ON DIRECT SUM BANACH SPACES AND UNIFORM NON-SQUARENESS

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Recently the strict convexity and the uniform convexity of the $\psi$-direct sum $X \oplus \psi Y$ of Banach spaces $X$ and $Y$ were characterized in [15, 12]. We shall characterize the uniform non-squareness of $X \oplus \psi Y$.

Let $N_a$ denote the family of all absolute normalized norms on $\mathbb{C}^2$, that is,

$$
\|(z, w)\| = \|(|z|, |w|)\| \quad \text{and} \quad \|(1, 0)\| = \|(0, 1)\| = 1,
$$

and let $\Psi$ denote the family of all continuous convex functions $\psi$ on $[0, 1]$ with $\psi(0) = \psi(1) = 1$ satisfying $\max\{1 - t, t\} \leq \psi(t) \leq 1$ ($0 \leq t \leq 1$). According to [3], the norms in $N_a$ and the convex functions in $\Psi$ correspond in a one-to-one way under the equation $\psi(t) = \|(1-t, t)\|$. Namely, for every element $\|\cdot\| \in N_a$ the function $\psi(t)$ defined by $\psi(t) = \|(1 - t, t)\|$ belongs to $\Psi$; and conversely for every element $\psi \in \Psi$, define

$$(1)\|(z, w)\|_\psi = \begin{cases} (|z| + |w|)\psi \left( \frac{|w|}{|z| + |w|} \right) & \text{if } (z, w) \neq (0, 0), \\ 0 & \text{if } (z, w) = (0, 0). \end{cases}$$

Then $\|(\cdot, \cdot)\|_\psi$ is a norm in $N_a$ and satisfies $\psi(t) = \|(1 - t, t)\|_\psi$.

In [15], the $\psi$-direct sum $X \oplus \psi Y$ of two Banach spaces $X$ and $Y$ was introduced as the direct sum $X \oplus Y$ with the norm $\|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi (x \in X, y \in Y)$. Recently the strict
convexity and the uniform convexity of $X \oplus_{\psi} Y$ were characterized in [15, 12]. In this note we characterize the uniform non-squareness of $X \oplus_{\psi} Y$. As an application we give an example of Banach spaces which are not uniformly convex but uniformly non-square.

Now recall that a Banach space $X$ is called uniformly non-square ([6]; cf. [2, 10]) provided there exists a $\delta$ $(0 < \delta < 1)$ such that, whenever $||(x - y)/2|| > 1 - \delta$, $||x|| = ||y|| = 1$, one has $||(x + y)/2|| \leq 1 - \delta$. $X$ is called strictly convex provided, if $||x|| = ||y|| = 1$, $x \neq y$, then $||\frac{x+y}{2}|| < 1$. $X$ is called uniformly convex if any $\epsilon > 0$ there is a $\delta$ $(0 < \delta < 1)$ such that, whenever $||x - y|| \geq \epsilon$, $||x|| \leq 1$, $||y|| \leq 1$, one has $||\frac{x+y}{2}|| < 1 - \delta$. As is well known, the notion of uniform non-squareness lies between uniform convexity and super-reflexivity. Also, it is well known that there exists a Banach space which is neither uniformly convex nor uniformly non-square but super-reflexive. (cf. [7], [1].) A function $\psi$ on $[0, 1]$ is called strictly convex if, for any $s, t \in [0, 1]$, $s \neq t$, and for any $c$ $(0 < c < 1)$, one has $\psi((1-c)s+ct) < (1-c)\psi(s) + c\psi(t)$.

**Theorem A** ([15, 12]). *Let $X$ and $Y$ be Banach spaces and let $\psi \in \Psi$. Then*

(i) *$X \oplus_{\psi} Y$ is strictly convex if and only if $X$ and $Y$ are strictly convex, and $\psi$ is strictly convex ([15, Theorem 1]).*

(ii) *$X \oplus_{\psi} Y$ is uniformly convex if and only if $X$ and $Y$ are uniformly convex, and $\psi$ is strictly convex ([12, Theorem 1]).*

Saito-Kato-Takahashi [13] gave the following characterization of the absolute norms on $\mathbb{C}^2$ which are uniformly non-square.
Proposition 1 ([13]). Let $\psi \in \Psi$. Then the following are equivalent.

(i) $(\mathbb{C}^2, \| \cdot \|_\psi)$ is uniformly non-square.
(ii) $\psi \neq \psi_1$ and $\psi \neq \psi_\infty$.

1. Monotonicity Property of Absolute Norms

We discuss the monotonicity property of absolute norms on $\mathbb{C}^2$ for later use. Recall the following fundamental facts. Proposition 2 played an essential role in the proof of Theorem A.

Lemma 1 ([2, p.36, Lemma 2]). Let $\| \cdot \| \in N_\alpha$.

(i) If $|p| \leq |r|$ and $|q| \leq |s|$, then $\|(p, q)\| \leq \|(r, s)\|$.
(ii) If $|p| < |r|$ and $|q| < |s|$, then $\|(p, q)\| < \|(r, s)\|$.

Proposition 2 (Takahashi, Kato and Saito [15]). Let $\psi \in \Psi$. Then the following assertions are equivalent:

(i) If $|z| \leq |u|$ and $|w| < |v|$, or $|z| < |u|$ and $|w| \leq |v|$, then $\|(z, w)\|_\psi < \|(u, v)\|_\psi$.
(ii) $\psi(t) > \psi_\infty(t)$ for all $t \in (0, 1)$.

A more precise (component-wise) result is given in [15]. Next we present a condition on $(z, w)$ and $(u, v)$ for which the above assertion (i) is valid (component-wise) for a general $\psi \in \Psi$.

Proposition 3. Let $\psi \in \Psi$ and let $(z, w)$, $(u, v) \in \mathbb{C}^2$.

(i) Let $|z| < |u|$ and $|w| = |v|$. Then $\|(z, w)\|_\psi = \|(u, v)\|_\psi$ if and only if $\|(z, w)\|_\psi = |w|$.
(ii) Let $|z| = |u|$ and $|w| < |v|$. Then $\|(z, w)\|_\psi = \|(z, v)\|_\psi$ if and only if $\|(z, w)\|_\psi = |z|$.

Proposition 3 is important in the proof of the uniform non-squareness of $X \oplus_\psi Y$. 
3. Uniform Non-squareness of $X \oplus_{\psi} Y$

We need the following lemma.

**Lemma 2.** Let $\{x_n\}$ and $\{y_n\}$ be sequences in a Banach space $X$ whose norms are convergent to non-zero limits.

(i) $\lim_{n \to \infty} \|x_n + y_n\| = \lim_{n \to \infty}(\|x_n\| + \|y_n\|)$.

(ii) $\lim_{n \to \infty} \left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| = 2$.

By Proposition 3 and Lemma 2, we obtain the following main theorem.

**Theorem 1.** Let $X$ and $Y$ be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.

(i) $X \oplus_{\psi} Y$ is uniformly non-square.

(ii) $X$ and $Y$ are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.

Now consider the Lorentz $\ell_{p,q}$-norm $\| \cdot \|_{p,q}$,

$$1 \leq q \leq p \leq \infty, \quad q < \infty$$

$$\|(z_1, z_2)\|_{p,q} = \left\{ z_1^{*q} + 2^{(q/p)-1}z_2^{*q} \right\}^{1/q},$$

where $\{z_1^*, z_2^*\}$ is the non-increasing rearrangement of $\{|z_1|, |z_2|\}$.

(Note that in case of $1 \leq p < q \leq \infty$, $\| \cdot \|_{p,q}$ is not a norm but a quasi-norm (cf. [8], [16, p.126]). Clearly $\| \cdot \|_{p,q}$ is an absolute normalized norm and the corresponding convex function $\psi_{p,q}$ is given by

$$\psi_{p,q}(t) = \left\{ \begin{array}{ll} \{(1 - t)^{q} + 2^{q/p-1}t^{q}\}^{1/q} & \text{if } 0 \leq t \leq 1/2, \\
(2t^{q} - 2^{q/p-1}(1 - t)^{q})^{1/q} & \text{if } 1/2 \leq t \leq 1. \end{array} \right.$$
Then $\psi_{p,q}$ yields the $\ell_{p,q}$-sum $X \oplus_{p,q} Y$:

$$
\|(x, y)\|_{p,q} = \left\{ \max(\|x\|^q, \|y\|^q) + 2^{(q/p)-1} \min(\|x\|^q, \|y\|^q) \right\}^{1/q}
$$

(3)

**Corollary 1.** Let $1 \leq q \leq p \leq \infty$ and not $p = q = 1, \infty$. Then, $\ell_{p,q}$-sum $X_1 \oplus_{p,q} X_2$ is uniformly non-square if and only if $X_1$ and $X_2$ are uniformly non-square.

In particular, $\ell_p$-sum $X_1 \oplus_p X_2, 1 < p < \infty$, is uniformly non-square if and only if $X_1$ and $X_2$ are uniformly non-square.

Theorem A and Theorem 1 easily gives an example of Banach spaces which are not uniformly convex but uniformly non-square.

**Example 1** (cf. [12, 13]). Let $X$ and $Y$ be uniformly convex Banach space and let $1/2 < \alpha < 1$. Now we define $\psi_\alpha \in \Psi$ by

$$
\psi_\alpha(t) = \begin{cases} \frac{\alpha-1}{\alpha} t + 1 & \text{if } 0 \leq t \leq \alpha, \\ t & \text{if } \alpha \leq t \leq 1. \end{cases}
$$

(4)

Then the norm of $X \oplus_{\psi_\alpha} Y$ is given by

$$
\|(x, y)\|_{\psi_\alpha} = \max\{\|x\| + (2 - \frac{1}{\alpha})\|y\|, \|y\|\}.
$$

(5)

$X \oplus_{\psi_\alpha} Y$ is an example of uniformly non-square Banach spaces without uniform convexity.
References


