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A relationship between the Riccati equation and the pivots in optimization
最適化における Riccati 方程式とピボットの関係

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Abstract The conjugate point is a global concept in the calculus of variations. In [4], the author proposed a conjugate point theory for an unconstrained nonlinear programming problem: minimize $f(x)$ on R^n. He introduced the Jacobi equation and (strict) conjugate points, and he described necessary- and sufficient optimality conditions in terms of (strict) conjugate points. In this paper, we introduce the Riccati equation for a nonlinear programming problem, and we clarify the relationship between the Jacobi equation and the Riccati equation. Furthermore, we show that the Riccati equation is nothing but the difference equation that the pivots of the Hesse matrix of $f(x)$ satisfy.

1 Introduction

The conjugate point was introduced by Jacobi to give a sufficient optimality condition for the simplest problem in the calculus of variations

$$\begin{array}{lll} (SP) & \text{Minimize} & F(x) := \int_0^T f(t, x(t), \dot{x}(t))dt \\ & \text{subject to} & x(0) = A, \ x(T) = B. \end{array}$$

It has been extended to optimal control problems and variational problems with state constraints in these two decades, see, for example, [6]-[14]. In those papers, the Jacobi equation and the Riccati equation played the central roles.

Recently, we proposed, in [4], a conjugate point theory for an elementary extremal problem

$$(P_0) \qquad \text{Minimize} \quad f(x) \quad \text{on} \quad R^n,$$

where $f : R^n \to R$is assumed to be twice continuously differentiable. In [4], we defined the Jacobi equation and (strict) conjugate points based on the insight in Gelfand and Fomin [2] by comparing (P_0) with (SP). We concluded that the recursion relation of the descending principal minors of the Hesse matrix $f''(x)$ is nothing but the Jacobi equation for (P_0), and that the conjugate point is defined as the size k of a principal submatrix $A_k := (f_{x_i x_j}(x))_{1 \le i,j \le k}$ such that

$$|A_1| > 0, \ |A_2| > 0, \ \ldots, |A_{k-1}| > 0, \ |A_k| \le 0.$$

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Furtheremore, we dealt with a constrained nonlinear programming problem in [5].

$$\begin{array}{llll} (P) & \text{Minimize} & f(x) & \\ & \text{subject to} & g_i(x) \leq 0 & i \in I := \{1, \ldots, \ell\}, \\ & & g_i(x) = 0 & i \in J := \{\ell+1, \ldots, m\}, \\ & & x \in R^n. & \end{array}$$

We showed that a projection of the Hesse matrix of the Lagrange function to the tangent space of the constraint set plays the same role with the Hesse matrix of $f(x)$ in (P_0).

On the other hand, the Riccati equation also plays an important role in the classical conjugate point theory, and it is deeply related to the Jacobi equation. However, we did not touch the Riccati equation in [4][5] at all. The aims of this paper are to define the Riccati equation for (P_0) and (P), to clarify the relationship between the Riccati equation and the Jacobi equation, and to show the algebraic meaning of the solution for the Riccati equation.

In Section 2, we briefly review the conjugate point theory presented in [4]. Section 3 is devoted to the classical Riccati equation. In Section 4, we define the Riccati equation for (P_0), and we clarify the relationship between the Riccati equation and the Jacobi equation. Furthermore, we describe a sufficient optimality conditions for (P_0) in terms of the Riccati equation. In Section 5, we clarify the algebraic meaning of the solution for the Riccati equation. Namely, we show that the Riccati equation is the difference equation that the pivots of the Hesse matrix satisfy. In Section 6, we deal with the constrained problem (P).

2 Preliminalies

In this section, we briefly review the conjugate point theory for (P_0) presented in [4].

The conjugate point theory given in [4] can be regarded as a new interpretation of the positive-definiteness of the Hesse matrix $f''(x)$. According to Sylvester's criterion, an $n \times n$-symmetric matrix $A = (a_{ij})$ is positive-definite if and only if its descending principal minors $|A_k|$ $(k = 1, \ldots, n)$ are positive, where $A_k = (a_{ij})_{1 \leq i,\ j \leq k}$.

The following lemma shows that the determinant of any matrix is expanded with respect to the descending principal minors.

LEMMA 2. 1 *([4]) For any $n \times n$-matrix $A = (a_{ij})$, its determinant is expanded as follows:*

$$|A| = \sum_{k=0}^{n-1} \sum_{\rho \in S(k+1,n)} \varepsilon(\rho) a_{k+1\rho(k+1)} a_{k+2\rho(k+2)} \cdots a_{n\rho(n)} |A_k| \tag{1}$$

where $|A_0| := 1$, $\varepsilon(\rho)$ denotes the sign of ρ, and $S(k+1, n)$ denotes the set of all permutations ρ on $\{k+1, \ldots, n\}$ satisfying that there is no $\ell > k$ such that ρ is closed on $\{\ell+1, \ldots, n\}$.

DEFINITION 2. 1 *([4]) For any $n \times n$-matrix $A = (a_{ij})$, we call the recursion relation on $y_0, \dots, y_n$*

$$y_k = \sum_{i=0}^{k-1} \sum_{\rho \in S(i+1,k)} \varepsilon(\rho) a_{i+1\rho(i+1)} a_{i+2\rho(i+2)} \cdots a_{k\rho(k)} y_i, \qquad k = 1, \dots, n \tag{2}$$

the Jacobi equation for A. We say that k is conjugate to 1 if the solution $\{y_i\}$ of the Jacobi equation with $y_0 > 0$ changes the sign from positive to non-positive at k. Namely,

$$y_0 > 0,\ y_1 > 0,\ \dots,\ y_{k-1} > 0,\ \text{and } y_k \leq 0. \tag{3}$$

Concerning the reason why we call the recursion relation (2) the Jacobi equation, readers may refer to [4, p. 57].

THEOREM 2. 1 *([4]) (1) For any $n \times n$-symmetric matrix A, $A > 0$ if and only if there is no point conjugate to 1. (2) A sufficient condition for an extremal $\bar{x}$ to be a minimum for (P_0) is that there is no point conjugate to 1 for the Hesse matrix $f''(\bar{x})$.*

3 The classical Riccati equation

In the classical conjugate point theory, although Legendre did not get to the goal, he also made important contributions to the conjugate point theory, see [2, p. 104]. In this section, we explain his idea.

For the simplest problem in the calculus of variations, he attempted to prove that a sufficient condition for $F(x)$ have a weak minimum at $\bar{x}(t)$ is the strengthened Legendre condition

$$P(t) := f_{\dot{x}\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) > 0 \qquad \forall t$$

in addition to the Euler equation

$$\frac{d}{dt} f_{\dot{x}}(t, \bar{x}(t), \dot{\bar{x}}(t)) = f_x(t, \bar{x}(t), \dot{\bar{x}}(t)) \qquad \forall t.$$

His approach was as follows. By integration by parts, the second variation

$$I(y) := \int_0^T \{P\dot{y}^2 + 2Qy\dot{y} + Ry^2\} dt \quad \text{for } y(0) = y(T) = 0$$

of $F(x)$ at $\bar{x}$ is expressed as

$$\int_0^T \{P\dot{y}^2 + (R - \dot{Q})y^2\} dt.$$

Since $y(0) = y(T) = 0$, it is written in the form

$$\begin{aligned} I(y) &= \int_0^T \{P\dot{y}^2 + (R - \dot{Q})y^2 + \frac{d}{dt}(wy^2)\} dt \\ &= \int_0^T \{P\dot{y}^2 + 2wy\dot{y} + (R - \dot{Q} + \dot{w})y^2\} dt, \end{aligned} \tag{4}$$

where $w(t)$ is any arbitrary piecewise smooth function. Next, he observed that the strengthened Legendre condition would be sufficient for $I(y) \geq 0$ if it were possible to find a function $w(t)$ for which the integrand in (4) is a perfect square. However, this is not always possible, since $w(t)$ would have to satisfy the Riccati equation

$$P(R - \dot{Q} + \dot{w}) = w^2, \tag{5}$$

and although the Riccati equation may not have a solution on the whole interval $[0, T]$. On the other hand, by the change of variable

$$w = -\frac{\dot{y}}{y} P, \tag{6}$$

the Riccati equation is transformed into the Jacobi equation

$$\frac{d}{dt}\{P\dot{y} + Qy\} = Q\dot{y} + Ry.$$

The change of variables (6) implies that the Riccati equation has a solution $w(t)$ except zero points of a non-trivial solution $y(t)$ of the Jacobi equation. Here we remark that $y(t)$ changes its sign at any zero point of $y(t)$.

4 The Riccati equation

In this section, we define the Riccati equation for (P_0), and we clarify the relationship between the Riccati equation and the Jacobi equation. As we saw in Section 3, the key point to derive the classical Riccati equation was the perfect square. This idea works very well when we define the Riccati equation for (P_0), too.

Let us first consider the case where A is a tridiagonal matrix

$$A = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix}. \tag{7}$$

Suppose that the quadratic form $x^T Ax$ ($x \in R^n$) is expressed as a summation of n perfect squares

$$\begin{aligned} x^T Ax &= a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2 + 2b_1 x_1 x_2 + 2b_2 x_2 x_3 + \cdots + 2b_{n-1} x_{n-1} x_n \\ &= \left(w_1 x_1 + \frac{b_1}{w_1} x_2\right)^2 + \cdots + \left(w_{n-1} x_{n-1} + \frac{b_{n-1}}{w_{n-1}} x_n\right)^2 + w_n^2 x_n^2 \end{aligned}$$

for some $w_1, \ldots, w_n \in R/\{0\}$. Then $\{w_k\}$ has to satisfy

$$w_1^2 = a_1, \quad w_k^2 = a_k - \frac{b_{k-1}^2}{w_{k-1}^2} \quad (k = 2, \ldots n). \tag{8}$$

On the other hand, the Jacobi equation (2) for the tridiagonal matrix reduces to

$$y_k = a_k y_{k-1} - b_{k-1}^2 y_{k-2}. \tag{9}$$

Dividing (9) by y_{k-1}, we get

$$\frac{y_k}{y_{k-1}} = a_k - b_{k-1}^2 \frac{y_{k-2}}{y_{k-1}}. \tag{10}$$

Comparing (10) with (8), we obtain a correspondence

$$w_k^2 = \frac{y_k}{y_{k-1}}. \tag{11}$$

Then, there is no nonzero $w_k \in R$ if $y_{k-1}y_k \leq 0$. This fact matches the classical result mentioned at the end of the preceding section.

Next, we deal with the general matrix A. Dividing the Jacobi equation (2) by y_{k-1}, we get

$$\begin{aligned}
\frac{y_k}{y_{k-1}} &= \sum_{i=0}^{k-1} \sum_{\rho \in S(i+1,k)} \varepsilon(\rho) a_{i+1\rho(i+1)} a_{i+2\rho(i+2)} \cdots a_{k\rho(k)} \frac{y_i}{y_{k-1}} \\
&= a_{kk} + \sum_{i=0}^{k-2} \sum_{\rho \in S(i+1,k)} \varepsilon(\rho) a_{i+1\rho(i+1)} \cdots a_{k\rho(k)} \frac{y_i}{y_{i+1}} \frac{y_{i+1}}{y_{i+2}} \cdots \frac{y_{k-2}}{y_{k-1}} \\
&= a_{kk} + \sum_{i=0}^{k-2} \sum_{\rho \in S(i+1,k)} \varepsilon(\rho) a_{i+1\rho(i+1)} \cdots a_{k\rho(k)} \frac{1}{w_{i+1}^2} \frac{1}{w_{i+2}^2} \cdots \frac{1}{w_{k-1}^2}.
\end{aligned}$$

By putting $\ell := i+1$, we obtain the definition of the Riccati equation.

DEFINITION 4. 1 *For any $n \times n$-matrix $A = (a_{ij})$, we define the Riccati equation as follows.*

$$\begin{aligned}
w_1^2 &= a_{11}, \\
w_k^2 &= a_{kk} + \sum_{\ell=1}^{k-1} \sum_{\rho \in S(\ell,k)} \frac{\varepsilon(\rho) a_{\ell\rho(\ell)} a_{\ell+1\rho(\ell+1)} \cdots a_{k\rho(k)}}{w_\ell^2 w_{\ell+1}^2 \cdots w_{k-1}^2}, \qquad k = 2, \dots, n.
\end{aligned} \tag{12}$$

The following theorem states the relationship between the conjugate point and the Riccati equation, and it is an immediate consequence of the definition of the Riccati equation and the change of variables (11).

THEOREM 4. 1 *If $k \geq 1$ is conjugate to 1, then the Riccati equation has a solution $w_1, \dots, w_{k-1} \in R/\{0\}$, but it has no solution $w_k \in R/\{0\}$. Conversely, if $w_1, \dots, w_{k-1} \in R/\{0\}$ satisfy the Riccati equation, and if there is no $w_k \in R/\{0\}$ satisfying the Riccati equation, then $k \geq 1$ is conjugate to 1.*

THEOREM 4. 2 *A sufficient condition for an extremal $\bar{x}$ to be a minimum for (P_0) is that the Riccati equation has non-zero real solution w_k $(k = 1, \dots, n)$ for the Hesse matrix $A = f''(\bar{x})$.*

5 Perfect squares

As we have seen at the beginning of Section 4, when A is a positive-definite tridiagonal matrix, the quadratic form x^TAx can be expressed as a summation of n perfect squares. This is true for an arbitrary positive-definite matrix A, see [8]. The aim of this section is to show that the solution $\{w_k\}$ of the Riccati equation shares coefficients of the perfect squares and that the Riccati equation is nothing but the recursion relation of the pivots of the Hesse matrix.

LEMMA **5. 1** *Let A be an $n \times n$-symmetric matrix, and divide A as follows.*

$$A = \begin{pmatrix} \alpha & a^T \\ a & B \end{pmatrix}, \tag{13}$$

where $\alpha \in R$, $a \in R^{n-1}$, and B is an $(n-1) \times (n-1)$-symmetric matrix. Then $A > 0$ if and only if $\alpha > 0$ and $B - aa^T/\alpha > 0$, and A is nonsingular if and only if $\alpha \neq 0$ and $B - aa^T/\alpha$ is nonsingular.

LEMMA **5. 2** *Let A be an $n \times n$-matrix, and divide A as follows.*

$$A = \begin{pmatrix} B & a \\ b^T & \alpha \end{pmatrix}, \tag{14}$$

where $\alpha \in R$, a, $b \in R^{n-1}$, and B is an $(n-1) \times (n-1)$-symmetric matrix. Assume that B is nonsingular. Then $|A| = |B|(\alpha - b^TB^{-1}a)$. Furthermore, if $\alpha - b^TB^{-1}a \neq 0$, then

$$A^{-1} = \begin{pmatrix} B^{-1} + \dfrac{B^{-1}ab^TB^{-1}}{\alpha - b^TB^{-1}a} & \dfrac{-B^{-1}a}{\alpha - b^TB^{-1}a} \\ \dfrac{-b^TB^{-1}}{\alpha - b^TB^{-1}a} & \dfrac{1}{\alpha - b^TB^{-1}a} \end{pmatrix}. \tag{15}$$

Lemma 5. 1 leads us to a method to express the quadratic form x^TAx as a summation of n perfect squares when A is positive-definite.

Step 1 Divide A and x as (13) and $x^T = (x_1,\ y^T) \in R \times R^{n-1}$, respectively.

Step 2 $x^TAx = \alpha\left(x_1 + \dfrac{a^Ty}{\alpha}\right)^2 + y^T\left(B - \dfrac{aa^T}{\alpha}\right)y.$

Step 3 Choose $B - \dfrac{aa^T}{\alpha}$ as A, and go to Step 1.

This procedure is rephrased as follows.

$$x^T Ax = \sum_{k=1}^{n} a_{kk}(k)\left\{x_k + \frac{\sum_{j=k+1}^{n} a_{kj}(k)x_j}{a_{kk}(k)}\right\}^2, \tag{16}$$

where $x^T = (x_1, \dots, x_n)$ and $a_{ij}(k)$ is inductively defined by

$$a_{ij}(1) := a_{ij}, \quad 1 \le i,\ j \le n \tag{17}$$

and

$$a_{ij}(k+1) := a_{ij}(k) - \frac{a_{ik}(k)a_{kj}(k)}{a_{kk}(k)}, \quad k+1 \le i,\ j \le n. \tag{18}$$

Indeed, it follows from Step 2 that

$$\begin{aligned} x^T Ax &= a_{11}\left(x_1 + \frac{\sum_{j=2}^{n} a_{1j}x_j}{a_{11}}\right)^2 + (x_2, \dots, x_n)\left(B - \frac{aa^T}{a_{11}}\right)\begin{pmatrix} x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= a_{11}\left(x_1 + \frac{\sum_{j=2}^{n} a_{1j}x_j}{a_{11}}\right)^2 + \sum_{i=2}^{n}\sum_{j=2}^{n}\left(a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}}\right)x_i x_j. \end{aligned}$$

Since the update rule (18) is same with that of the Gaussian elimination, $a_{kk}(k)$ equals to k-th pivot of A, that is,

$$a_{kk}(k) = \frac{|A_k|}{|A_{k-1}|}, \tag{19}$$

where $A_k := (a_{ij})_{1 \le i,j \le k}$, see [8]. Hence we obtain

$$w_k^2 = a_{kk}(k). \tag{20}$$

The following theorem provides an explicit representation of $a_{ij}(k)$.

Theorem 5. 1 *Let $A = (a_{ij})$ be a nonsingular symmetric matrix of size n, let $a_{ij}(k)$ be the sequence defined by (17) and (18). Then*

$$a_{ij}(k) = \frac{\begin{vmatrix} A_{k-1} & a_j \\ a_i^T & a_{ij} \end{vmatrix}}{|A_{k-1}|} \tag{21}$$

for $k = 1, \dots, n$, where $|A_0| := 1$, $A_k := (a_{ij})_{1 \le i,j \le k}$, and $a_i^T := (a_{i1}, \dots, a_{ik-1}) = (a_{1i}, \dots, a_{k-1i})$. In particular, $a_{kk}(k) = w_k^2$. So the Riccati equation is the recursion relation of the pivots of A.

6 Contrained problem

In this section, we extend the previous results to the constrained problem (P). The following technique was presented in [5].

Let $\bar{x}$ be a feasible solution for (P), and $I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}$. Assume that

$$\{g_i'(\bar{x}) : i \in I(\bar{x}) \cup J\} \quad \text{are linearly independent.} \tag{22}$$

Then, a sufficient condition for a feasible solution $\bar{x}$ be a minimum is that there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in R^m$ such that

$$L'(\bar{x}) = 0, \tag{23}$$

$$\lambda_i > 0 \quad \forall i \in I(\bar{x}), \tag{24}$$

and

$$\xi^T L''(\bar{x})\xi > 0 \quad \forall \xi \neq 0 \quad \text{satisfying} \quad (g_i'(\bar{x})\xi = 0 \quad \forall i \in I(\bar{x}) \cup J), \tag{25}$$

where $L(x) := f(x) + \sum_{i=1}^m \lambda_i g_i(x)$, see Fiacco and McCormick[1].

Next, let $k := |I(\bar{x}) \cup J|$ and G' denote the $k \times n$-matrix whose row vectors are $\{g_i'(\bar{x}) : i \in I(\bar{x}) \cup J\}$. Then it follows from (22) that rank$G' = k$, so that G' can be divided as $G' = (B, N)$, where B is a $k \times k$-nonsingular matrix and N a $k \times (n-k)$-matrix. Similarly, we divide $\xi \in R^n$ as $\xi = (y, z) \in R^k \times R^{n-k}$. Then $G'\xi = 0$ is equivalent to $y = -B^{-1}Nz$, so that

$$\xi^T L''(\bar{x})\xi = z^T(-N^T B^{-T}, I)L''(\bar{x}) \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix} z. \tag{26}$$

Hence, (25) is equivalent to

$$M := (-N^T B^{-T}, I)L''(\bar{x}) \begin{pmatrix} -B^{-1}N \\ I \end{pmatrix} > 0. \tag{27}$$

Therefore, All the results for (P_0) in Section 4 can be extended to (P) by replacing A with M.

THEOREM 6. 1 *A sufficient condition for a feasible solution of (P) be a minimum is that there exists $\lambda \in R^m$ satisfying (23), (24), and that the Riccati equation for M has a non-zero real solution $\{w_k\}$ $(k = 1, \ldots, n)$.*

7 Conclusion

We close this paper with giving a table that shows the correspondence between the simplest problem in the calculus of variations (SP) and the unconstrained nonlinear programming problem (P_0). This paper has added the last two rows. In the following table, $A = (a_{ij})$ denotes the Hesse matrix $f''(\bar{x})$, $A_k := (a_{ij})_{1 \leq i,j \leq k}$, and $y_k := |A_k|$.

(SP)	(P_0)
$x = x(t)$	$x = (x_1, \dots, x_n)$
$t \in [0, T]$	$k \in \{1, \dots, n\}$
Sufficently small neighborhood of t	$\{k\}$
Euler equation	$f_{x_k}(\bar{x}) = 0 \quad \forall k = 1, \dots, n$
Legendre condition	$f_{x_k x_k}(\bar{x}) \geq 0 \quad \forall k = 1, \dots, n$
Strengthened Legendre condition	$f_{x_k x_k}(\bar{x}) > 0 \quad \forall k = 1, \dots, n$
Jacobi equation: $y(t)$	Difference equaiton of y_k
Conjugate point	$y_1 > 0, \dots, y_{k-1} > 0,\ y_k \leq 0$
Riccati equation: $w(t)$	Difference equation of the pivots of A
$w = -\frac{P\dot{y}}{y}$	$w_k^2 = \frac{y_k}{y_{k-1}}$

Table 8.1

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