ are equivalent each other:

(a) $u(\cdot)$ is fuzzy convex;
(b) $L_\alpha(u)$ is convex for any $\alpha \in I$;
(c) $u(\cdot)$ is non-decreasing on $(-\infty, m]$ and that $u(\cdot)$ is non-increasing on $[m, \infty)$;
(d) $L_\alpha(u) \subset L_\beta(u)$ for $\alpha > \beta$.

From (a) it is clear that (b) holds. If we suppose that (a) doesn't hold but (b) hold, this leads to a contradiction. It can be seen that (c) leads to (d) and the converse holds. Suppose that for any $m_1 \in \mathbb{R}$ with $m_1 > m$ there exist $y_1 < y_2 \leq m_1$ such that $u(y_1) > u(y_2)$ under Condition (ii) and (a). Then it leads to a contradiction. From (c), it follows that (a) holds.

In the following definition we give the quasiconvexity of functions.

Definition 2 Let $C$ be a convex set in a linear space and $f$ a mapping from $C$ to $\mathbb{R}$. It is said that $f$ is quasiconvex if $f(\lambda y_1 + (1 - \lambda)y_2) \geq \min\{f(y_1), f(y_2)\}$ for $y_i \in C, i = 1, 2$ and $0 \leq \lambda \leq 1$. It is said that $f$ is quasiconvex if

$$f(\lambda y_1 + (1 - \lambda)y_2) \leq \max\{f(y_1), f(y_2)\}$$

for $y_i \in C, i = 1, 2$ and $0 \leq \lambda \leq 1$.

Remark 2 In the same way as in Remark 2.1 it is easily seen that $f : C \rightarrow \mathbb{R}$ is quasiconvex if and only if the lower level set $L(f; \gamma) = \{x \in C : f(x) \leq \gamma\}$ is convex for any $\gamma \in \mathbb{R}$.

Next we consider the following linear optimization problem (e.g. [4]):

$$a_0^T x \leq b_0 \quad \text{subject to} \quad a_i^T x \leq b_i, \quad i = 1, 2, \cdots, m, \quad x \geq 0,$$

where the symbol $\leq$ denotes a relaxed or fuzzy version of the ordinary inequality $\leq$. The first fuzzy inequality (fuzzy goal) means that the objective function $a_0^T x$ should be essentially smaller than or equal to an aspiration level $b_0 \in \mathbb{R}$ of the decision maker (DM) and the second (fuzzy constraints of DM) means that the constraints $a_i^T x$ should be essentially smaller than or equal to $b_i \in \mathbb{R}, i = 1, \cdots, m$. Membership functions $u_i \in \mathcal{F}(\mathbb{R}), i = 0, 1, \cdots, m$, and it follows that $u_i(y)$ is non-decreasing in $y \in [C_i, b_i]$, non-increasing in $y \in [b_i, D_i]$ and $u_i(y) \equiv 0$ elsewhere. Here $C_i \leq b_i \leq D_i$ are constants. Let $u_i$ be concave on the set $[C_i, D_i]$. Put $S_i = \{x \in \mathbb{R}^n : C_i \leq a_i^T x \leq D_i\}$ and $S = \bigcap_{i=0}^m S_i$.

Then, in order to solve the above problem, we have the following optimization problem:

$$\maximize \ u(x),$$

where $u(x) = \min_{0 \leq i \leq m} |u_i(a_i^T x)|$.\hspace{1cm} (2.4)

In [5] we showed the existence criterion for optimal solutions of fuzzy optimization problems as follows:

Theorem 1 Let $u_i(\cdot) \in \mathcal{F}$ for $i = 0, 1, \cdots, m$.

The following statements (i) and (ii) hold:

(i) Let $\mu_0 = \max_{x} \min_{i} u_i(a_i^T x)$. Then we have

$$\mu_0 = \max \{0 < \alpha \leq 1 : \bigcap_{i=0}^m L_\alpha(u_i) \neq \emptyset\} = \sup \{0 < \alpha \leq 1 : \bigcap_{i=0}^m L_\alpha(u_i) \neq \emptyset\}.$$
Let \( \gamma_{i=0}^{n} L_{\alpha_{0}}(u_{i}) \neq \emptyset \).

The above condition (ii) can be reduced to another type of condition by applying Ky Fan's theorem in [2] as follows:

**Theorem K** Let \( C \) be a compact and convex set in a topological linear space. Suppose that functions \( f_{i} : C \rightarrow \mathbb{R}, i = 1, 2, \ldots, n \), are lower semi-continuous and convex. Let \( d \in \mathbb{R} \).

Then the following (i) and (ii) are equivalent each other:

(i) There exists an \( x_{0} \in C \) such that

\[
f_{i}(x_{0}) \leq d
\]

for \( i = 1, 2, \ldots, n \);

(ii) for \( c = (c_{1}, \ldots, c_{n}) \) such that \( c_{i} \geq 0, i = 1, 2, \ldots, n \), and \( \sum_{i=1}^{n} c_{i} = 1 \), there exists a \( y_{c} \in C \) satisfying

\[
\sum_{i=1}^{n} c_{i} f_{i}(y_{c}) \leq d.
\]

From the above theorem, Problem ((2.3),(2.4)) has an optimal solution \( x^{*} \) if and only if there exist \( 0 < \alpha_{0} \leq 1 \) and \( x_{0} \) such that

\[
u_{i}(a_{i}^{T} x_{0}) \geq \alpha_{0}
\]

for \( i = 0, 1, \ldots, m \).

**Theorem 2** Let \( S = \cap_{i=0}^{n} S_{i} \) be non-empty and \( u_{i}() \) concave on \([C_{i}, D_{i}]\) for \( i = 0, 1, \ldots, m \). Then Problem ((2.3),(2.4)) has an optimal solution \( x^{*} \), if and only if for some \( \alpha_{0} \) with \( 0 < \alpha_{0} \leq 1 \) and \( c = (c_{0}, \ldots, c_{m}) \in \mathbb{R}^{m+1} \) with \( c_{i} \geq 0, i = 0, 1, \ldots, m \), there exists a \( y_{c} \in S \) such that

\[
\sum_{i=0}^{m} c_{i} u_{i}(a_{i}^{T} y_{c}) \geq \alpha_{0}.
\]

### 3 Quasiconvex Functions

In this section we suppose the quasiconvexity of membership functions and we show an extension of Ky Fan’s theorem by applying the following lemma.

**Lemma 1** Let \( C \) be a compact and convex set in a topological linear space \( E \). Suppose that a set \( A \subset C \times C \) satisfies the following conditions (a) - (c):

(a) The set \( \{x \in C : (x, y) \in A\} \) is closed for any \( y \in C \);

(b) the set \( \{y \in C : (x, y) \notin A\} \) is convex for any \( x \in C \);

(c) for \( x \in C \), the point \( (x, x) \in A \).

Then there exists some \( x_{0} \in C \) such that \( \{x_{0}\} \times C \subset A \).

The above Lemma can be proved by applying the following type of fixed points theorem for a class of set-valued mappings (e.g., Theorem 10.3.6 in [1]).

**Theorem 3** Let \( E \) be a topological linear space and \( C \) a non-empty, compact and convex set in \( E \). Let \( T \) be a mapping from \( C \) to the set of all subsets of \( C \). Assume that the image \( T(x) \) is non-empty and convex for
each \( x \in C \). If for each \( y \in C \), the inverse \( T^{-1}(y) = \{ x \in C : T(x) \ni y \} \) is open, then \( T \) has a fixed point in \( C \), i.e., there exists an \( x_0 \in C \) such that \( x_0 \in T(x_0) \).

**Proof of Lemma 1**

Suppose that for any \( x \in C \) there exists a \( y \in C \) such that \( (x, y) \not\in A \). Denote a set-valued mapping \( T \) from \( C \) to the set of all subsets of \( C \) by \( T(x) = \{ y \in C : (x, y) \not\in A \} \). The image \( T(x) \subset C \) is non-empty and convex from Condition (b) for any \( x \in C \). From Condition (a) the set \( T^{-1}(y) = \{ x \in C : (x, y) \not\in A \} \) is open set in \( E \). Then, by applying Theorem 3, \( T \) has a fixed point \( x_0 \in C \), i.e., \( x_0 \in T(x_0) \).

It follows that \( (x_0, x_0) \not\in A \), which contradicts Condition (c). Thus the conclusion holds.

Q.E.D.

By utilizing the above lemma we think that the following results of an extension of Theorem K can be shown as the below outline of proof.

**Extension of Theorem K(ETK)**

- Let \( f_i : C \rightarrow \mathbf{R} \) for \( i = 1, \ldots, n \), be lower semi-continuous and quasiconvex, where \( C \) is a compact and convex set in a topological linear space \( E \) and let \( d \in \mathbf{R} \). Then the following (i) and (ii) are equivalent each other:
  (i) There exists an \( x_0 \in C \) such that
      \[ f_i(x_0) \leq d \]
      for \( i = 1, 2, \ldots, n \);
  (ii) for \( c = (c_1, \cdots, c_n) \) such that \( c_i \geq 0, i = 1, 2, \cdots, n \), and \( \sum_{i=1}^{n} c_i = 1 \), there exists a \( y_c \in C \) such that
      \[ \sum_{i=1}^{n} c_i f_i(y_c) \leq d. \]

In the similar way to the discussion of Chapter 6 in [7], we expect that we can prove the above extension.

**4 Extensions of Minimax Theorems**

[7] gives definitions of convexlike or concavelike functions, which play an important role in proving an extension of minimax theorems under that ETK holds.

**Definition 3** Let \( C, D \) be two sets and \( F \) a mapping from \( C \times D \) to \( \mathbf{R} \). It is said that \( F \) is concavelike on \( D \) for \( x \in C \) if for each \( y_1, y_2 \in D \) and \( 0 < \lambda < 1 \), there exists an \( y_0 \in D \) such that \( F(x, y_0) \geq \lambda F(x, y_1) + (1 - \lambda) F(x, y_2) \). It is said that \( F \) is convexlike on \( C \) for \( y \in D \) if for each \( x_1, x_2 \in C \) and \( 0 < \lambda < 1 \), there exists an \( x_0 \in C \) such that \( F(x_0, y) \leq \lambda F(x_1, y) + (1 - \lambda) F(x_2, y) \).

In what follows we show an extension of minimax theorems concerning concavelike functions.

**Extension of Minimax Theorems (EMT)**

- Let \( C \) be a convex and compact set in a topological linear space and \( D \) an arbitrary non-empty set. A function \( F : C \times D \rightarrow \mathbf{R} \) satisfies the following conditions (i) and (ii).
  (i)
(i) $F(\cdot, y)$ is lower semi-continuous quasiconvex on $C$ for $y \in D$.

(ii) $F(x, \cdot)$ is concave-like on $D$ for $x \in C$.

Then it follows that

$$\sup_{y \in D} \min_{x \in C} F(x, y) = \min_{x \in C} \sup_{y \in D} F(x, y).$$

Proof. From (i) and the compactness of $C$ there exists $\min_{x \in C} F(x, y)$. Let $c = \sup_{y \in D} \min_{x \in C} F(x, y) < +\infty$. For any $x \in C$, \{y_1, y_2, \ldots, y_n\} \subset D and \{\lambda_i \geq 0 : \sum_{i=1}^{n} \lambda_i = 1\}, Condition (ii) means that there exists a $y_0 \in D$ such that $\sum_{i=1}^{n} \lambda_i F(x, y_i) \leq F(x, y_0)$. From (i) there exists an $x_0 \in C$ such that $F(x_0, y_0) = \min_{x \in C} F(x, y_0) \leq c$ and also we have $\sum_{i=1}^{n} \lambda_i F(x, y_i) \leq c$ for any $x \in C$.

By Condition (i) and ETK, there exists an $x_1 \in C$ such that $F(x_1, y_i) \leq c$ for any $i$. Then we get $\cap_{i=1}^{n} \{x \in C : F(x, y_i) \leq c\} \neq \emptyset$. From the compactness of $C$, we have $\cap_{y \in D} \{x \in C : F(x, y) \leq c\} \neq \emptyset$, which means that there exists an $x_2 \in C$ and any $y \in D$ such that $F(x_2, y) \leq c$, or $\min_{x} \sup_{y} F(x, y) \leq \sup_{y} \min_{x} F(x, y)$. Since $F(x, y) \geq \min_{x} F(x, y)$, for $y \in D$, we have $\sup_{y} F(x, y) \geq \sup_{y} \min_{x} F(x, y)$ and also $\min_{x} \sup_{y} F(x, y) \geq \sup_{y} \min_{x} F(x, y)$. Therefore $\sup_{y} \min_{x} F(x, y) = \min_{x} \sup_{y} F(x, y)$.

If $\sup_{y \in D} \min_{x \in C} F(x, y) = \infty$, it can be seen that the conclusion holds.

Q.E.D.

The above theorem is an extension of Sion’s minimax theorem and Tuy’s one. In the following remark an example illustrates EMT.

Remark 3 (a) In [6] Sion assumes that $F$ is upper semi-continuous and quasiconcave on $D$ under the condition that $D$ is compact, in addition to the conditions of EMT. He gets the conclusion that $\min_{x} \max_{y \in D} F(x, y) = \max_{y \in D} \min_{x} F(x, y)$.

Thus EMT is an extension of Sion’s theorem.

(b) Tuy [8] assumes that $C$ and $D$ are convex. He shows that the conclusion $\inf_{x \in C} \sup_{y \in D} F(x, y) = \sup_{y \in D} \inf_{x \in C} F(x, y)$ under the condition that $F$ is upper semicontinuous in $y$ in addition to conditions of EMT.

(c) Let $F(x, y) = f(x)g(y)$ for $(x, y) \in [-n, n] \times (-1, 1)$, where $n \geq 1$ is integer, $f$ denotes the largest integer which is less than $|x|$. Here

$$g(y) = y^2 + |y \sin \frac{\pi}{2y}|,$$

where $y \in (-1, 1)$. Then function $F$ satisfies Conditions (i) and (ii) of EMT. Since $\min_{y \in [-1, 1]} F(x, y) = 0$ and $\sup_{y \in [-1, 1]} F(x, y) = 2f(x)$, it follows that the conclusion of EMT holds.

References


